Numerical Solution of a Delay-Advanced Equation from Acoustics

M. Filomena Teodoro, *Member, CEMAT/CINAV*

Abstract—It is introduced a numerical scheme which approximates the solution of a particular non-linear mixed type functional differential equation from physiology, the mucosal wave model of the vocal oscillation during phonation. The mathematical equation models a superficial wave propagating through the tissues. The numerical scheme is adapted from the work developed previously by the author and collaborators.

Keywords—Mixed-type functional differential equations, non linear equations, vibration of elastics tissues, numerical approximation, method of steps, human phonation, mechanical system.

I. Introduction

IN applied sciences, many mathematical models show up functional differential equations with delayed and advanced arguments, the mixed type functional differential equations (MTFDEs).

Functional differential equations of mixed type appear in numerous and distinct areas of knowledge such as optimal control [1, 2], economic dynamics [3], nerve conduction [4, 5, 6, 7] and traveling waves in a spatial lattice [8, 9], quantum photonic physics [10].

We are particularly interested in the numerical approximation of the multi-delay-advance differential equation

$$x'(t) = F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)), \qquad (1)$$

where the shifts τ_i may take negative or positive values.

Some recent numerical methods to approximate the solution of a particular case of (1), a linear MTFDE with symmetric shifts (2), were introduced in [11, 12] and improved in [13, 14, 15].

$$x'(t) = F(t, x(t), x(t-\tau), x(t+\tau)), \quad \tau > 0. \qquad (2)$$

More recently, these algorithms were adapted and used to solve numerically a nonlinear MTFDE [7, 16], the FitzHugh-Nagumo equation that models the nervous conduction in an myelinated axon.

Presently, we pretend to calculate the numerical solution of a particular case of a nonlinear MTFDE which describes the dynamical behavior (vibration) of some elastics tissues, by the interaction of a flowing fluid (air, blood,...) with an elastic structure tissue, the aero-elastic oscillatory phenomena (AOP). The AOP occurs frequently in physiology. Particularly, the considered model characterizes the oscillation of a superficial wave propagating through the tissues in the direction of the flow. This model was initially introduced at the late eighties of 20th century by the author of [17, 18]. He proposed the mucosal wave model where a surface wave represents the motion of vocal tissues. Later, some variants of the this model where introduced in several studies in phonation dynamics (e.g. see [19]).

In [20], a preliminary approach to the mucosal wave model was introduced where a numerical scheme was adapted from algorithms resulting from the earlier work described in [7, 16, 21]. In [22], this work was extended using a non-uniform mesh. Using some ideas presented in [23, 24], where homotopic analytical method (HAM) is used to solve linear and non linear delay differential equations, the authors of [25] presented a preliminary approach where HAM was applied to get the solution of the mixed type differential equation under study. In this article, we extend the previous results using a collocation method (COL), a finite element method (FEM), method of steps (MS) and Newton method (NM) to obtain the approximate solution of the mucosal wave model.

The outline of this work consists in six sections and two appendixes. Section 2 describes the problem and analyzes the equation to solve. The numerical approach is described in Section 3. Section 4 displays some details about NM. In fifth Section, we obtain some numerical results. By last we get some conclusions.

II. The problem

We intend to compute the numerical solution of an equation from acoustics which is associated to mucosal wave model of the vocal oscillation during phonation.

We consider the mechanical behavior of tissues focused at mid point of glottis. The mathematical model that describes the displacement of tissue $x(t)$ at the midpoint of the glottis is obtained imposing the following assumptions:

(i) The pressure at exit of glottis (P_g) equals the atmospheric pressure;
(ii) The sub-glottal pressure equals the lung pressure (P_L);
(iii) The air flow is incompressible, frictionless and stationary;
(iv) The glottis is open ($a_1 > 0$).

We can get the equation of motion, a nonlinear MTFDE with deviating arguments, given by

$$Mx''(t) + Bx'(t) + Kx(t) = P_g \qquad (3)$$

with P_g, the average glottal pressure. In Appendix A can be found some details about the mucosa wave model. For complete description of this model, please consult [18, 17].

Prof. Teodoro is with CEMAT, Center for Computational and Stochastic Mathematics, Instituto Superior Técnico, Lisbon University, Av. Rovisco Pais, 1, 1048-001 Lisboa, Portugal and CINAV, Center of Naval Research, Portuguese Naval Academy, Portuguese Navy, Base Naval de Lisboa, Alfeite, 1910-001 Almada, Portugal e-mail: maria.alves.teodoro@marinha.pt; alternative e-mail: mteodoro64@gmail.com.

Manuscript received April 30, 2017.

Equation (3) can be rewritten in form

$$Mx''(t)+Bx'(t)+Kx(t)=\frac{P_L}{k_t}\frac{x(t-\tau)-x(t+\tau)}{x_0+x(t+\tau)}, \qquad (4)$$

where $x_0+x(t+\tau)>0$ (glottis is open) and τ is the time that the wave travels to the edges of vocal fold at $z=\pm T/2$ (upper and lower edges respectively), is given by $\tau=T/2c$, where c is the velocity of the wave and T the vocal fold thickness.

The parameters M, B, K and k_t are, respectively, the effective mass, damping and stiffness per area unit of vocal fold medial surface and the transglottal pressure coefficient. The model (4) is also applied in other physiological systems such as avian syrinx, snore, or a flow passing a constricted channel (artery, lips, soft palate, nostrils).

Equation (4) can be transformed in a non-dimensional model after an adequate change of variable introduced in [19], $u=x/x_0$,

$$u''(t)+\alpha u'(t)+\omega^2u(t)=p\frac{u(t-\tau)-u(t+\tau)}{1+u(t+\tau)}, \qquad (5)$$

where $1+u(t+\tau)>0$, $p=\frac{P_L}{k_tx_0M}$, $\alpha=B/M$ and $\omega=\sqrt{K/M}$.

The model (5) can also be represented as a bidimensional form

$$\begin{cases} u'(t)=v(t), \\ v'(t)=-\alpha v(t)-\omega^2u(t)+p\frac{u(t-\tau)-u(t+\tau)}{1+u(t+\tau)}, \end{cases} \qquad (6)$$

where $1+u(t+\tau)>0$.

As a brief note, an advantage of the rectangular glottis configuration is that it can clarify the importance of prephonatory glottal width x_0 for oscillation threshold through the following relation

$$P_L=\frac{k_t}{T}B_cx_0. \qquad (7)$$

The closer the vocal folds are brought together, the easier is to begin of small amplitude oscillation.

Another detail is that Titze [18] assumes small values of τ, making possible to consider the first order (linear) approximation of the expansion of solution (14) in a Taylor series around glottis midpoint $z=0$ (see 22 in appendix A).

When we consider that mucosal wave has a small time delay [18], equation (5) becomes an autonomous ordinary differential equation, analytically solvable for some values of parameters. This model is similar to the one introduced in [26]. The author of [19, 27] considered a more realistic issue: an arbitrary time delay for mucosal wave.

III. NUMERICAL METHODS

Some of the work introduced previously by the author and collaborators in [14, 15, 28], presenting different algorithms to solve numerically some autonomous and non-autonomous linear MTFDEs (1) with symmetric delay and advance, using collocation (COLL), least squares and finite element method (FEM) was considered. In [7, 16, 21], it was taken into account the numerical solution of a nonlinear MTFDE with deviating arguments arising from nerve conduction theory, taking the form (2).

In particular, to solve numerically (5), it is developed a numerical scheme based on work presented in [14, 28, 21]. Before proceed, we notice that (5) is a second order equation, so we can use the bi-dimensional formulation (6), and adapt the method of steps presented in [14] to system (6), similarly to the approach introduced in[21].

For the bi-dimensional case, the formula (7) presented in Section 2 of [14] and recently verified in [29], is presented is the appendix B, where are exposed the details of teorem B (method of steps theorem). It is based on Bellman's method of steps for differential equations. In the linear case [14], one solves the equation over successive intervals of unitary length. In the case of [21], the equation is solved for successive intervals of length τ. Doing some algebraic manipulation and simplification, we get the formula

$$u(t+\tau)=-p_n(t)(u''(t)+\alpha u')+u(t-\tau)+g(u(t)), \quad t\in\mathbb{R} \qquad (8)$$

where $g(u(t))=-p_n(t)\omega^2u(t)$ and $p_n(t)$ polinomial function with order $n\in\mathbb{N}$.

Supposing that all the derivatives of u exist in $(a-2\tau,a]$, in order to simplify the calculations, we can use the simpler formula (8) to extend the solution for equation (5) on an interval $[a,a+k\tau]$ (where k is an integer and a some adequate value), starting from its initial values on $[a-2\tau,a]$; these starting values are calculated using the solution of equation (5) or (6) taking into account the small amplitude approximation, which can be found in formula (25) of [18]. After some computation, we may obtain explicitly the expressions for the solution successively in intervals $(a,a+\tau]$, $(a+\tau,a+2\tau)$, ... starting with

$$\begin{aligned} u(t+\tau) &= -p_n(t)(u''(t)+\alpha u'(t))+u(t-\tau)+g(u(t)), \\ & t\in(a,a+\tau]. \end{aligned} \qquad (9)$$

Using this process, we can extend the solution to any interval, provided that the initial functions in the first two intervals with length τ are smooth enough functions and satisfy some simple relationships.

IV. NEWTON'S METHOD

The problem is reduced to a BVP on a limited interval, using the solution of equation (5) under the approximation proposed by Titze (small amplitude delay) in formula (25) of [18] as a boundary function. A numerical solution of the problem (5) subject to some natural constrains is computed.

The nonlinear problem can be reduced to a sequence of linear problems by means of the NM. Can be found in [21] a detailed description about the NM iterative process. In order to enable the convergence of the Newton iteration process, we consider different values of a set of parameters. We also impose regularity conditions and boundary conditions. The system and all parameters can be updated for each iterate of the NM. The values of u in this system are computed, using the MS, and assuming that u satisfies the boundary conditions.

Then we can define u on a specific limited interval and extend it to the closest intervals using a recurrence formula (9).

The numerical schemes described here are generalizations from the algorithm presented in [7, 21], using a uniform mesh. They can be summarized in three steps:

- 1^{st} - The knowledge of initial boundary functions is essential to proceed and implement the numerical scheme. The initial step consists in the determination of the boundary conditions using the Titze approximation [18]:

$$\begin{cases} u(t) = \phi_0(t), & t \in [-R-\tau, -R]; \\ u(t) = \phi_1(t), & t \in [R, R+\tau], \end{cases} \tag{10}$$

  where $\phi_0(t)$ and $\phi_1(t)$ are the boundary functions, the solution of (5) using the small amplitude delay approximation (formula (25) in [18]), R is some positive real multiple of τ. Once the boundary functions are defined, we are able to apply the more recent approaches and techniques using the adapted method of steps (MS) for the nonlinear case (5). Using MS, we can extend the solution to any interval, provided that the initial functions in the first two intervals with a specific length (τ) are smooth enough functions and satisfy some simple relationships.

- 2^{nd} - Reduction the nonlinear Equation (5) to a sequence of linear equations using NM;

- 3^{rd} - The COLL and FEM were applied separately to linearized equation obtained by NM application.

V. Numerical results

The parameters of model were chosen accordingly with [18], page 1548. In table I are presented the absolute error ε_N (2-norm) and the estimated order of convergence $p = log_2\varepsilon_{2N}/log_2\varepsilon_N$ of approximate solution of (5) by COLL, when it is considered an uniform mesh, $x_0 = 0.04$ cm and $x_0 = 0.16$ cm.

N	$\varepsilon^{(1)}$	$p^{(1)}$	$\varepsilon^{(2)}$	$p^{(2)}$
16	$1.26e-2$		$1.00e-2$	
32	$3.15e-3$	1.98	$2.50e-3$	1.99
64	$7.86e-4$	1.99	$6.06e-4$	1.99
128	$1.96e-4$	2.00	$1.50e-4$	2.01

TABLE I
Absolute error ε and estimated order of convergence p for estimate solution of (5) by COL, using Titze approximation. Parameters defined in [18], page 1548; $^{(1)}$ $x_0 = 0.04$ cm and$^{(2)}$ $x_0 = 0.16$ cm. Partition size: N subintervals.

In table II are also computed the absolute error and the estimated order of convergence of approximate solution of (5), but using FEM, with $x_0 = 0.04$ cm and $x_0 = 0.16$ cm. At each table, the results are accurate.

When the COL method is applied, the absolute error is of order 2×10^{-4} for both set of parameters, with a partition of 128 sub-intervals. The estimated order of convergence p is compatible with the expected one, $p \approx 2$.

By other hand, when we apply the FEM, the results are more accurate. The absolute error is of order 4×10^{-6} for first set of parameters and 1×10^{-5} for the second set of parameter, when we take a partition with 128 sub-intervals. The estimated order of convergence p is lower than the expected one, $p \approx 2$ for the two set of parameters.

n	$\varepsilon^{(1)}$	$p^{(1)}$	$\varepsilon^{(2)}$	$p^{(2)}$
16	$2.9e-4$	2.00	$8.55e-3$	1.89
32	$7.26e-5$	2.02	$2.18e-4$	1.97
64	$1.80e-5$	2.02	$5.48e-5$	1.99
128	$4.46e-6$	2.09	$1.37e-5$	2.00

TABLE II
Absolute error ε and estimated order of convergence p for estimate solution of (5) by FEM, using Titze approximation. Parameters defined in [18], page 1548; $^{(1)}$ $x_0 = 0.04$ cm and$^{(2)}$ $x_0 = 0.16$ cm. Partition size: N subintervals.

The figures 1 and 2 illustratethe absolute error for each case (COL and FEM).

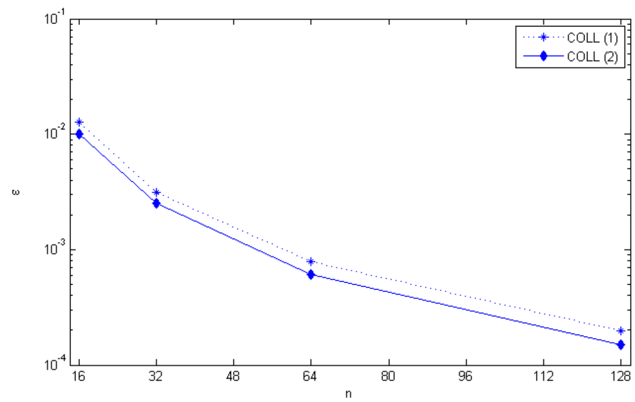

Fig. 1. Absolute error ε for estimate solution of (5) by COLL, using Titze approximation. Parameters defined in [18], page 1548; $^{(1)}$ $x_0 = 0.04$ cm and$^{(2)}$ $x_0 = 0.16$ cm. Partition size: N subintervals.

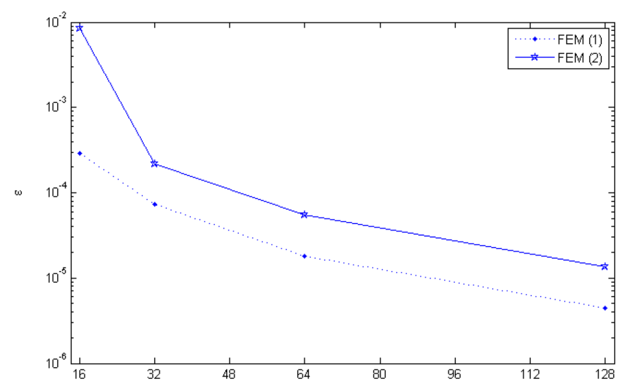

Fig. 2. Absolute error ε for estimate solution of (5) by FEM, using Titze approximation. Parameters defined in [18], page 1548; $^{(1)}$ $x_0 = 0.04$ cm and$^{(2)}$ $x_0 = 0.16$ cm. Partition size: N subintervals.

VI. Conclusion

Our initial proposal was to apply the more recent approaches and techniques using an adapted method of steps for the nonlinear case (5).

The method introduced previously in [16] and extended from [7], using a numerical scheme based on an adapted method of steps, was rebuilt and re-adapted, using a uniform mesh. Using MS, we could extend the solution to any interval, and provided that the initial functions in the first two intervals were smooth enough.

To solve numerically the equation on study, we consider an issue about a symmetrical system. Two different sets of parameters, the same that Titze used in [18], were tested and, in general, the results obtained by COLL and by FEM were accurate. The COLL method gave results consistent with the expected order when the convergence is guaranteed. The FEM conduced to order of convergence estimates lower than expected. It is still necessary to test another set of parameter values. As it happens when we consider the linear case, in larger intervals, the numerical solution is less accurate. The computation of numerical solution with a nonuniform mesh by COLL, FEM and finite differences have already preliminaries results which are under an evaluation and a validation process.

Based on the work in [30, 31], where radial basis functions are used as a tool to solve some nonlinear functional differential equations, in [32] we have made a first approach using such basis of functions but we still are in a very initial stage.

APPENDIX A
BASIC PRINCIPLES: MUCOSAL WAVE MODEL

A. Preliminaries

The first principle of vocal fold vibration is presented in [17] where the vocal fold oscillation is seen as a flow induced mechanical system. The glottal airstream and the yielding duct wall, the vocal folds, form a mechanical system that can show some instability under some flow conditions. In such case, a transfer of energy from the glottal airstream to the tissue will overcome frictional energy losses. The oscillation extent depends on the combination of inertial and elastic properties (mass and stiffness) and the geometry of vocal folds. When the aerodynamic driving force has a component in phase with tissue velocity, there is a positive flow of energy from the airstream. If we take a system which consists in a mass-spring oscillator

$$M\xi'' + B\xi' + K\xi = f(\xi', \xi, t) \tag{11}$$

where t is the time, M, B and K are mass, damping and stiffness, respectively, ξ, ξ' and ξ'' are the displacement, velocity and acceleration, f the diving force. The situation of most interest is when we get an autonomous differential equation. It occurs when f is not time dependent, meaning that the system oscillates by itself. We can notice that the oscillation depends on the relation between f and ξ': If f and ξ' have the save direction, as illustrated in figure (3), energy is transmitted to the mass, in opposite situation the energy is taken out the mass.

Whether the system has self oscillation or driven oscillation (whether if the hand in figure 3 is considered part of the system) the dynamic of system its described by an autonomous equation. In other case, we get a non-autonomous differential equation.

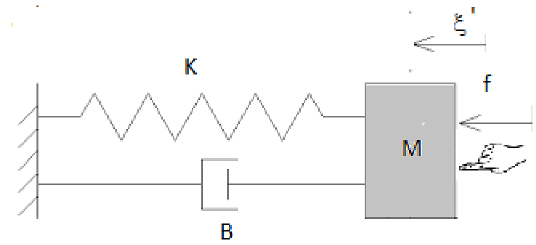

Fig. 3. Mechanical oscillator. Velocity ξ' and driving force f in same direction.

In [33], Libermann presents some of the pictures where the glottal airstream can provide a driving force which depends on velocity. In some way, the system needs to change the effective driving force on alternate cycles. The same force sucks the vocal folds together prior to closure works and invert direction so it can cancel partially the impulse resulting from prior to closure. In figure 3, the force applied by the hand needs to be reduced or reversed on the return, when the velocity direction invert.

This process of reverting the driving force direction is done using different mechanisms which can be simultaneous, such as deforming the glottal geometry so can exist different intra-glottal pressure distributions or making use of the oppositely phased the supraglottal and subglottal pressures.

B. The model

Returning to our objective, we intend to compute the numerical solution of the equation which is associated to mucosal wave model of the vocal oscillation during phonation.

This model was proposed by Titze in [18] and it can be represented by the geometric scheme presented in Figure 4.

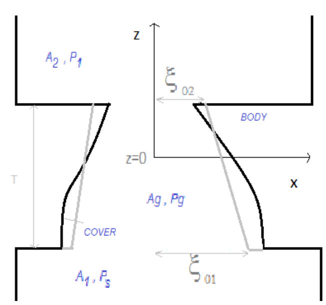

Fig. 4. Trapezoidal glottal configuration. Geometric scheme associated to mucosal wave model of the vocal fold oscillation during phonation. (Adapted from [18])

In figure 4 is represented the trapezoidal glottal configuration, the geometric scheme associated to mucosal wave model of the vocal fold oscillation during phonation, introduced in [18].

A_1 and A_2 are the subglottal and supraglottal areas, a_1, a_g and a_2 are the glottal areas at entry, midglottis and exit, respectively. A_1 and A_2 are constant, but glottal areas are space and time dependent.

We can see easily that the cross section area along glottis is given by

$$a(z,t) = 2L[\xi_0(z) + \xi(z,t)], \tag{12}$$

where t is the time, z is the vertical distance from the midpoint of the glottis in the direction of the airflow, $\xi(z,t)$ is the time-dependent displacement of tissue, ξ_0 (prephonatory position) is the half width at rest point (the vocal fold displacement at rest), and L is the vocal fold length (normal to the plane defined by the sheet of paper).

It is assumed left-right symmetry and the motion of tissues is done in the horizontal direction. In [34, 35] was shown that the wave propagates through the superficial tissues, in the upward direction of the airflow. In the simplest case, these waves can be represented using an one-dimensional wave equation with wave velocity c:

$$\frac{\partial^2\xi}{\partial t^2} = c^2\frac{\partial^2\xi}{\partial z^2}. \tag{13}$$

The solution of (13) is given by the general d'Alembert solution. The expression of tissue displacement is

$$\xi(z,t) = x(t-\frac{z}{c}), \tag{14}$$

where $x(t) = \xi(0,t)$ is the tissue displacement at midpoint of glottis. We can notice from (14) that the propagation of mucosal wave causes a time delay from bottom to top of vocal fold. Titze verifies in [18] that this delay helps to get some necessary instabilities for the oscillation of vocal fold.

If the prephonatory glottis has a linear (trapezoidal) dependence, see the grey line of scheme represented in figure 4, we get

$$\xi_0(z) = \frac{(\xi_{01}+\xi_{02})}{2} - (\xi_{01}-\xi_{02})\frac{z}{T}, \tag{15}$$

where ξ_{01} and ξ_{02} are the inferior and superior glottal half widths; T is the vocal fold thickness.

The time delay τ so the wave travels to the edges of vocal fold at $z = \pm T/2$, upper and lower edges respectively, is given by $\tau = T/2c$, which conduces to glottal upper and lower areas

$$\begin{aligned} a_1 &= 2L[\xi_{01} + x(t+\tau)], \\ a_2 &= 2L[\xi_{02} + x(t-\tau)]. \end{aligned} \tag{16}$$

Hence the equations (16) are defined at glottis midpoint, the bio-mechanical properties of the tissues are lumped at midpoint of the glottis.

In figure 5 is represented the rectangular glottal configuration, a geometric scheme associated to mucosal wave model of the vocal fold oscillation during phonation, introduced in [18, 19]. It is a particular case of figure 4, where a general trapezoidal glottal configuration is replaced by a particular case, the rectangular one.

This fact implies that the prephonatory glottis has no linear dependence on z, being constant in any point of glottis. So $\xi_{01} = \xi_{02} = x_0$

Geometrically, it is assumed a very simple case, where the vocal fold width is constant along glottis when in rest position, with cross sectional area given by $a = 2L(x_0 + \xi)$.

When it is considered the rectangular glottal configuration, the equations (16) are rewritten such as (17). The time delay τ, so the wave travels to the edges of vocal fold at $z = \pm T/2$ (upper and lower edges respectively), is given by $\tau = T/2c$, which conduces to glottal upper and lower areas

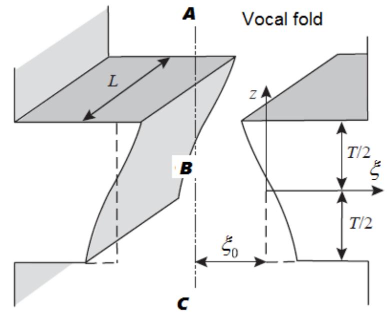

Fig. 5. Rectangular glottal configuration. Geometric scheme associated to mucosal wave model of the vocal fold oscillation during phonation. A-Pharynx; B-Glottis; C-Trachea; ξ-tissue displacement. (Adapted from[18, 19])

$$\begin{aligned} a_1 &= 2L[x_0 + x(t+\tau)], \\ a_2 &= 2L[x_0 + x(t-\tau)]. \end{aligned} \tag{17}$$

The mathematical model which describes the displacement of tissue is obtained imposing the following assumptions:

(i) The pressure at exit of glottis (P_g) equals the atmospheric pressure;
(ii) The sub-glottal pressure equals the lung pressure (P_l);
(iii) The air flow is incompressible, frictionless and stationary;
(iv) The glottis is open ($a_1 > 0$).

Remembering that equations (16) and (17) are defined in the mid point of glottis, we consider mechanical behavior of tissues focused at mid point of glottis.

$x(t)$ is the displacement of tissues at the midpoint of the glottis, so we can get the equation of motion, a nonlinear MTFDE with deviating arguments, with the form (3)

$$Mx''(t) + Bx'(t) + Kx(t) = P_g \tag{18}$$

where P_g, the average glottal pressure, is given by

$$P_g = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} P(z)dz, \tag{19}$$

if an intraglottal pressure $P(z)$ is known.

Under the conditions enumerate above, we can establish that the average glottal pressure (P_g) is given by

$$P_g = \frac{P_l}{k_t}\left(1 - \frac{a_2}{a_1}\right), \tag{20}$$

where k_t is a transglottal pressure coefficient.

Consequentely, (3) can be rewritten in form (4)

$$Mx''(t) + Bx'(t) + Kx(t) = \frac{P_L}{k_t}\frac{x(t-\tau) - x(t+\tau)}{x_0 + x(t+\tau)}, \tag{21}$$

where $x_0 + x(t+\tau) > 0$.

The parameters M, B, K, are, respectively, the effective mass, damping and stiffness per area unit of vocal fold medial surface. The model (4) is also applied in other physiological systems such as avian syrinx, snore, or a flow passing a constricted channel (artery, lips, soft palate, nostrils).

Another detail is that Titze [18] assumes small values of *tau*, making possible to consider the first order (linear) approximation of the expansion of solution (14) in a Taylor series around glottis midpoint ($z=0$):

$$x(t\pm\tau)\approx x(t)\pm\tau x'(t) \tag{22}$$

where $x(t)$ and $x'(t)$ are the midpoint displacement and velocity of the fold.

With approximation (22), equations (16) and (17) are given by (23)

$$\begin{aligned} a_1 &= 2L[\xi_{01}+x(t+\tau)] = 2L[\xi_{01}+x(t)+\tau x(\tau)],\\ a_2 &= 2L[\xi_{02}+x(t-\tau)] = 2L[\xi_{02}+x(t)-\tau x(\tau)]. \end{aligned} \tag{23}$$

and (24)

$$\begin{aligned} a_1 &= 2L[x_0+x(t)+\tau x(\tau)],\\ a_2 &= 2L[x_0+x(t)-\tau x(\tau)]. \end{aligned} \tag{24}$$

respectively. When we consider that mucosal wave has a small time delay [18], equation (5) becomes an autonomous ordinary differential equation, analytically solvable for some values of parameters. This model is similar to the one introduced in [26]. The author of [19, 27] considered a more realistic issue: an arbitrary time delay for mucosal wave.

Appendix B
Method of steps theorem

In present section, we revisit the method of steps for a linear non-autonomous MTFDE with the form (25),

$$x'(t)=\alpha(t)x(t)+\beta(t)x(t-1)+\gamma(t)x(t+1), \tag{25}$$

where x is the unknown function, α, β and γ are known functions. In order to analyze and solve this BVP of (25) which satisfies the boundary conditions (26)

$$x(t)=\begin{cases}\varphi_1(t), & \text{if } t\in[-1,0],\\ f(t), & \text{if } t\in(k-1,k],\end{cases} \tag{26}$$

where φ_1 and f are smooth real-valued functions, defined on $[-1,0]$ and $(k-1,k]$, respectively ($1<k\in\mathbb{N}$), one solves the equation over successive intervals of unitary length. We need to assume the non-degeneracy condition that $\gamma(t)\neq 0$, for $t\geq 0$, so that equation (25) can be rewritten in the form (27)

$$x(t+1)=a(t)x'(t)+b(t)x(t-1)+c(t)x(t), \quad t\geq 0 \tag{27}$$

where $a(t)=\frac{1}{\gamma(t)}$, $b(t)=-\frac{\beta(t)}{\gamma(t)}$ and $c(t)=-\frac{\alpha(t)}{\gamma(t)}$.

MS is used usually in delay differential equations (DDEs) which extend a known solution of equation in an interval to a larger interval. Its a way to increase our knowledge about the solutions of (25) as well as it provides us sufficient conditions for the existence of solution for this kind of MTFDE. We have looked for a differentiable solution x on an interval $[-1,k]$, $k\in\mathbb{N}$, given its values on the intervals $[-1,0]$ and $(k-1,k]$. In next theorem is formulated this result in more precise terms. However, the solution of the non-autonomous IVP (27) subject to (29), the main idea of the theorem B is to get a particular solution of equation (25).

$$x(t)=\varphi(t), \quad t\in[-1,1], \tag{28}$$

where the function φ is defined by

$$\varphi(t)=\begin{cases}\varphi_1(t), & \text{if } t\in[-1,0],\\ \varphi_2(t), & \text{if } t\in(0,1].\end{cases} \tag{29}$$

constructed using the method of steps, becomes *less smooth* as time increases. The conclusions on smoothness for the solution of the constructed using the method of steps is summarized in the Theorem B.

Theorem B (Method of Steps): Let x be the solution of problem (27),(29), where

$$\begin{aligned} &\alpha(t),\quad \beta(t),\quad \gamma(t)\in C^{2L}([-1,2L+1]),\quad \gamma(t)\neq 0,\\ &\qquad\qquad t\in[-1,2L+1],\\ &\varphi_1(t)\in C^{2L+1}([-1,0]),\quad \varphi_2(t)\in C^{2L+1}([0,1])\\ &\qquad\qquad for\ some\ L\in\mathbb{N}. \end{aligned} \tag{30}$$

Moreover, suppose that

$$\begin{aligned} \varphi_1^{(\ell)}(0^-) &= \varphi_2^{(\ell)}(0^+),\\ \varphi_2(1) &= a(0)\varphi_1'(0^-)+b(0)\varphi_1(-1)+c(0)\varphi_1(0);\\ \varphi_2^{(\ell)}(1^-) &= \frac{d^\ell}{dt^\ell}\left(a(t)\varphi_1'(t)+b(t)\varphi_1(t-1)+c(t)\varphi_1(t)\right)|_{t=0^-},\\ &\ell=0,1,2,\dots,2L+1. \end{aligned} \tag{31}$$

Then there exist functions $\delta_{i,l}$, $\varepsilon_{i,l}$, $\bar{\delta}_{i,l}$, $\bar{\varepsilon}_{i,l}\in C([-1,2L+1])$, $l=1,\dots,L$,
$i=0,1,\dots,2l$, such that the following formulae are valid:

$$\begin{aligned} x(t)&=\sum_{i=0}^{2l-1}\delta_{i,l}(t)\varphi_1^{(i)}(t-2l)+\sum_{i=0}^{2l-1}\varepsilon_{i,l}(t)\varphi_2^{(i)}(t-2l+1),\\ &\qquad t\in[2l-1,2l];\\ x(t)&=\sum_{i=0}^{2l}\bar{\varepsilon}_{i,l}(t)\varphi_2^{(i)}(t-2l)+\sum_{i=0}^{2l-1}\bar{\delta}_{i,l}(t)\varphi_1^{(i)}(t-2l-1),\\ &\qquad t\in[2l,2l+1]\ \ l=1,2,\dots. \end{aligned} \tag{32}$$

Moreover, the solution x, constructed according to the formulae (32), belongs to the class

$$C^{2L+1}([-1,1))\bigcap C^{2L}([-1,2))\bigcap\cdots\bigcap C^1([-1,2L+1)). \tag{33}$$

A detailed proof by induction was provided in [29].

Acknowledgment

This work was supported by Portuguese funds through the *Center for Computational and Stochastic Mathematics* (CEMAT), *The Portuguese Foundation for Science and Technology* (FCT), University of Lisbon, Portugal, project UID/Multi/04621/2013, and *Center of Naval Research* (CINAV), Naval Academy, Portuguese Navy, Portugal.

References

[1] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Process*. New York: Interscience, 1962.

[2] A. Rustichini, “Functional differential equations of mixed type: The linear autonomous case,” *Journal of Dynamics and Differential Equation*, vol. 1, pp. 121–143, 1989.

[3] ——, "Hopf bifurcation for functional differential equations of mixed type," *Journal of Dynamics and Differential Equation*, vol. 1, pp. 145–177, 1989.

[4] J. Bell, "Behaviour of some models of myelinated axons," *IMA Journal of Mathematics Applied in Medicine and Biology*, vol. 1, pp. 149–167, 1984.

[5] J. Bell and C. Cosner, "Threshold conditions for a diffusive model of a myelinated axon," *Journal of Mathematical Biology*, vol. 18, pp. 39–52, 1983.

[6] H. Chi, J. Bell, and B. Hassard, "Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction," *Journal of Mathematical Biology*, vol. 24, pp. 5583–601, 1986.

[7] S. Pinelas, M. Chipot, and Z. Dosla, Eds., *Analysis and Computational Approximation of a Forward-Backward Equation Arising in Nerve Conduction*, ser. Springer Proc. in Math. & Stat., no. 47. New York: Springer Science+Business Media, 2013.

[8] K. Abell, C. Elmer, A. Humphries, and E. V. Vleck, "Computation of mixed type functional differential boundary value problems," *SIADS - Siam Journal on Applied Dynamical Systems*, vol. 4, pp. 755–781, 2005.

[9] J. Mallet-Paret, "The global structure of traveling waves in spatially discrete dynamical systems," *Journal of Dynamics and Differential Equation*, vol. 11, pp. 49–128, 1999.

[10] U. Alvarez-Rodriguez, A. Perez-Leija, I. Egusquiza, M. Grfe, M. Sanz, L. Lamata, A. Szameit, and E. Solano, "Advanced-retarded differential equations in quantum photonic systems," *Scientific Reports*, vol. 7, 2017.

[11] V. Iakovleva and C. Vanegas, "On the solution of differential equations withe delayed and advanced arguments," *Electronic Journal of Differential Equations*, vol. Conference 13, pp. 57–63, 2005.

[12] N. Ford and P. Lumb, "Mixed-type functional differential equations: a numerical approach," *Journal of Computational and Applied Mathematics*, vol. 229(2), pp. 471–479, 2009.

[13] M. Teodoro, P. Lima, N. Ford, and P. Lumb, "Numerical modelling of a functional differential equation with deviating arguments using a collocation method," in *Num. Anal. and App. Math.*, ser. AIP Conf. Proc., T. Simos, G. Psihoyios, and C. Tsitouras, Eds., no. 1048. Melville, NY: AIP, 2008, pp. 553–557.

[14] P. Lima, M. Teodoro, N. Ford, and P. Lumb, "Analytical and numerical investigation of mixed type functional differential equations," *Journal Computational and Applied Mathematics*, vol. 234, pp. 2732–2744, 2010.

[15] M. Teodoro, P. Lima, N. Ford, and P. Lumb, "New approach to the numerical solution of forward-backward equations," *Frontiers of Mathematics on China*, vol. 4, pp. 155–168, 2009.

[16] M. Teodoro, P. Lima, N. J. Ford, and P. Lumb, "Numerical approximation of a nonlinear delay-advance functional differential equations by a finite element method," in *Num. Anal. and App. Math.*, ser. AIP Conf. Proc., T. Simos, G. Psihoyios, C. Tsitouras, and Z. Anastassi, Eds., no. 1479. Melville, NY: AIP, 2012, pp. 406–409.

[17] I. Titze, *Principles of Voice Production*. Englewood Cliffs: Prentice-Hall, 1994.

[18] ——, "The physics of small amplitude oscillation of the vocal folds," *The Journal of the Acoustical Society of America*, vol. 83, pp. 1536–52, 1988.

[19] J. Lucero, "Advanced-delay equations for aerolastics oscillations in physiology," *Biophysical Reviews and Letters*, vol. 3, pp. 125–133, 2008.

[20] M. Teodoro, "Numerical approximation of a delay-advanced equation from acoustics," in *Proc. of CMMSE, Spain, 2015*, ser. Math. Meth. in Sc. and Eng., J. V. A. et all, Ed., no. ISBN:978-84-617-2230-3; ISSN=2312-0177. CMMSE, 2015, pp. 1086–1089.

[21] ——, "Numerical solution of a forward-backward equation from physiology," (submitted).

[22] ——, "Modelling a nonlinear mtfde from acoustics," in *Num. Anal. and App. Math.*, ser. AIP Conf. Proc., T. Simos and C. Tsitouras, Eds., no. 1738. Melville, NY: AIP, 2016.

[23] B. Caruntu and C. Bota, "Analytical approximate solutions for a general class of nonlinear delay differential equations," *The Scientifique World Journal*, vol. 2014, 2014.

[24] F. Shakery and M. Dehghan, "Solution of delay differential equations via a homotopy perturbation method," *Mathematical and Computer Modelling*, vol. 48, pp. 486–498, 2008.

[25] M. Teodoro, "Approximating a nonlinear advanced-delayed equation from acoustics," in *Num. Anal. and App. Math.*, ser. AIP Conf. Proc., Y. D. Sergeyev, D. E. Kvasov, F. D. Accio, and M. S. Mukhametzhanov, Eds., no. 1776. Melville, NY: AIP, 2016.

[26] K. Ishizaka and M. Matsudaira, *Fluid Mechanical Considerations of Vocal Cord Vibration*. Speech Communications Research Laboratory, 1972, monography 8.

[27] J. Lucero, "A lumped mucosal wave model of vocal folds revisited: Recent extensions and oscillation hysteresis," *The Journal of the Acoustical Society of America*, vol. 129, pp. 1568–1579, 2011.

[28] P. Lima, M. Teodoro, N. Ford, and P. Lumb, "Finite element solution of a linear mixed-type functional differential equation," *Numerical Algorithms*, vol. 55, pp. 301–320, 2010.

[29] M. Teodoro, "An issue about the existence of solutions for a linear non-autonomous mtfde," in *Differential and Difference Equations with Applications*, ser. Springer Proceedings in Mathematics & Statistics, S. Pinelas, Z. D. a, O. D. y, and P. Kloeden, Eds., no. 164. Springer International Publishing AG.: Springer, Cham, 2017.

[30] C. Tiago and V. Leitão, "Utilização de funções de base radial em problemas unidimensionais de análise estrutural," in *Métodos Numéricos em Engenieria*, J. M. Goicolea, C. M. Soares, M. Pastor, and G. Bugeda, Eds., no. V. SEMNI, 2002, (in Portuguese).

[31] M. Buhmann, *Radial Basis Functions: Theory and Implementations*. Cambridge: Cambridge University Press, 2003.

[32] M. Teodoro, "Numerical approach of some delayed-

advanced differential equations," in *Proceedings of International Conference on Numerical and Symbolic Computation: Developments and Applications, Guimarães, 2017*, ser. Proceedings of ECCOMAS Thematic Conferences, M. Loja and et al., Eds. European Community on Computational Methods in Applied Sciences, ECCOMAS, 2017, (in press).

[33] K. Ishizaka and M. Matsudaira, *Speech Physiology and Acoustic Phonetics*. New York: MacMillan, 1977.

[34] M. Hirano, *Phonosurgery: Basic and Clinical Investigation*. Fukuoka, kioto University: Oto-RhinoLaryngological Society of Japan, 1975, oficial Report of the 78th Annual Convention of the Oto-RhinoLaryngological Society of Japan.

[35] ——, *Clinical examination of voice*. New York: Springer-Verlag, 1981.

M. Filomena Teodoro is Technological Physics Engineer from Instituto Superior Técnico, Lisbon University, has a Master Degree in Statistics and Information Management from NovaIMS, New Lisbon University and a Ph.D. in Computational Mathematics from Instituto Superior Técnico, Lisbon University. She has developed several numerical algorithms to get the approximate solution of mixed type functional differential equations which arise from different ares of science. Complementary research interests are computational statistics and its applications in medicine, environment, economics and other sciences.