Modelling of resonance and stability of drill string nonlinear dynamics

Askat K. Kudaibergenov, Askar K. Kudaibergenov, Lelya A. Khajiyeva

***Abstract*—**This paper studies nonlinear vibrations and stability of a rotating drill string applied in shallow drilling. It is supposed that the drill string is under the effect of a variable compressive axial load with consideration of finite deformations. Taking the drill string as a pinned-pinned rod its lateral vibrations and resonance regimes on basic and higher (the third) frequencies are modelled. The classical Galerkin technique and the method of harmonic balance are utilized. The investigations show considerable nonlinear effects and bifurcation phenomena on the amplitude-frequency characteristics (AFC) of the drill strings that may indicate instability of the studied process. Instability zones of the resonance on the basic frequency, which correspond to a frequency range of bifurcation effects on the AFC, are determined. Numerical analysis of the mathematical model is performed and recommendations for choosing optimal constructive parameters of the drill strings and their operating regimes are provided.

***Keywords*—**Drill string, nonlinear model, lateral vibrations, resonance, stability.

I. Introduction

One of the main problems of machine dynamics is a problem of dynamic stabilization of various mechanical structures. It is of utmost value for rod elements, including rotating drill strings applied in oil and gas extracting industry.

Investigation of drilling system stability has a great importance for improvement of efficiency of drilling operations, protection of expensive drill string components and avoidance of damage to borehole walls. Torsional stick-slip oscillations, high-amplitude lateral vibrations and axial bit-bounce motions of drill strings caused by their complicated dynamic behavior during the drilling process may cause severe technical failures of the drilling equipment [1, 2]. In order to minimize dangerous vibrations of drill strings it is necessary to recommend such combinations of an external load, rotation speed of the drill string and other system parameters, under which resonance regimes of vibration are not observed.

Furthermore, the drill string dynamics is highly nonlinear by its nature. It is due largely to flexibility of the drill string that can result in finite deformations of the string in view of its large length and effects of variable external loadings, in particular, the compressing axial force. Therefore, when investigating the drill string dynamics, it is necessary to consider its deformability to determine amplitudes of displacements with detecting dangerous resonant oscillating regimes of the drill string. In Al-Hiddabi's paper [3] research of nonlinear controller to decrease lateral and torsional vibrations accompanying the nonlinear drill string dynamics was carried out. It was established that application of the controller enabled one to eliminate torsional vibrations and considerably reduced lateral motions.

Problems of analyzing dynamic stability of mechanical systems and modelling of resonance regimes of their nonlinear vibrations mostly reduce to determination of unknown functions of displacements u_{ij}, strains ε_{ij} and stresses σ_{ij}, satisfying boundary conditions given and minimizing some functional $\Phi\left(u_{ij}, \varepsilon_{ij}, \sigma_{ij}\right)$. The classical Galerkin method [4, 5], the Rayleigh-Ritz approach [6, 7] and the finite element method [8, 9] are widely used amongst different variation methods for solution of these problems.

In [4] Vaz and Patel indicated that approximation by Galerkin's method could be successfully applied to research of the drill string dynamics. Importance of considering inertial forces to investigate stability of the drill string when performing drilling of vertical holes was also shown.

Amongst the early works contributing significantly to the development of the theory of elastic systems dynamic stability it is worth indicating the classical books of Bolotin, Hayashi, Vol'mir, Timoshenko and others. In [10] problems of dynamic instability of rods, rod systems and shells, and methods of their analysis are studied. In the work [11] methods for definition of stability of nonlinear systems and oscillations are considered in depth, and also their comprehensive analysis is carried out. A great deal of attention is given to determine stability of periodic vibrations. Various problems on stability of rods, plates and shells, particularly, at their dynamic loading are studied in [12]. The general theory of bending and stability of thin-walled rods, stability of a flat form of bending of straight

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and curvilinear beams, dynamic stability of homogeneous rectangular plates and circular cylindrical shells for various cases of external loadings and boundary conditions are presented in [13].

Some authors mainly focus only on stability of the drill string bottom-hole assembly (BHA) whereas research of stability of the whole drill string is still insufficiently studied. In [14] authors model stability of the lower part of a drill string, using the finite element method to integrate governing equations.

Other authors in their research on stability of drill strings apply the Lagrange technique with restrictions on the degrees of freedom, widely used in machine dynamics. In [15] a nonlinear dynamic model of lateral and axial vibrations of a rotating drill string, developed by the use of the Lagrange approach, was investigated. Modelling results pinpointed that the parametric resonance and wirl vibration phenomena in drilling machines, resulting in high-amplitude lateral vibrations of the drill strings, might occur simultaneously at appropriate work regimes of drilling.

Direct usage of the harmonic balance method for solution of nonlinear equations of vibration motion relative to one generalized time variable was shown in [16]. The authors presented benefits of application of the harmonic balance technique in combination with the nonlinear normal vibration modes approach and the modified Rauscher method under the analysis of multi-degree of freedom (DOF) systems having internal and external resonances.

In this paper nonlinear vibrations and stability of rotating drill strings for shallow drilling under the assumption of finiteness of their deformations are investigated. Modelling of the nonlinear system of the drilling machine as a system with distributed parameters can allow to enhance description of the drill string dynamics and the results obtained herein.

II. Statement of the Problem

One of the major problems of the drill string dynamics are resonance phenomena and the need to eliminate the operating frequencies of drill string motion from resonance regions in order to maintain stability of drilling.

Here the resonance phenomena of drill strings and their stability taking into account nonlinear complicating factors are examined. It is known that lateral vibrations of a drill string, modelled as a rotating elastic isotropic rod of symmetric cross-section, make the main contribution to the general oscillatory process, whereas the contribution of longitudinal and torsional vibrations is negligible in comparison with lateral ones. The largest amplitudes of longitudinal and torsional vibrations are several orders of magnitude less than the amplitude of lateral vibrations. Therefore, a mathematical model of lateral vibrations of the elastic rod with initial curvatures to analyze the resonance phenomena of drill strings under the influence of an external compressive load is used here.

Let us consider a global Cartesian coordinate system *Oxyz*. The axis of the drill string is assumed to be bent only in the *Oyz*-plane, i.e. flat bending of the elastic isotropic rod of length *l* with symmetric cross-section is examined (see Fig. 1).

A nonlinear mathematical model of the drill string lateral vibrations, based on the V.V. Novozhilov theory of finite deformations [17], is:

$$\rho A \frac{\partial^2 v}{\partial t^2} + EI_x \frac{\partial^4 v}{\partial z^4} - \rho I_x \frac{\partial^4 v}{\partial z^2 \partial t^2} + \frac{\partial}{\partial z}\left(N(z,t) \frac{\partial (v+v_0)}{\partial z} \right) - \\ - \frac{EA}{1-\nu} \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z} \right)^3 - \rho A \omega^2 v = 0, \quad (1)$$

where ρ is the mass density, A is the cross-section area of the drill string, $v(z,t)$ is the displacement of the flexural center of the cross-section along the y-axis owing to bending, E is Young's modulus, $I_x = \int_{A(z)} y^2 dA$ is the axial inertia moment, ν is Poisson's ratio, ω is the angular speed of the rod.

Boundary conditions for the rod with hinged ends are written as

$$v(z,t) = 0 \qquad (z = 0,\ z = l), \\ EI_x \frac{\partial^2 v(z,t)}{\partial z^2} = 0 \qquad (z = 0,\ z = l). \quad (2)$$

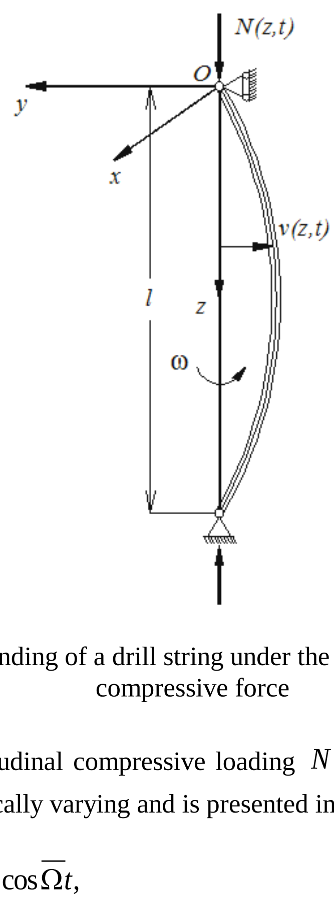

Fig. 1 flat bending of a drill string under the effect of an axial compressive force

The longitudinal compressive loading $N(z,t)$ is supposed to be periodically varying and is presented in the form:

$$N = N_0 + N_t \cos \overline{\Omega} t, \quad (3)$$

where N_0 and N_t denote constant and variable in time components, respectively; $\overline{\Omega}$ is the frequency of external effects.

III. MODELLING OF THE RESONANCE REGIMES

Modelling of the resonance in elastic systems can be reduced to investigation of the equations of motion in the following form [18]:

$$\ddot{f}_i + \Omega_{0i}{}^2 f_i + \Phi_i(\boldsymbol{f}) = F_i \cos\Omega t, \quad i = \overline{1, n}, \tag{4}$$

where $\boldsymbol{f} = \left(f_1(t), f_2(t), \ldots, f_n(t)\right)^{\mathrm{T}}$ is the vector of generalized (modal) parameters, Ω_{0i} is the i^{th} natural frequency, $\Phi_i(\boldsymbol{f})$ is the nonlinear term of the i^{th} equation of motion, and Ω is the frequency of external effects.

Nonlinear systems like (4) are widely applied to model motion of separate or coupled elements of different constructions and machines, describing nonlinear oscillations of systems with one DOF or with discrete masses. Besides, such equations can be used to simulate nonlinear oscillations of systems with distributed parameters.

Let us define the dimensionless time:

$$\tau = t \cdot \Omega_0, \tag{5}$$

where Ω_0 is the frequency of the drill string natural vibrations.

Then applying the Galerkin method, the lateral displacement $v(z,t)$ in the *Oyz*-plane is given by

$$v(z,t) = \sum_{i=1}^{n} f_i(t) \sin\left(\frac{i\pi z}{l}\right). \tag{6}$$

The initial curvature of the drill string has a smooth form. Hence, it can be presented in the form of a periodic trigonometric function:

$$v_0(z) = f_0 \sin\left(\frac{\pi z}{l}\right) \tag{7}$$

Considering the lateral vibrations of the drill string on the general form of bending of its axis, i.e. at $n = 1$ in (6), and taking into account (5), (7) we obtain an ordinary differential equation for the generalized time function $f(\tau)$:

$$\frac{d^2 f}{d\tau^2} + \left(1 + \beta\cos\Omega\tau\right) f + \alpha f^3 = F_0 + F_1 \cos\Omega\tau, \tag{8}$$

where

$$\beta = \frac{a_3}{a_2}, \quad \alpha = \frac{a_4}{a_2}, \quad F_0 = \frac{d_1}{a_2}, \quad F_1 = \frac{d_2}{a_2}, \quad \Omega = \frac{\overline{\Omega}}{\Omega_0}, \quad \Omega_0 = \sqrt{\frac{a_2}{a_1}}, \tag{9}$$

and

$$\begin{gathered} a_1 = \frac{\rho}{2l}\left(Al^2 + I_x \pi^2\right), \\ a_2 = \frac{1}{2l^3}\left[EI_x \pi^4 - N_0 \pi^2 l^2 - \rho A \omega^2 l^4\right], \\ a_3 = -\frac{N_t \pi^2}{2l}, \quad a_4 = \frac{3EA\pi^4}{8l^3(1-\nu)}, \\ d_1 = f_0 \frac{N_0 \pi^2}{2l}, \quad d_2 = f_0 \frac{N_t \pi^2}{2l}. \end{gathered} \tag{10}$$

Hereinafter the index “1” of the function $f(\tau)$ is omitted.

Investigation of the resonance regimes of the drill string motion can be reduced to analysis of amplitude-frequency characteristics of their lateral vibrations.

In nonlinear system (8) along with vibrations, which frequency coincides with frequency of the external force, higher and subharmonic oscillations can arise. The general method to solve such a system is expansion of the function $f(\tau)$ into the Fourier series with undefined coefficients. In the resonance case difference of phases between natural vibrations and external effects may have a great impact on the magnitude of amplitudes and the frequency of vibrations.

A. Basic Resonance

Considering the resonance on the basic frequency a solution of (8) can be approximated by a simple harmonic with frequency Ω:

$$f(\tau) = r_0 + r_1 \cos(\Omega\tau - \varphi_1). \tag{11}$$

On substituting (11) into (8) and applying the method of harmonic balance, we obtain the following system of equations defining the dependence between the amplitudes r_0, r_1 and the frequency Ω:

$$\begin{gathered} r_1^2\left(\lambda(r_0, r_1) - \Omega^2\right)^2 = \left(F_1 - \beta r_0\right)^2, \\ \frac{\beta r_1^2}{2\left(F_1 - \beta r_0\right)}\left(\lambda(r_0, r_1) - \Omega^2\right) \\ + r_0\left(\lambda(r_0, r_1) - \alpha\left(2r_0^2 - \frac{3r_1^2}{4}\right)\right) = F_0, \end{gathered} \tag{12}$$

where

$$\lambda(r_0, r_1) = 1 + 3\alpha\left(r_0^2 + \frac{r_1^2}{4}\right). \tag{13}$$

Assuming that there is no constant component in the expression for the axial load $N(z,t)$ (3) and allowing for only the first term in the Fourier expansion, i.e. $f = r_1\cos(\Omega\tau - \varphi_1)$, and taking into account resistance forces of the rock in (1), the nonlinear system (12) becomes identical to the amplitude-frequency dependence obtained in [19].

B. Resonance on Higher Frequencies

In nonlinear dynamic systems in view of existence nonlinear quadratic or cubic terms the resonance on higher frequencies can occur. Therefore, to analyze the resonance phenomena in details an approximate solution of (8) is written as follows:

$$f(\tau) = r_0 + r_1\cos(\Omega\tau - \varphi_1) + r_3\cos(3\Omega\tau - \varphi_3). \tag{14}$$

Substituting (14) into (8), using the harmonic balance method and eliminating the unknown phase angles φ_1 and φ_3 through some trigonometric transformations, we get a system of equations for the unknown amplitudes of vibrations r_0, r_1, r_3 and the frequency Ω:

$$\begin{aligned} & r_1^2\left[A_1\left(A_1 - 3A_3\frac{r_3^2}{r_1^2}\right) + \frac{3\alpha^2}{16}r_1^2\left(3r_3^2 - \frac{A_1}{A_3}r_1^2\right)\right] = F_1^2, \\ & A_3^2 r_3^2 = \frac{\alpha^2}{16}r_1^6, \\ & \frac{\beta}{2}r_1^2\left[A_1 - \frac{3\alpha}{32}\left(\frac{\alpha^2}{A_3}r_1^6 + 16A_3r_3^2\right)\right] = F_1(F_0 - A_0r_0), \end{aligned} \tag{15}$$

where

$$\begin{aligned} & A_0 = 1 + \beta + \alpha r_0^2 + \frac{3\alpha}{2}\left(r_1^2 + r_3^2\right), \\ & A_1 = -\Omega^2 + 1 + 3\alpha r_0^2 + \frac{3\alpha}{4}\left(r_1^2 + 2r_3^2\right), \\ & A_3 = -9\Omega^2 + 1 + 3\alpha r_0^2 + \frac{3\alpha}{4}\left(2r_1^2 + r_3^2\right). \end{aligned} \tag{16}$$

The amplitude-frequency characteristics (12), (15) depend on geometrical and physical parameters of the dynamic system. It allows to examine the effects of these parameters on the resonance regimes of the drill string lateral vibrations to separate the resonant frequencies from drilling operating frequencies or to control them.

IV. Numerical Results and Discussions

Numerical analysis of the basic and higher resonances of the nonlinear dynamic system (8), based on the amplitude-frequency relations obtained above, is conducted in the Wolfram Mathematica computational package. The influences of the drill string length, thickness of its wall, angular speed of rotation, axial compressive load and the magnitude of its initial curvature on the branches of resonance curves are investigated.

The dimensions and material properties of a pinned supported steel drill string are: $D = 0.2\,\text{m}$ (outer diameter of the drill string), $d = 0.12\,\text{m}$ (inner diameter), $E = 2.1\times10^5\,\text{MPa}$, $\rho = 7800\,\text{kg/m}^3$, $\nu = 0.28$.

In Fig. 2-7 resonance curves for various values of the drill string length, namely $l = 100\,\text{m}$ (tiny points), $l = 250\,\text{m}$ and $l = 500\,\text{m}$ (bold points) with the angular speed of rotation $\omega = 1.05\,\text{rad/s}$ are shown. The constant and variable parts of the longitudinal compressive load $N(z,t)$ are given as $N_0 = 1.7\,\text{kH}$, $N_t = 0.5\cos(\Omega\tau)\,\text{kH}$, respectively.

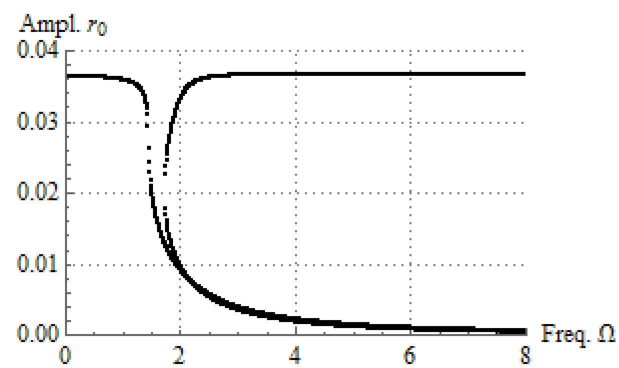

Fig. 2 Amplitude-frequency dependence $r_0(\Omega)$ for the case of basic resonance, $l = 100\,\text{m}$

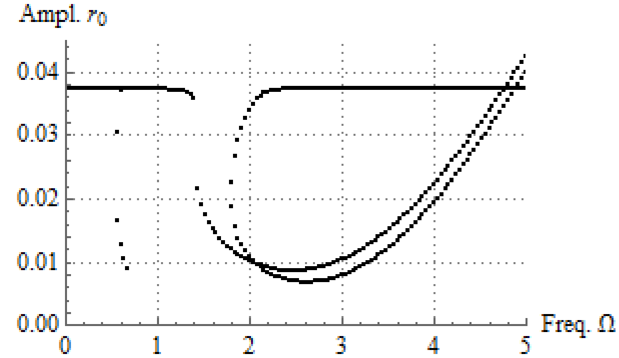

Fig. 3 Amplitude-frequency dependence $r_0(\Omega)$ for the resonance on higher frequencies, $l = 100\,\text{m}$

As shown in Fig. 4, resonance curves ($l = 100\,\text{m}$) stretch out to the right because of existence of geometrical nonlinearity in the system; meanwhile, shifting of the resonances curves towards the growth of external vibration frequency Ω takes place due to the initial curvature of the drill string axis. It is worth noting that the increase in the drill string length results in stretching the resonance curves out to the left, which is typical for mechanical systems with softening characteristics, and leads to instability of the system in the lower frequencies region. Moreover, the anomaly in the loop form for the simple harmonic appears at the range of amplitudes from 0.17 to 0.23 m.

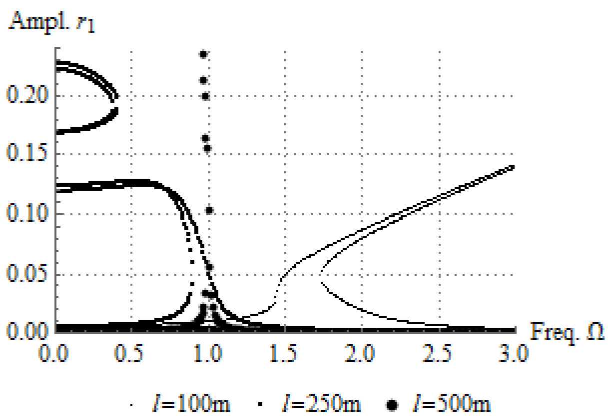

Fig. 4 the influence of the drill string length on the resonance curves of its nonlinear lateral vibrations on the first harmonic, $f_0 = 0.3\,\text{m}$

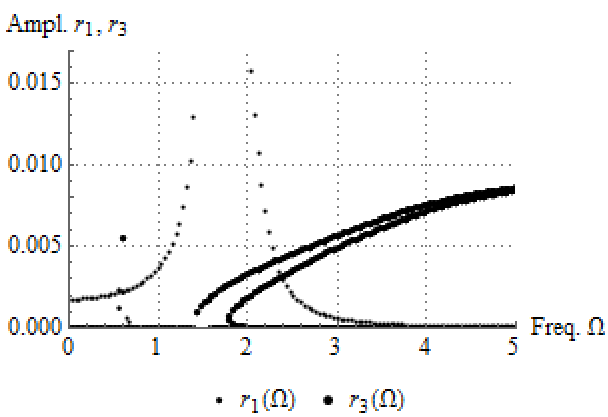

Fig. 5 resonance curves of 1st and 3rd harmonic vibrations of the drill string at the following values of parameters: $l = 100\,\text{m}$, $f_0 = 0.3\,\text{m}$

Allowing for the third harmonic in the approximate solution (14), the increase in the external frequency Ω causes the sharp bias of the resonance curves $r_3(\Omega)$ in the higher frequencies direction, which corresponds to much smaller amplitudes compared to the resonance curves $r_1(\Omega)$ (Fig. 5). In addition, one more resonance curve $r_3(\Omega)$ appears to the left of the basic resonance (Ω changes from 0.5 to 0.7) due to the influence of the third harmonic on the oscillatory process.

However, such a high value of the initial curvature of the drill string can be considered only in theoretical research, and if neither friction nor rigid contacts with borehole walls is taken into account.

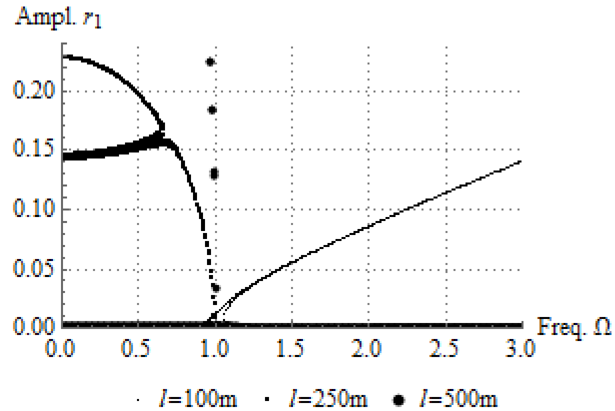

Fig. 6 the influence of the drill string length on the resonance curves of its nonlinear lateral vibrations on the first harmonic, $f_0 = 0.01\,\text{m}$

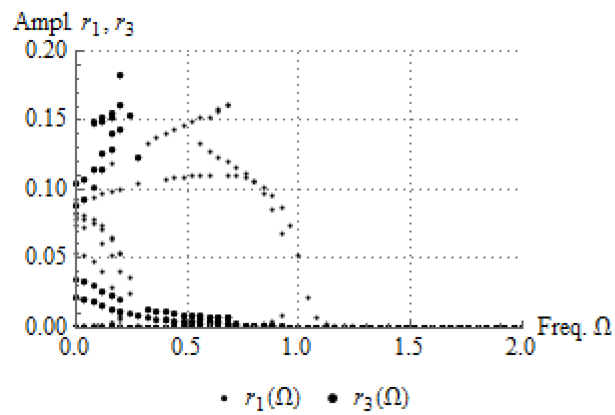

Fig. 7 resonance curves of 1st and 3rd harmonic vibrations of the drill string at the following values of parameters: $l = 250\,\text{m}$, $f_0 = 0.01\,\text{m}$

When the value of the initial curvature $f_0 \leq 0.01\,\text{m}$, no shifting of the resonance curves to a zone of higher frequencies of the external effect is observed, as illustrated in Fig. 6.

As can be seen from Fig. 7, when the amplitude-frequency characteristics of the basic resonance drop down, oscillations on the third harmonic take place, that is the rise in amplitude-frequency characteristics of the resonance on higher frequencies is observed in the bifurcation zones of the basic resonance.

Fig. 8 demonstrate the impact of the increase in the drill string angular speed of rotation on the resonance phenomena arising in the considered dynamic system. The results show that the resonance appears in the system on lower frequencies of the external effect when the angular speed of rotation of the drill string ($l = 100\,\text{m}$) increases significantly.

Resonance curves for different values of thickness of the drill string wall are illustrated in Fig. 9. Only the outer diameter of the drill string changes, the inner diameter remains unchanged $d = 0.12\,\mathrm{m}$. It is obvious that the decrease in the drill string thickness results in reduction of its cross-section area A and the value of its axial inertia moment I_x.

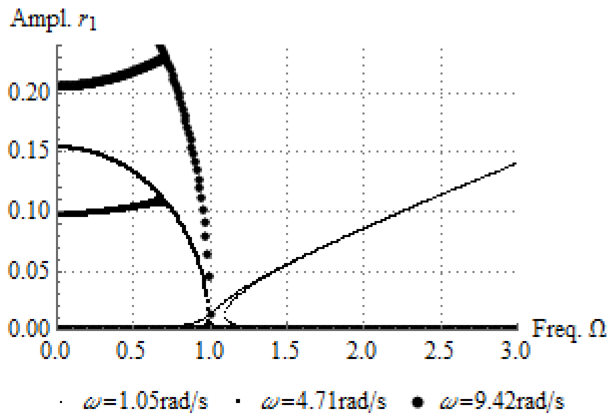

Fig. 8 the influence of the drill string angular speed of rotation on the resonance curves of its vibrations on the first harmonic at the following values of parameters: $l = 100\,\mathrm{m}$, $f_0 = 0.02\,\mathrm{m}$

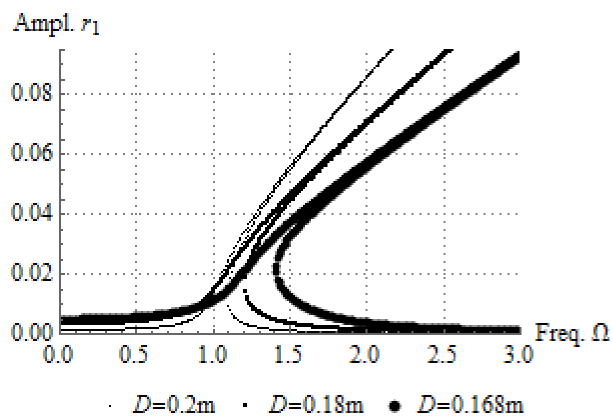

Fig. 9 the influence of the drill string wall thickness on the resonance curves of its vibrations on the first harmonic at the following values of parameters: $l = 100\,\mathrm{m}$, $\omega = 1.05$ rad/s, $f_0 = 0.02\,\mathrm{m}$

If follows from Fig. 9 that the branches of the resonance curves stretch out towards higher frequencies Ω with diminution of the drill string outer diameter D. Hence, the less thickness of the drill string walls, the greater influence of the geometrical nonlinearity on the resonance curves of its lateral vibrations.

Analysis of the resonance regimes of the drill string lateral vibrations depending on different values of the constant in time component N_0 of the axial compressive load $N(z,t)$ affecting the drill string is also carried out. The system parameters are given as $l = 100\,\mathrm{m}$, $D = 0.18\,\mathrm{m}$, $\omega = 1.05$ rad/s, $f_0 = 0.02\,\mathrm{m}$.

Fig. 10-11 show that the increase in the axial compressive load up to $5.5\,\mathrm{kN}$ results in stretching the branches of the resonance curves out to the right with simultaneous considerable shift of the curves to the region of higher frequencies. Similar results were obtained at high value of the initial curvature f_0 (Fig. 4-5).

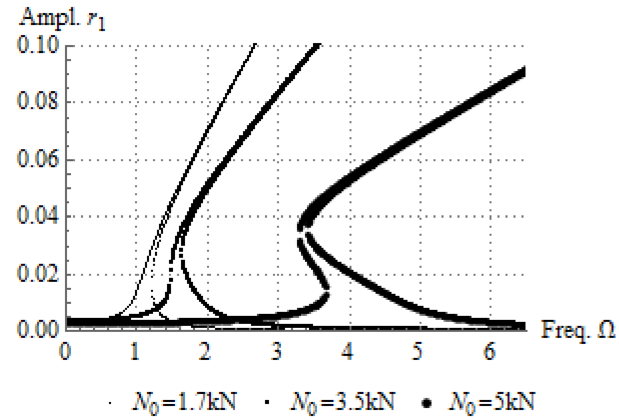

Fig. 10 the impact of the axial compressive load on the resonance curves of the drill string nonlinear vibrations on the first harmonic

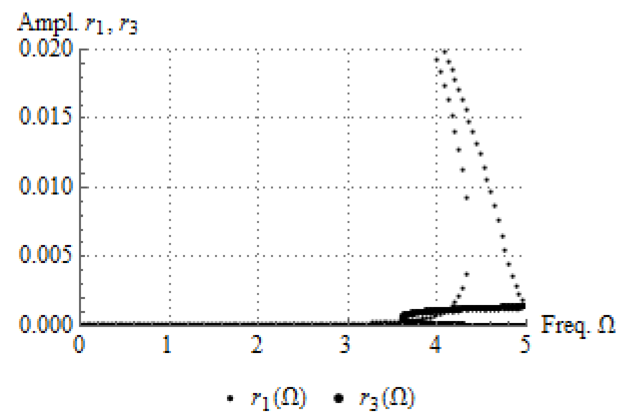

Fig. 11 resonance curves of 1st and 3rd harmonic vibrations of the drill string under the effect of the axial force $N(z,t) = 5.5\,\mathrm{kN}$

Consequently, in the bifurcation points of the amplitude-frequency characteristics of the drill string vibrations, presented on the constructed figures, one can determine instability zones of the resonance on basic and higher frequencies of the external effect.

The results of this research provide deeper insight into the nature of the drill string behaviour under the lateral vibrations with determination of their amplitude-frequency characteristics.

V. Stability of the Resonance on Basic Frequency

In order to study stability of resonance on a basic frequency we carried out the analysis of stability of harmonic solution (11). Considering small deviation δf of the periodic equilibrium state $f_0(\tau)$, an equation of the perturbed state of the dynamic state is found

$$\frac{d^2\delta f}{d\tau^2}+\left(1+\beta\cos\Omega\tau+3\alpha f_0^{\,3}\right)\delta f=0. \quad (17)$$

This equation is known in literature as the Mathieu equation [11]. Stability of $f_0(\tau)$ depends on the behaviour of the small parameter δf in time. This behaviour defines stability or instability of the basic resonance in accordance with the Lyapunov stability theory. The solution $f_0(\tau)$ is said to be unstable if the quantity δf increases indefinitely or "tends to infinity" at $\tau\to\infty$, and the solution $f_0(\tau)$ is said to be stable if the magnitude δf remains limited at $\tau\to\infty$.

According to the Floquet theory [11], the solution (17) is defined as:

$$\delta f=e^{\mu\tau}\eta(\tau). \quad (18)$$

Type of the quantity $\eta(t)$ indicates a zone of vibration instability. For the basic resonance

$$f_0=r_1\cos\left(\Omega\tau-\phi_1\right), \quad (19)$$

the generalized Hill equation in variations was obtained:

$$\frac{d^2\delta\eta}{d\tau^2}+\left(\theta_0+\sum_{n=1}^{2}\theta_{ns}\sin n\Omega\tau+\sum_{n=1}^{2}\theta_{nc}\cos n\Omega\tau\right)\delta\eta=0, \quad (20)$$

where

$$\theta_0=1+\frac{3}{2}\alpha r_1^{\,2},$$
$$\theta_{1c}=\beta,\quad \theta_{1s}=0, \quad (21)$$
$$\theta_{2c}=\frac{3}{2}\alpha r_1^{\,2}\cos 2\varphi_1,\quad \theta_{2s}=\frac{3}{2}\alpha r_1^{\,2}\sin 2\varphi_1.$$

Studying the behavior of the Hill equation solution one can determine stable or unstable states of the system. In this case the resonance on the basic frequency is under consideration.

VI. NUMERICAL ANALYSIS OF THE BASIC RESONANCE INSTABILITY ZONES

When determining the first instability zone of the basic resonance it is supposed that a frequency of the small perturbation δf coincides with the frequency of the periodic solution (11). For that η in formula (18) is given in the form:

$$\eta=b_1\cos\left(\Omega\tau-\psi_1\right). \quad (22)$$

Applying the method of harmonic balance, the characteristic determinant of the fourth order defining the boundaries of the first zone of instability on resonance curves on the basic frequency was constructed. In view of nonlinearity of the model it is necessary to construct the boundaries of the instability zones on the higher frequencies. Then, taking η as a series on multiple harmonics in the form

$$\eta=b_1\cos\left(\Omega\tau-\psi_1\right)+b_3\cos\left(3\Omega\tau-\psi_3\right), \quad (23)$$

the characteristic determinant of the eighth order describing the third instability zone was obtained.

Numerical analysis of the instability zones of the basic resonance of the drill string is conducted at the same parameter values that used when analyzing the resonance curves.

It was established that geometrical nonlinearity of the system has a significant influence on stretching the characteristic curves out to the region of higher frequencies at relatively small length of the drill string ($l=100\,\text{m}$). When increasing the drill string length, characteristic curves stretch out to the left ($l=250\,\text{m}$, $l=500\,\text{m}$), and unstable state of the system is observed on the lower frequencies of the external effect (Fig. 12). Similar results are obtained with increase in the rotation speed of the drill string (Fig. 13).

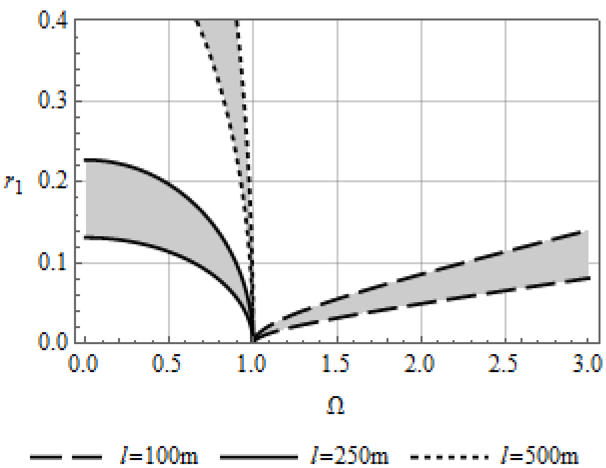

Fig. 12 the influence of the drill string length on the boundaries of 1st instability zone of the basic resonance

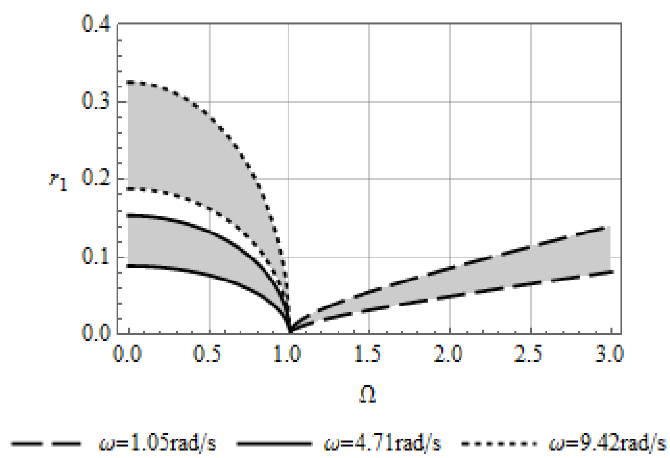

Fig. 13 the influence of the drill string speed of rotation on the boundaries of 1st instability zone of the basic resonance

Change of the drill string wall thickness causes instability of the harmonic solutions at high amplitudes of vibrations (Fig. 14).

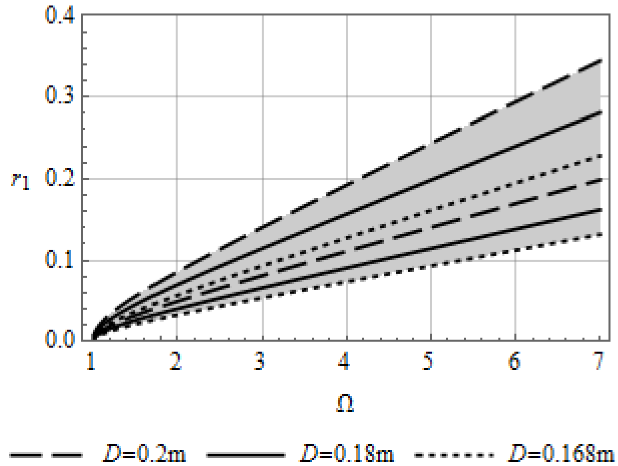

Fig. 14 the influence of the drill string wall thickness on the boundaries of 1st instability zone

More precise definition of the instability zones of the basic resonance (the third zone) is the evidence of essential correction to the earlier obtained results for the first instability zone. It also shows possibility of appearing the parametric resonance on multiple frequencies, which may occur in nonlinear systems (Fig. 15-16).

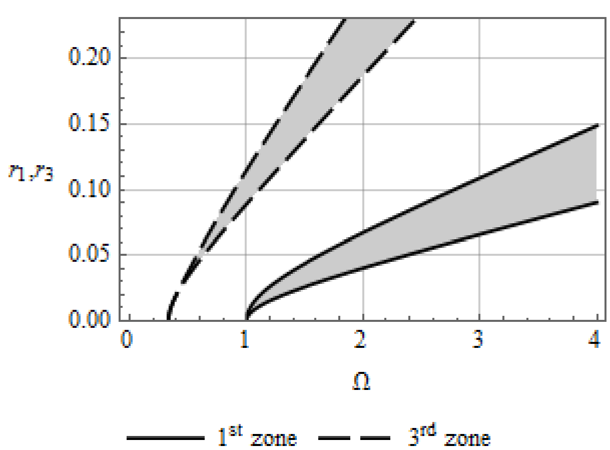

Fig. 15 instability zones of the basic resonance at $l = 100\,\text{m}$, $\omega = 1.05$ rad/s, $D = 0.18\text{m}$

Results of the numerical analysis for the boundaries of the first and the third instability zones of the resonance on the basic frequency bring into accord with the results of the numerical analysis of the amplitude-frequency characteristics of the drill string basic resonance, which are given above.

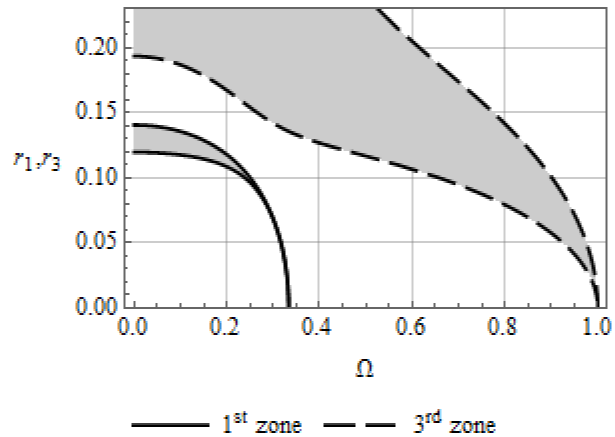

Fig. 16 instability zones of the basic resonance at $l = 250\,\text{m}$, $\omega = 1.05$ rad/s, $D = 0.2\text{m}$

VII. Conclusion

As a result of the qualitative and quantitative analysis of the nonlinear model of the drill string motion it was established that emergence of the resonance on higher harmonics in the system has a considerable impact on stability of the oscillatory process. The rise in the amplitude-frequency characteristics of the resonance on the third harmonic in bifurcation zones of the amplitude-frequency characteristics of the basic resonance was observed. Especially sharp leap of the amplitude of the basic resonance takes place when the axial compressive load several times increases.

It is worth indicating that the significant increase in the drill string length and the angular speed of its rotation causes occurrence of considerable nonlinear effects of the drill string amplitude-frequency characteristics, which is typical generally for the dynamic systems with softening characteristics.

The results of this research show that geometrical nonlinearity of the models, describing the drill string dynamics, make a great contribution to the results of dynamic analysis of the drill string stability.

By these reasons, modelling of resonance regimes of the drill string dynamics along with the analysis of its stability has a great importance for development of drilling equipment and improving its dynamic characteristics. In doing so, it is essential to take into account the geometrical nonlinearity of the system and the initial curvature of the drill string.

Analysis of the instability zones of the basic resonance showed the possibility of the resonance occurrence on multiple frequencies (the third, in particular). At that, "jump" phenomena and the increase in the AFC on the third harmonic were observed. The investigations allowed to find the resonance zones on multiple frequencies when the basic resonance was studied. This correction for the range of resonance frequencies might enable to eliminate them from the drill string operating regimes.

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