Axisymmetric deformation analysis of thick-walled cylinders and rotating-disks using an improved Adomian decomposition method

Safa Bozkurt Coşkun and Zuhal Elif Kara

***Abstract*—**In this study, a problem arising in advanced engineering mechanics governed by a singular differential equation is solved using an improved Adomian decomposition method. The technique was previously used in literature for the solution of two-point singular boundary value problems in applied mathematics. Authors extend the use of the mathematical technique to presented problem and the results show that improved Adomian decomposition method can be successfully used in the axisymmetric deformation and stress analysis of thick-walled cylinders and rotating disks.

***Keywords*—**Adomian decomposition method, thick-walled cylinder, rotating disk, axisymmetric deformation.

I. Introduction

MANY problems in applied mathematics, physics and engineering governed by a singular differential equation of the form

$$u''(x)+p(x)u(x)+q(x)f\left(u(x)\right)=r(x) \tag{1}$$

Subject to boundary conditions

$$u(a)=\alpha \quad \text{and} \quad u(b)=\beta \tag{2}$$

where at least one of the functions *p(x), q(x)* and *r(x)* has a singular point and $x \in (a,b)$.

Ebaid[1] proposed a solution technique to solve singular two-point boundary value problems (BVPs) for which a general form of governing equation is given in Eq.(1). In this study, the operator proposed by Ebaid [1] is used for the axisymmetric deformation analysis of thick-walled cylinders and rotating-disks. To this aim, first the physical problem is reviewed and then the proposed method is explained. Obtained results and exact solutions are compared through a number of figures to show the efficiency of the method.

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II. Problem

A. Thick-Walled Cylinder

Thick-walled cylinders are extensively used in industry as pipes, heat exchanger tubes, pressure vessels, etc. In most of the cases, the cylinder has a constant wall thickness and subjected to a uniform internal and/or external pressure. The isotropy assumption leads to axisymmetrical deformation of the cylinder.

Consider a thick-walled cylinder with open and unconstrained edges. Then, the cylinder may be assumed to be under plane stress conditions ($\sigma_z = 0$). Such a cylinder subjected to internal pressure and stress components are shown in *Fig. 1*.

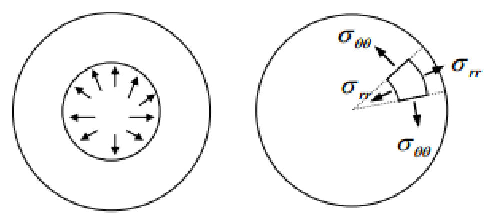

Fig. 1 cylinder under internal pressure and stress components

According to Hooke's law the strains are

$$\varepsilon_r=\frac{du}{dr}=\frac{1}{E}\left(\sigma_r-\nu\sigma_\theta\right) \tag{3}$$

$$\varepsilon_\theta=\frac{u}{r}=\frac{1}{E}\left(\sigma_\theta-\nu\sigma_r\right) \tag{4}$$

From Eqs.(3-4) stress components can be obtained as follows:

$$\sigma_r=\frac{E}{1-\nu^2}\left(\frac{du}{dr}+\nu\frac{u}{r}\right) \tag{5}$$

$$\sigma_\theta=\frac{E}{1-\nu^2}\left(\frac{u}{r}+\nu\frac{du}{dr}\right) \tag{6}$$

In the absence of radial body forces, polar equation of equilibrium reduces to

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{7}$$

Substituting Eqs.(5-6) into (7) results in the following equation.

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0 \tag{8}$$

Eq.(8) has a solution of the form [2],

$$u=c_1r+\frac{c_2}{r} \tag{9}$$

Consider the cylinder is subjected to internal and external pressures p_i and p_o respectively. Then the boundary conditions becomes

$$\left(\sigma_r\right)_{r=a}=-p_i \tag{10}$$

$$\left(\sigma_r\right)_{r=b}=-p_o \tag{11}$$

Introducing Eq.(9) into Eqs.(10-11) constants c_1 and c_2 can be obtained The exact solution for the problem is then produced as follows [2]:

$$\sigma_r=\frac{a^2p_i-b^2p_o}{b^2-a^2}-\frac{(p_i-p_o)a^2b^2}{(b^2-a^2)r^2} \tag{12}$$

$$\sigma_\theta=\frac{a^2p_i-b^2p_o}{b^2-a^2}+\frac{(p_i-p_o)a^2b^2}{(b^2-a^2)r^2} \tag{13}$$

$$u=\frac{1-\nu}{E}\frac{(a^2p_i-b^2p_o)r}{b^2-a^2}+\frac{1+\nu}{E}\frac{(p_i-p_o)a^2b^2}{(b^2-a^2)r} \tag{14}$$

Eqs.(12-14) simplify for internal pressure only as follows:

$$\sigma_r=\frac{a^2p_i}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) \tag{15}$$

$$\sigma_\theta=\frac{a^2p_i}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) \tag{16}$$

$$u=\frac{a^2p_ir}{E(b^2-a^2)}\left[(1-\nu)+(1+\nu)\frac{b^2}{r^2}\right] \tag{17}$$

Eqs.(12-14) reduce to following equations for external pressure only.

$$\sigma_r=-\frac{b^2p_o}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \tag{18}$$

$$\sigma_\theta=-\frac{b^2p_o}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \tag{19}$$

$$u=-\frac{b^2p_or}{E(b^2-a^2)}\left[(1-\nu)+(1+\nu)\frac{a^2}{r^2}\right] \tag{20}$$

B. Rotating-Disk

In the case of *rotating-disk*, equation of equilibrium includes centrifugal inertia force.

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\omega^2r=0 \tag{21}$$

where ρ is the mass density, ω is the constant angular speed of the disk in *rad/sec*. Introducing Eqs.(5-6) into (21) leads to following singular differential equation.

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=-(1-\nu^2)\frac{\rho\omega^2r}{E} \tag{22}$$

The general solution of Eq.(22) is of the following form [2].

$$u=-\frac{\rho\omega^2r^3(1-\nu^2)}{8E}+c_1r+\frac{c_2}{r} \tag{23}$$

An annular rotating disk with no pressure at inner and outer boundaries is shown in Fig.2.

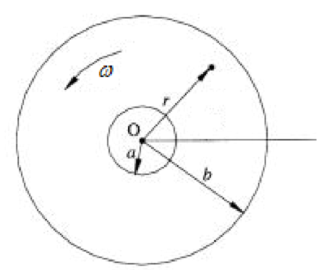

Fig. 2 annular rotating disk

The solution for stresses and displacement for annular disk shown in Fig.2 with no pressure can be obtained in the same way and the procedure leads to following exact solutions [2].

$$\sigma_r=\frac{3+\nu}{8}\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\rho\omega^2 \tag{24}$$

$$\sigma_\theta=\frac{3+\nu}{8}\left(a^2+b^2-\frac{1+3\nu}{3+\nu}r^2+\frac{a^2b^2}{r^2}\right)\rho\omega^2 \tag{25}$$

$$u=\frac{(3+\nu)(1-\nu)}{8E}\left(a^2+b^2-\frac{1+\nu}{3+\nu}r^2+\frac{1+\nu}{1-\nu}\frac{a^2b^2}{r^2}\right)\rho\omega^2 \tag{26}$$

In the following sections improved Adomian decomposition method will be explained and applied to Eqs.(8) and (22) which are singular differential equations in terms of radial displacement.

III. IMPROVED ADOMIAN DECOMPOSITION METHOD

Adomian decomposition method (ADM) was developed by Adomian [3] and has been effectively used in the solution of linear/nonlinear ordinary/partial differential equations over the past three decades. The reader may refer to [3] for the details of the technique.

Ebaid[1] improved ADM for the solution of singular two-point boundary value problems such as Bessel equation, Emden-Fowler equation, Thomas-Fermi equation, singular boundary value problem of Cauchy-Euler type.

Ebaid [1] proposed the following operator to solve a singular two-point BVP based on the work by Lesnic [4].

$$L_{xx}^{-1}(\cdot)=\int_a^x dx'\int_c^{x'}(\cdot)\,dx''-\frac{x-a}{b-a}\int_a^b dx'\int_c^{x'}(\cdot)\,dx'' \qquad (27)$$

Applying Eq.(26) to (1) we obtain the solution as

$$u(x)=u(a)+\frac{x-a}{b-a}\left[u(b)-u(a)\right]+L_{xx}^{-1}\left[r(x)\right]-L_{xx}^{-1}\left[p(x)u'(x)\right]-L_{xx}^{-1}\left[q(x)f\left(u(x)\right)\right] \qquad (28)$$

According to ADM the solution *u(x)* can be computed using the following recurrence relation

$$u_0(x)=u(a)+\frac{x-a}{b-a}\left[u(b)-u(a)\right]+L_{xx}^{-1}\left[r(x)\right] \qquad (29)$$

$$u_{n+1}(x)=-L_{xx}^{-1}\left[p(x)u_n'(x)+q(x)A_n(x)\right],\quad n\geq 0 \qquad (30)$$

where A_n's are Adomian's polynomials for the nonlinear term *f(u(x))* and can be produced from

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}f\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \qquad (31)$$

If *f(u(x))* is linear, the recurrence relation (29) becomes

$$u_{n+1}(x)=-L_{xx}^{-1}\left[p(x)u_n'(x)+q(x)u_n(x)\right],\quad n\geq 0 \qquad (32)$$

Finally, the N^{th} order solution to the problem can be computed from

$$u(x)=\sum_{n=0}^{N}u_n(x) \qquad (33)$$

IV. CASE STUDIES

A. Thick-walled Cylinder Subjected to Internal Pressure

In the case of internal pressure, exact solutions are given between Eqs.(15-17). In the model equation (*Eq.(1)*) boundary conditions are given in terms of dependent variable of the problem. However, in thick-walled cylinder and rotating-disk problems the boundary conditions are given in terms of radial stresses. Hence *u(a)* and *u(b)* appearing in initial approximation given in Eq.(29) are also unknowns and they are determined by inserting the approximate solution into Eq.(5) and by applying radial stress boundary conditions.

We consider a thick-walled cylinder with internal diameter of *0.3 m*, external diameter of *1.2 m*, Poisson's ratio of *0.3* and Young modulus of *200 GPa*. Then the initial approximation for the solution according to Eq.(29) becomes

$$u_0(r)=u(0.3)+\frac{10r-3}{9}\left[u(1.2)-u(0.3)\right] \qquad (34)$$

Recurrence relation for the problem is given in Eq.(32) and based on governing equation (8) it takes the following form.

$$u_{n+1}(r)=-L_{rr}^{-1}\left[\frac{1}{r}u_n'(r)-\frac{1}{r^2}u_n(r)\right],\quad n\geq 0 \qquad (35)$$

where the inverse operator is

$$L_{rr}^{-1}(\cdot)=\int_{0.3}^{r}dr'\int_c^{r'}(\cdot)\,dr''-\frac{10r-3}{9}\int_{0.3}^{1.2}dr'\int_c^{r'}(\cdot)\,dr'' \qquad (36)$$

According to Eq.(35) $u_1(r)$ is computed as

$$u_1(r)=\left(-2.22143+2.05377r-1.33333\ln r\right)u(0.3)+\left(0.555357-0.513442r+0.333333\ln r\right)u(1.2) \qquad (37)$$

Second-order, fourth-order and sixth-order approximations are obtained and these approximations are compared including exact solutions in following figures.

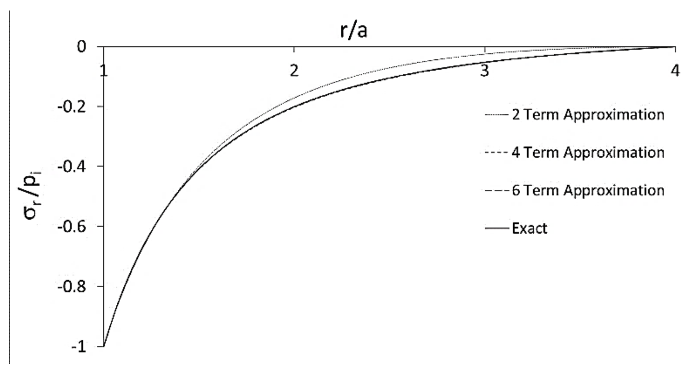

Fig. 3 comparison of radial stresses for inner pressure case

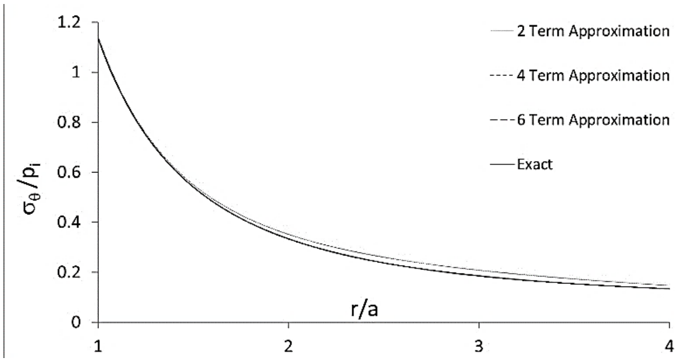

Fig. 4 comparison of tangential stresses for inner pressure case

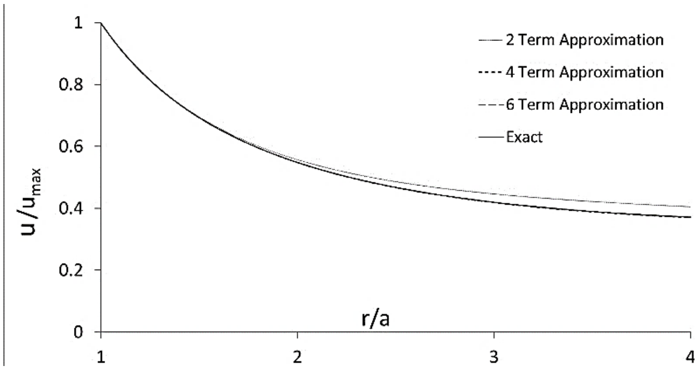

Fig. 5 comparison of radial displacements for inner pressure case

Figs.3-5 show that, even the 2-term approximation may be assumed as a good approximation. 4-term approximation shows very good agreement and 6-term approximation shows excellent agreement with the exact solution.

B. Thick-walled Cylinder Subjected to External Pressure

We consider the thick-walled cylinder with the same properties used in previous internal pressure case. Then the initial approximation for the solution according to Eq.(29) becomes the same as in previous case as follows:

$$u_0(r) = u(0.3) + \frac{10r-3}{9}\left[u(1.2) - u(0.3)\right] \tag{38}$$

Below, comparison for radial and tangential stresses and radial displacements for external pressure case are given.

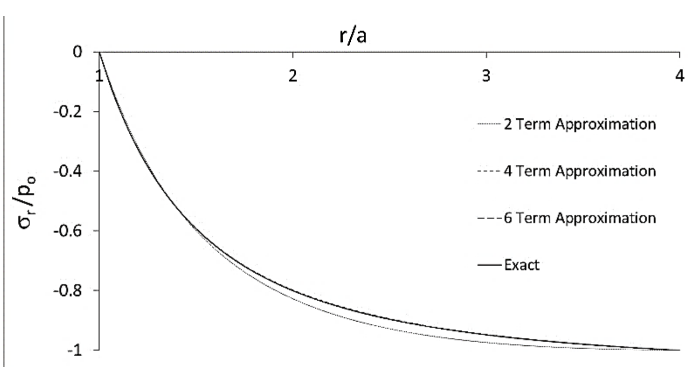

Fig. 6 comparison of radial stresses for external pressure case

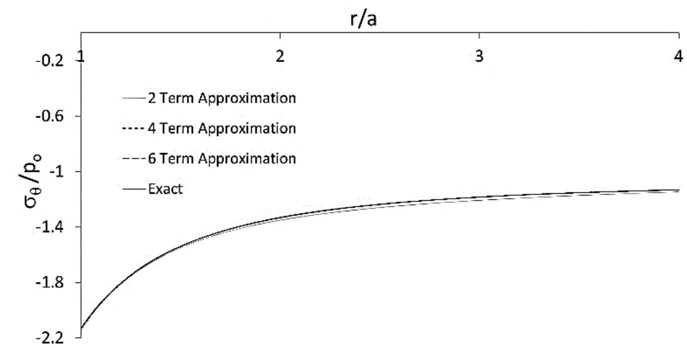

Fig. 7 comparison of tangential stresses for external pressure case

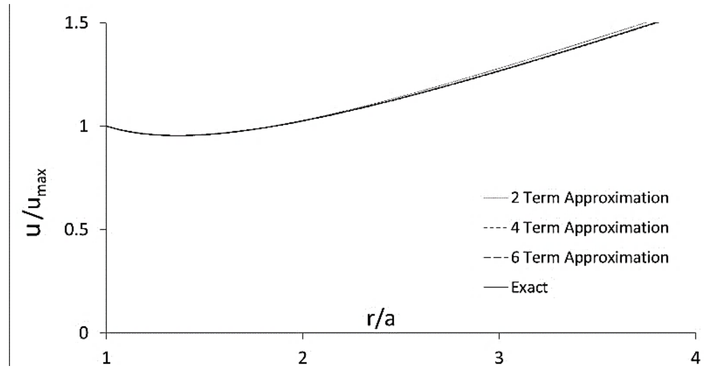

Fig. 8 comparison of radial displacements for external pressure case

As in previous case, 2-term approximation is still a good approximation for the problem. Figs.6-8 show that, 4-term approximation has very good agreement and 6-term approximation has an excellent agreement with the exact solution.

C. Rotating Disk with Constant Thickness

In this case an annular rotating disk with constant thickness is assumed. For the sake of simplicity no pressure is assumed around the inner and outer surfaces.

A rotating annular disk with internal diameter of *0.3 m*, external diameter of *1.2 m* is considered. Poisson's ratio of *0.3* and Young modulus of *200 GPa* as in thick-walled cylinder problem.

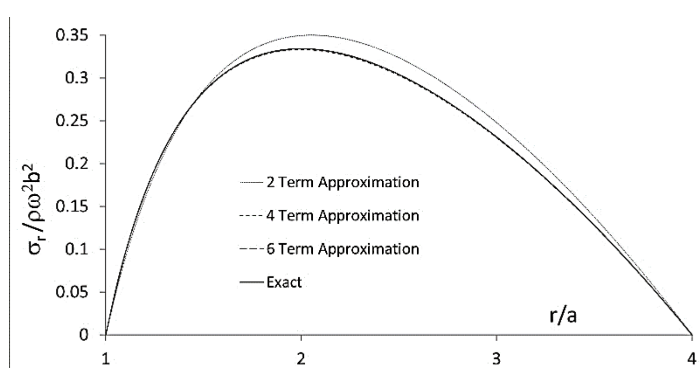

Fig. 9 comparison of radial stresses for rotating disk case

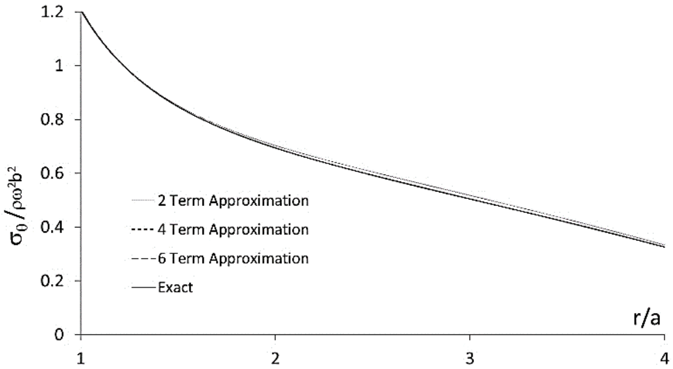

Fig. 10 comparison of tangential stresses for rotating disk case

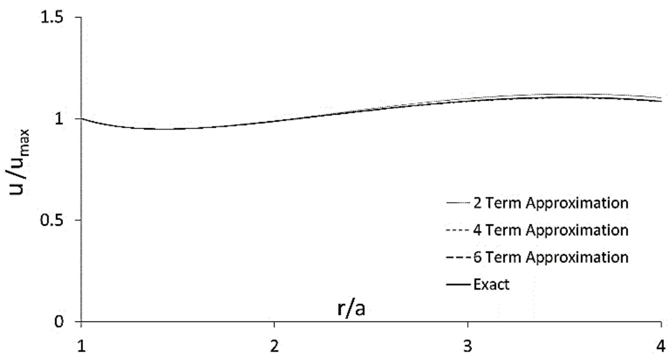

Fig. 11 comparison of radial displacements for rotating disk case

From Figs.9-11 it can be observed that 2-term approximation is in good agreement with the exact solution for tangential stress and radial displacement. For radial stress 2-term approximation is also a good approximation to the problem. 4-term approximation is in very good agreement and 6-term solution is in excellent agreement with the exact solution.

V. CONCLUSION

In this study, axisymmetric deformation analysis of thick-walled cylinders and rotating disks is conducted using an improved Adomian decomposition technique. The method is very effective for handling singular differential equations and may be applied for the problems governed by these types of equations.

As case studies, thick-walled cylinder with internal pressure, thick-walled cylinder with external pressure and a rotating annular disk with constant thickness are considered. Solutions to the problem, produced by the presented technique, are second, fourth and sixth order approximations. These approximations are compared with the available analytical solutions.

In all three cases, 2-term approximations become a good approximation for the problem at hand even it is a low-order approximation. 4-term approximations are in very good agreement with the exact solutions and 6-term approximations show excellent agreement with the exact solutions.

Presented technique is very effective and easy-to-apply technique in the solution of the problem handled in this study. Another advantage is that the method can also treat the simple modifications in the governing equation for which an exact solution may not be easily obtained.

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