Complete study for solving Navier-Lamé equation with new boundary condition using mini element method

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Abstract—The objective of our article is to solve the Navier-Lamé equation with a new boundary $C_{A,B}$ condition using the mixed finite elements method . We compare between mini- element method and the ordinary finite element method by the other side. We compute the displacement and its divergence simultaneously by using an extra unknown. We prove the existence and uniqueness of the weak and discrete solution by proving the discrete $\inf-\sup$ and coerciveness conditions. We expose two ways of comparison, that the first way we calculate the rate α called speed of convergence found by each of the two numerical methods, all this will be done by the use of the linear regression. An analytical example is used to validate the accuracy, convergence and robustness of the present mixed finite elements method for elasticity. In order to evaluate the performance of the method, and to confirm our method, the numerical results of mini element method are compared with others coming from commercial code like Abaqus system.

Keywords—Elasticity, Mini-element, Matrices computing, Linear regression, Comparison, Matlab, Abaqus

I. Introduction

Elasticity theory is an important component of continuum mechanics and has had widely spread applications in science and engineering. This theory is primary for isotropic, linearly elastic materials subjected to small deformations. All governing equations in this theory are linear partial differential equations, which means that the principle of superposition may be applied: The sum of individual solutions to the set of equations is also a solution to the equations.

The aim of our project is to compare several numerical schemes, like the ordinary finite element and the mixed finite element to solve the Navier-Lamé system with a new boundary generalizes the well known basis conditions, especially the Dirichlet and the Neumann conditions. We computed the displacement u_{app} for each methods. We program the two methods by using Matlab, and we needed to program again the functions to estimate the error between the computed solution and the reference one, which is whether the analytical solution. When we calculate the solution of the system $-\mu\Delta u-(\lambda+\mu)\nabla\nabla.u=f$ on a given mesh, we get an approximate value u_{app} of the solution. Of course, the finer will be the mesh and better will be the solution. We want to know for a schema numerical given how evolves the quality of the solution according to the step of mesh.

We know the theoretical relationship

$$\parallel u-u_{app}\parallel_{1,\Omega}=\beta h^{\alpha}, \tag{1}$$

Where β is a constant, h is the step of the mesh and α the speed of convergence.

For the calculation of $\parallel u-u_{app}\parallel_{1,\Omega}$ we will use the the norm $\parallel . \parallel_{1,\Omega}$ which will be defined later. Knowing $\parallel u-u_{app}\parallel_{1,\Omega}$ and the mesh step h we want to calculate α.

For this, the simplest way to proceed is to go to the logarithm in formula (1). We get

$$\log(\parallel u-u_{app}\parallel_{1,\Omega})=\log(\beta)+\alpha.\log(h), \tag{2}$$

Note that $\log(\parallel u-u_{app}\parallel_{1,\Omega})$ is an affine function of $\log(h)$ where the slope is α.

To find α we compute $(\parallel u-u_{app}\parallel_{1,\Omega})$ on different meshes, then we plot the graph of the log of $(\parallel u-u_{app}\parallel_{1,\Omega})$ according to the log of the step h. We obtain the slope straight line. In practice the points are not exactly aligned, to get the value of α in fact, we perform a linear regression in the least squares sense, that is, to say that we take for α the slope of the line that goes closer to all points.

Since 2002, the article [12] entitled by Matlab implementation of the finite element method in Elasticity, thanks to the authors of this work J.Alberty, Kiel, C.Carstensen, Vienna, SA Funken, Kiel and R.Klose, that have a great contribution in computing the numerical solution that is the approximation of the exact solution which is the unknown in the Navier-Lamé equation, using the ordinary finite element method programmed by Matlab. In fact, nobody has thought to apply the mixed finite element Method to the equation Navier-Lamé, that will be the subject of our research. This study is based on the calculation of the numerical solution using the mixed finite element method $(P1-bubble,P1)$ and to do this, we have to create another new unknown by setting ψ equal to the divergence of the displacement,getting a couple of unknown (u,ψ). comparing the numerical results found in the article cited above mentioned in article [12], we will prove that the new method is more accurate and efficient. We propose the numerical Method employs the mixed finite element $(P1-bubble,P1)$ to calculate the numerical solution of the displacement, and its discrete divergence, to the following 2D Navier -Lamé problem.

As we know there are three types of boundary conditions for the problem of linear elasticity: traction or natural boundary conditions (Neumann): For tractions imposed on the portion

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of the surface of the body $\partial\Omega$, There is displacement or essential boundary conditions (Dirichlet): for displacements u imposed on the portion. of the surface of the body $\partial\Omega$, this includes the supports for which we have $u = 0$ or $u = g$, and there is mixed (Robin) boundary conditions, physically, this implies that the traction which the elastic foundation exerts on the body is proportional to the boundary displacement.

In this paper we propose a new formulation of boundary conditions $\mathbf{C_{A,B}}$, this formulation generalizes all type of boundary conditions (neumann, Dirichlet, Robin), with $\mathbf{C_{A,B}}$ we will not need to implement a solving program for every type of problem, just produce a single code and using two square matrices A, B and we assign them values we will get the three type of boundary conditions mentioned above. The rest of this paper is organized as follows. The basic setting and governing of elasticity equation is presented in Section 2. The linear elasticity equations, constitutive laws are discussed in detail to facilitate further consideration. Sections 3 and 4 are devoted to existence and uniqueness of weak solution, and the construction of the mixed elements $(P1bubble - P1)$ and the proof of the existence and uniqueness of the approach solution. General schemes are proposed for elasticity problem. Our method is extensively validated by analytical tests with membrane with hole geometry in Section 6. This paper ends with a conclusion.

II. GOVERNING EQUATION

Linear elasticity is the mathematical study of how solid objects deform and become internally stressed due to prescribed loading conditions. Linear elasticity models materials as continua. Linear elasticity is a simplification of the more general nonlinear theory of elasticity and is a branch of continuum mechanics. The fundamental ”linearizing” assumptions of linear elasticity are: infinitesimal strains or ”small” deformations (or strains) and linear relationships between the components of stress and strain. In addition linear elasticity is valid only for stress states that do not produce yielding. These assumptions are reasonable for many engineering materials and engineering design scenarios. Linear elasticity is therefore used extensively in structural analysis and engineering design, often with the aid of finite element analysis.

Let’s consider $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary condition Γ which will be presented in a new form that generalizes the Neumann and Dirichlet boundaries conditions. Given $f \in L^2(\Omega)$, $\mathbf{A}, \mathbf{B} \in \mathbf{L}^\infty(\mathbf{\Gamma})^{\mathbf{2\times 2}}$, $g \in H^{\frac{1}{2}}(\Gamma)$ and as well as the positive parameters λ and μ.

When solid objects are subjected to external or internal loads, they deform and led to stress. If the deformation of the solid is relatively small, linear relationships between the components of stress and strain are maintained. Consequently, linear elasticity theory is valid. In practice, linear elasticity theory is applicable to a wide range of natural and engineering materials, and thus extensively used in structural analysis and engineering design. The equation of Navier Lamé below is governed as follows .

Solid object is deformed under the action of forces applied. A point in the solid, originally in (x, y), after sometime it will come into (X, Y), the vector $u = (u_1, u_2) = (X - x, Y - y)$ is called displacement. When the movement is small and the solid is elastic, then HOOK’s law gives a relationship between the stress tensor and the strain tensor. $\sigma = \lambda tr(\varepsilon)I_2 + 2\mu\varepsilon$ is the stress tensor, $\varepsilon = \frac{1}{2}(\nabla u + (\nabla U)^T)$ is the strain tensor, I_2 is the identity matrix, μ is the shear modulus (or rigidity), where λ is Lam’s first parameter. Navier Lamé equation is given by the law of conservation moment $\rho a = div\sigma$ with a is the acceleration and ρ is the density of material, On the other hand

$$div\sigma = \lambda div(tr(\varepsilon)I_2) + 2\mu div\varepsilon, \tag{3}$$

then we have

$$div\sigma = \lambda div(tr(\varepsilon)I_2) + \mu div(gradu) + \mu div(gradu)^t, \tag{4}$$

with a simple calculation, we find that

$$div(tr(\varepsilon)I_2) = div(gradu)^t = grad(div(u)), \tag{5}$$

Then we get

$$\rho a = \mu\Delta u + (\lambda + \mu)grad(divu), \tag{6}$$

If the solid is in dynamic equilibrium then we have $\rho a + f = 0$, f are the external forces applied to the solid. Finally, we find out the equation

$$f = -\mu\Delta u - (\lambda + \mu)grad(divu), \tag{7}$$

We refer the reader to [17], [18] for more information of the elasticity problems.

We create a new unknown $\psi = \nabla.u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$ that is equal to divergence of the displacement.

The equation of Navier-Lamé become

$$\begin{cases} -\mu\Delta u - (\lambda + \mu)\nabla\psi = f \ in\ \Omega, \\ \psi - \nabla.u = 0 \ in\ \Omega, \\ \mathbf{Au} + \mathbf{B}(\mu\,\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda\nabla.\mathbf{un}) = \mathbf{g\ on\ \Gamma}, \end{cases} \tag{8}$$

Our mathematical model is the Navier-Lamé system with a new boundary condition noted $\mathbf{C_{A,B}}$ such as $\mathbf{A}$is called Dirichlet matrix and $\mathbf{B}$ is Neumann matrix

There are two strictly positive constants α and β, such that

$$\alpha < u^t\mathbf{B^{-1}Au} < \beta\ ,\ \forall \mathbf{u} \in \mathbb{R}^\mathbf{2} \tag{9}$$

With $|||\ .\ |||$ is a matrix norm that will be defined below.

If $|||\ \mathbf{A}\ ||| \ll |||\ \mathbf{B}\ |||$, then $\mathbf{C_{A,B}}$ is the Neumann boundary condition and if

$|||\ \mathbf{B}\ ||| \ll |||\ \mathbf{A}\ |||$ then $\mathbf{C_{A,B}}$ is the Dirichlet boundary

We need functional spaces and norms

$$h^1(\Omega) = \{u : \Omega \to \mathbb{R} \setminus u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)\}, \tag{10}$$

$$V(\Omega) = H^1(\Omega) = [h^1(\Omega)]^2, \tag{11}$$

$$M(\Omega) = L^2_0(\Omega) = \{q \in L^2(\Omega) \setminus \int_\Omega q = 0\}, \tag{12}$$

$$\|\ v\ \|_{1,\Omega} = \{\int_\Omega \nabla v : \nabla v d\Omega + \int_\Omega v.v\, d\Omega\}^{\frac{1}{2}}, \tag{13}$$

$$\|\ v\ \|_{0,\Omega} = \{\int_\Omega v.v\, d\Omega\}^{\frac{1}{2}}, \tag{14}$$

$$||| A |||= \max | a_{i,j} | \quad i = 1.2, j = 1.2 \tag{15}$$

The variational formulation of the Navier-Lamé problem (8) is as follows
Find $(u, \psi) \in V(\Omega) \times M(\Omega)$ such that

$$\begin{cases} \int_\Omega \mu \nabla u : \nabla v d\Omega + \int_\Gamma \mathbf{B^{-1} A u . v d\Gamma} \\ + \int_\Gamma \mu\psi\, n.v d\Gamma + \int_\Omega (\lambda+\mu)\psi \nabla .v d\Omega \\ = \int_\Omega f.v d\Omega + \int_\Gamma \mathbf{B^{-1} g.v\, d\Gamma} \\ \int_\Omega (\lambda+\mu) q \nabla .u d\Omega - \int_\Omega (\lambda+\mu)\psi q d\Omega = 0 \end{cases} \tag{16}$$

The weak formulation (16) may be restated as:
Find $(u, \psi) \in V(\Omega) \times M(\Omega)$

$$\begin{cases} a(u,v) + b_\Gamma(v,\psi) = L(v)\ \forall v \in V_0(\Omega), \\ b(u,q) - d(\psi,q) = 0\ \forall q \in M(\Omega), \end{cases} \tag{17}$$

With the bilinear forms

$$\begin{cases} a(u,v) = \int_\Omega \mu \nabla u : \nabla v d\Omega + \int_\Gamma \mathbf{B^{-1} A u . v d\Gamma}, \\ b(v,q) = \int_\Omega (\lambda+\mu) q \nabla .v d\Omega, \\ b_\Gamma(v,q) = b(v,q) + \int_\Gamma \mu q n.v d\Gamma, \\ d(\psi,q) = \int_\Omega (\lambda+\mu)\psi q d\Omega, \\ L(v) = \int_\Omega f.v d\Omega + \int_\Gamma \mathbf{B^{-1} g.v d\Gamma}, \end{cases} \tag{18}$$

III. Existence and uniqueness of weak solution

In this section we will study the existence and uniqueness of the solution of problem, for realizing this work we need the following norms that are provided for the standard spaces. a is a bilinear which operates on $E \times F$ with E and F two vector spaces, the bilinear form a has a norm like

$$\| a \| = \sup_{\|u\|_E \leq 1\ ,\ \|v\|_F\ \leq 1} a(u,v) \tag{19}$$

L is a form linear has the following norm

$$\| L \| = \sup_{\|u\|_E \leq 1} L(u) \tag{20}$$

The norm produced by inner product

$$\| v \| = (v.v)^{\frac{1}{2}} \tag{21}$$

The norm of the space $L^2_0(\Omega)$ is $\| v \|^2_{0,\Omega}$ such as

$$\| v \|^2_{0,\Omega} = \{\int_\Omega v.v\, d\Omega\}^{\frac{1}{2}} \tag{22}$$

The norm of the space $H^1(\Omega)$ is $\| v \|_{1,\Omega}$ such as

$$\| v \|_{1,\Omega} = \{\int_\Omega \nabla v : \nabla v d\Omega + \int_\Omega v.v\, d\Omega\}^{\frac{1}{2}} \tag{23}$$

$$= | v |^2_{1,\Omega} + \| v \|^2_{0,\Omega}]^{\frac{1}{2}} \tag{24}$$

With the semi norm for $H^1(\Omega)$ and it is a norm for the space $H^1_0(\Omega)$

$$| v |^2_{1,\Omega} = \int_\Omega \nabla v : \nabla v d\Omega = \| \nabla v \|^2_{0,\Omega} \tag{25}$$

The norm of the space $L^\infty(\Gamma)^{2\times 2}$ is

$$||| A |||= \max | a_{i,j} | \quad i = 1.2, j = 1.2 \tag{26}$$

$(V(\Omega), \| \ v \ \|_{1,\Omega})$ is a Hilbert space Let consider some assumptions satisfying by the bilinear and linear forms $a,\ b_\Gamma,\ b,\ d,\ L,\ G$
There exist positives strictly constants $\alpha,\ \beta,\ \gamma,\ \delta,\ \theta,\ \xi,\ \rho,$ such that

$$\begin{cases} | a(u,v) | \leq \alpha \| u \|_{1,\Omega} \| v \|_{1,\Omega}, \\ | b(u,q) | \leq \beta \| u \|_{1,\Omega} \| q \|_{0,\Omega}, \\ | b_\Gamma(u,q) | \leq \gamma \| u \|_{1,\Omega} \| q \|_{0,\Omega}, \\ | d(q,\chi) | \leq \theta \| q \|_{0,\Omega} \| \chi \|_{0,\Omega}, \\ | L(v) | \leq \zeta \| v \|_{1,\Omega}, \\ | G(q) | \leq \rho \| q \|_{0,\Omega}, \end{cases} \tag{27}$$

$\forall (u,v) \in V(\Omega) \times V(\Omega),\ \forall (\chi, q) \in L^2_0(\Omega) \times L^2_0(\Omega)$
Assume that for some constant, a satisfy the condition of coerciveness such as. There exist δ is a constant positive such as

$$a(v,v) \geq \delta \| v \|^2_{1,\Omega} \quad for\ all\ v \in V(\Omega) \tag{28}$$

b satisfying the $\inf - \sup$ condition such as there exist a constant $\varrho > 0$

$$\sup_{v \in V(\Omega)} \frac{b(v,q)}{\| v \|_{1,\Omega}} \geq \varrho \| q \|^2_{0,\Omega}\ \ \forall q \in L^2_0(\Omega) \tag{29}$$

b_Γ satisfying the $\inf - \sup$ condition such as there exist a constant $\vartheta > 0$

$$\sup_{v \in V(\Omega)} \frac{b_\Gamma(v,q)}{\| v \|_{1,\Omega}} \geq \vartheta \| q \|^2_{0,\Omega}\ \ \forall \in L^2_0(\Omega) \tag{30}$$

The bilinear form d satisfy the weak coerciveness such as there exist a constant $\varepsilon \leq 0$ such as

$$d(\chi,\chi) \geq -\varepsilon \| \chi \|^2_{0,\Omega}\ \ \forall \chi \in L^2_0(\Omega) \tag{31}$$

We consider the following generalized variational problem
$\forall (v,q) \in V(\Omega) \times L^2_0(\Omega)$
$\forall (L,G) \in (V(\Omega))' \times (L^2_0(\Omega))'$

$$\begin{cases} a(u,v) + b_\Gamma(v,\chi) = L(v), \\ b(u,q) - d(q,\chi) = G(q), \end{cases} \tag{32}$$

Let consider two special cases of problem (32)
Case $b_\Gamma = b$, $d \neq 0$,

$$\begin{cases} a(u,v) + b(v,\chi) = L(v), \\ b(u,q) - d(q,\chi) = G(q), \end{cases} \tag{33}$$

Case $d = 0$, $b_\Gamma \neq b$,

$$\begin{cases} a(u,v) + b_\Gamma(v,\chi) = L(v), \\ b(u,q) = G(q)\ \forall q \in L^2_0(\Omega), \end{cases} \tag{34}$$

When $b_\Gamma = b$ and $d \neq 0$ we reduce to problem (33), this problem has been studies in [6], [8], [14], [15].

The existence and uniqueness of the solutions to the system (33) are shown under some standard conditions : (27) - (31) and the theorem 2.2 and its proof in [14] explain that in detail. While the well-possedness for the case $d = 0$ and , $b_\Gamma \neq b$ we are restricted to problem (34) this problem has an unique solution using the proof of theorem 2.1 in [14], it was established too in [4], [16]

Theorem 1: with the assumptions (27) ... (31) the generalized variational problem (32) has a unique solution $(u, \chi) \in V(\Omega) \times L_0^2(\Omega)$
for any $L \in (V(\Omega))'$ and $G \in (L_0^2(\Omega))'$ as long as

$$\xi = \frac{\delta^2 \parallel b_\Gamma \parallel \parallel b_\Gamma - b \parallel}{\delta \parallel a \parallel^{-2} \vartheta^2 - \varepsilon} < 1 \tag{35}$$

With $\delta \parallel a \parallel^{-2} \vartheta^2 > \varepsilon$
Further, the solution has the stability estimates

$$\parallel u \parallel_{1,\Omega} \leq \frac{1}{1-\xi} \parallel u_0 \parallel_{1,\Omega} \tag{36}$$

$$\parallel \chi \parallel_{0,\Omega} \leq \parallel \chi_0 \parallel_{0,\Omega} + \frac{\xi\delta}{\parallel b_\Gamma \parallel (1-\xi)} \parallel u_0 \parallel_{1,\Omega} \tag{37}$$

Where (u_0, χ_0) the bounded solution of (33) such as

$$\parallel \chi_0 \parallel_{0,\Omega} \leq \frac{\delta^{-1} \parallel b_\Gamma \parallel \parallel L \parallel + \parallel G \parallel}{\delta \parallel a \parallel^{-2} \vartheta^2 - \varepsilon} \tag{38}$$

$$\parallel u_0 \parallel_{0,\Omega} \leq \delta^{-1} (\parallel L \parallel + \parallel b_\Gamma \parallel \parallel \chi_0 \parallel_{0,\Omega}) \tag{39}$$

Proof: The proof of the theorem 1 is similar to the proof of the theorem 3.1 in [14]

Remark 2: The weak problem (17) is a special case $(G = 0)$ of the problem (32), see ref. [14]
By the remark 3.2 in [14] prove that only $b_\Gamma(v, q)$ not $b(v, q)$ is required to satisfy the $\inf - \sup$ condition. Similar results hold when $b(v, q)$ satisfies the $\inf - \sup$ condition but, $b_\Gamma(v, q)$ does not

Proposition 3: There exist positives strictly constants α, β, γ, δ, θ, ζ, ρ such that
For all $(u, v) \in V(\Omega) \times V(\Omega)$,
For all $(p, q) \in L_0^2(\Omega) \times L_0^2(\Omega)$

$$| a(u, v) | \leq \alpha \parallel u \parallel_{1,\Omega} \parallel v \parallel_{1,\Omega} \tag{40}$$

$$| b(u, q) | \leq \beta \parallel u \parallel_{1,\Omega} \parallel q \parallel_{0,\Omega} \tag{41}$$

$$| b_\Gamma(u, q) | \leq \gamma \parallel u \parallel_{1,\Omega} \parallel q \parallel_{0,\Omega} \tag{42}$$

$$| d(q, p) | \leq \theta \parallel q \parallel_{0,\Omega} \parallel p \parallel_{0,\Omega} \tag{43}$$

$$| L(v) | \leq \zeta \parallel v \parallel_{1,\Omega} \tag{44}$$

$$| G(q) | \leq \rho \parallel q \parallel_{0,\Omega} \tag{45}$$

Proof:

Lemma 4: We define the extension of u and v in $\mathbb{R}^2$ as follows. It is assumed that Ω is C^1 with Γ is bounded. With these conditions there is a prolongation operator P that is linear and continuous

$$P : H^1(\Omega) \longrightarrow H^1(\mathbb{R}^2), u \longmapsto P\,u \in H^1(\mathbb{R}^2) \tag{46}$$

Such as

$$Pu = \begin{cases} u & \text{if} \quad in\,\Omega \\ 0 & \text{if} \quad in\,\mathbb{R}^2 \setminus \Omega \end{cases} \tag{47}$$

$$Pu\,|_\Omega = u \tag{48}$$

$$\parallel Pu \parallel_{H^1(\mathbb{R}^2)} \leq c \parallel u \parallel_{1,\Omega} \tag{49}$$

$$\parallel Pu \parallel_{L^2(\mathbb{R}^2)} \leq c \parallel u \parallel_{0,\Omega} \tag{50}$$

Proof: Look at proof of theorem IX.7 in [2]

Lemma 5: Let v two elements of $V(\Omega)$ and χ from $L_0^2(\Omega)$, there exist a constant c positive nonzero such as For all $(v, \chi) \in V(\Omega) \times L_0^2(\Omega)$

$$\int_\Gamma | \mu\chi\, n.v \,|\, d\Gamma \leq c \parallel \chi \parallel_{0,\Omega} \parallel v \parallel_{1,\Omega} \tag{51}$$

For all $(u, v) \in V(\Omega)^2$

$$\int_\Gamma \mathbf{B^{-1}Au.vd\Gamma} \leq \mathbf{c_0} \parallel \mathbf{u} \parallel_{\mathbf{1,\Omega}} \parallel \mathbf{v} \parallel_{\mathbf{1,\Omega}} \tag{52}$$

Proof: Ω is bounded domain, it means that $\Gamma \subset \mathbb{R}^2$. $\Gamma \subset \mathbb{R}^2$ imply that

$$\int_\Gamma | \mu\chi\, n.v \,|\, d\Gamma \leq \int_{\mathbb{R}^2} | \mu(P\chi)\,(Pv.n) \,| \tag{53}$$

By applying Hölder

$$\int_{\mathbb{R}^2} | \mu(P\chi)(Pv.n) | \leq (\int_{\mathbb{R}^2} | \mu P\chi |^2)^{\frac{1}{2}} \tag{54}$$

$$\times(\int_{\mathbb{R}^2} | Pv.n |^2)^{\frac{1}{2}} \tag{55}$$

$$\leq (\int_{\mathbb{R}^2} | \lambda | \,(P\chi)^2\,)^{\frac{1}{2}} \tag{56}$$

$$\times(\int_{\mathbb{R}^2} 2\max(n_1^2, n_2^2)(Pv)^2)^{\frac{1}{2}} \tag{57}$$

$$\leq c_0 \parallel P\chi \parallel_{L^2(\mathbb{R}^2)} \parallel Pv \parallel_{L^2(\mathbb{R}^2)} \tag{58}$$

$$\leq c_1 \parallel P\chi \parallel_{0,\Omega} \parallel Pv \parallel_{H^1(\mathbb{R}^2)} \tag{59}$$

$$\leq c_2 \parallel \chi \parallel_{0,\Omega} \parallel v \parallel_{1,\Omega} \tag{60}$$

with c_0, c_1, c_2 are positives constants and by according to lemma 4 the proof of the lemma 5 is completed We have $a(u, v) = \int_\Omega \mu\nabla u : \nabla v d\Omega + \int_\Gamma \mathbf{B^{-1}Au.vd\Gamma}$
Let prove that the bilinear form a is continuous, by using Hölder we find that

$$| a(u, v) | \leq \int_\Omega | \mu\nabla u : \nabla v \,|\, d\Omega \tag{61}$$

$$\leq \mu \,| u |_{1,\Omega} | v |_{1,\Omega} \tag{62}$$

$$\leq \mu \parallel u \parallel_{1,\Omega} \parallel v \parallel_{1,\Omega} \tag{63}$$

By using the relation (52) of the lemma 5
We can take $\alpha = \mu + \mathbf{c_0}$
Secondly we will prove that

$$| b(u, q) | \leq \beta \parallel u \parallel_{1,\Omega} \parallel q \parallel_{0,\Omega} \;\; \forall u \in V(\Omega) \,\forall q \in L_0^2(\Omega) \tag{64}$$

We know that

$$| b(u, q) | \leq \int_\Omega | \lambda + \mu \,|\, | q \,|\, | \nabla.u \,|\, d\Omega \tag{65}$$

Hölder gives

$$| b(u,q) | \leq \quad | \lambda+\mu | \parallel q \parallel_{0,\Omega} [\int_\Omega 2(\frac{\partial u_1^2}{\partial x}+\frac{\partial u_2^2}{\partial y})]^{\frac{1}{2}} \quad (66)$$

$$\leq \quad | \lambda+\mu | \parallel q \parallel_{0,\Omega} [\int_\Omega 2 \mid \nabla u \mid^2 d\Omega]\frac{1}{2} \quad (67)$$

$$\leq \quad \sqrt{2} \mid \lambda+\mu \mid \parallel q \parallel_{0,\Omega} \parallel \nabla u \parallel_{0,\Omega} \quad (68)$$

$$\leq \quad \sqrt{2} \mid \lambda+\mu \mid \parallel q \parallel_{0,\Omega} \parallel u \parallel_{1,\Omega} \quad (69)$$

$$\leq \quad c \parallel q \parallel_{0,\Omega} \parallel u \parallel_{1,\Omega} \quad (70)$$

It will be good if we take c such as $c=\sqrt{2} \mid \lambda+\mu \mid$
Now let prove that the form bilinear b_Γ is continuous we have

$$| b_\Gamma(v,q) | = \quad | b(v,q)+\int_\Omega \mu\, q\, n.v\, d\Gamma | \quad (71)$$

$$\leq \quad | b(v,q) | + \int_\Gamma | \mu\, q\, n.v \ | d\Gamma \quad (72)$$

$$\leq \quad c \parallel q \parallel_{0,\Omega} \parallel u \parallel_{1,\Omega} + \int_\Gamma | \mu\, q\, n.v \ | d\Gamma \quad (73)$$

On the other hand we have by the relation (51) of lemma 5 there exist a constant c_1 positive such as

$$\int_\Gamma | \mu\, q\, n.v \ | d\Gamma \leq c_1 \parallel q \parallel_{0,\Omega} \parallel u \parallel_{1,\Omega} \quad (74)$$

Then we deduce that

$$| b_\Gamma(v,q) | \leq c_2 \parallel v \parallel_{1,\Omega} \parallel q \parallel_{0,\Omega} \quad (75)$$

Such as $c_2 = c + c_1$
Hölder is used,it may be easily that the form bilnear $d(,)$ is continuous like this

$$| d(p,q) | \leq c \parallel p \parallel_{0,\Omega} \parallel q \parallel_{0,\Omega} \quad (76)$$

We can take $c = | \lambda+\mu |$
Let prove that L is continuous like that there is $\zeta > 0$

$$| L(v) | \leq \zeta \parallel v \parallel_{1,\Omega} \ \ \forall v \in V(\Omega) \quad (77)$$

$$L(v) = \int_\Gamma g.v\, d\Gamma + \int_\Omega f.v d\Omega \quad (78)$$

$$L(v) \leq | \int_\Gamma g.v\, d\Gamma | + | \int_\Omega f.v\, d\Omega | \quad (79)$$

$$L(v) \leq \int_\Gamma | g.v | \ d\Gamma + \int_\Omega | f.v | \ d\Omega \quad (80)$$

$$L(v) \leq \sup_\Omega g \parallel v \parallel_{0,\Gamma} + \sup_\Omega f \parallel v \parallel_{1,\Omega} \quad (81)$$

From the lemma 5 then

$$| L(v) | \leq \sup_\Omega gc \parallel v \parallel_{1,\Omega} + \sup_\Omega f \parallel v \parallel_{1,\Omega}$$

$$\leq (c \sup_\Omega g + \sup_\Omega f) \parallel v \parallel_{1,\Omega}$$

with taking $\zeta = c \sup_\Omega g + \sup_\Omega f$

Lemma 6: It exist $\rho > 0$ such as

$$\parallel v \parallel_{0,\Omega}^2 \leq \rho(| \nabla v |_{1,\Omega}^2 + \parallel v \parallel_{0,\Gamma}^2) \quad (82)$$

Proof: The proof of lemma 6 exist in [1] a satisfy the condition of coerciveness such as there exist δ is a constant positive

Proposition 7: Let define $a_0(u,u) = \int_\Omega \mu \nabla u : \nabla u\, d\Omega$, then $a(u,u) = a_0(u,u) + \int_\Gamma \mathbf{B^{-1}Au.ud\Gamma}$

$$a_0(u,u) \ \geq \ \delta \parallel u \parallel_{1,\Omega}^2 \ \ \forall u \in V(\Omega) \quad (83)$$

Proof: In fact, according the lemma 6

$$\exists \rho > 0 \ \parallel u \parallel_{0,\Omega}^2 \leq \rho\{\parallel \nabla u \parallel_{0,\Omega}^2 + \parallel u \parallel_{0,\Gamma}^2\} \ \forall u \in V(\Omega)$$

And from the theorem 1.2 in the chapter 1 [V.Girault ,P.A Raviard 1986] there exist a constant c positive

$$\parallel u \parallel_{0,\Gamma} \leq c \parallel u \parallel_{1,\Omega}$$

so we will have

$$\parallel u \parallel_{0,\Omega}^2 \leq \rho\{\parallel \nabla u \parallel_{0,\Omega}^2 + c^2 \parallel u \parallel_{1,\Omega}^2\}$$

$\forall u \in V(\Omega)$, then

$$\parallel \nabla u \parallel_{0,\Omega}^2 + \parallel u \parallel_{0,\Omega}^2 \leq \quad (84)$$

$$\rho\{\frac{1}{\mu} a(u,u) + c^2 \parallel u \parallel_{1,\Omega}^2\} + \frac{1}{\mu} a(u,u) \quad (85)$$

$\forall u \in V(\Omega)$, that imply

$$(1-\rho c^2) \parallel u \parallel_{1,\Omega}^2 \leq (\frac{\rho}{\mu}+\frac{1}{\mu}) a(u,u)$$

finally :

$$a_0(u,u) \geq \frac{1-\rho c^2}{\frac{\rho}{\mu}+\frac{1}{\mu}} \parallel u \parallel_{1,\Omega}^2$$

As soon as $1-\rho c^2 > 0$ we take $\delta = \frac{1-\rho c^2}{\frac{\rho}{\mu}+\frac{1}{\mu}}$

$$a_0(u,u) \geq \delta \parallel u \parallel_{1,\Omega}^2 \ \ \forall u \in V(\Omega)$$

According to the lemma (5), relation (52) we obtain

$$a(u,u) \geq (\delta - c_0) \parallel u \parallel_{1,\Omega}^2 \ \ \forall u \in V(\Omega) \quad (86)$$

As long as, $\delta - c_0 > 0$
The result (86) will be explain in the proof of the theorem 13 that will come later
b satisfying the $\inf - \sup$ condition. There exist constant positive ϱ such as

Proposition 8:

$$\sup_{v \in V(\Omega)} \frac{b(v,q)}{\parallel v \parallel_{1,\Omega}} \geq \varrho \parallel q \parallel_{0,\Omega}^2 \ \forall q \in L_0^2(\Omega) \quad (87)$$

b_Γ, satisfying the $\inf - \sup$ condition.There exist constant positive ϑ

$$\sup_{v \in V(\Omega)} \frac{b_\Gamma(v,q)}{\parallel v \parallel_{1,\Omega}} \geq \vartheta \parallel q \parallel_{0,\Omega}^2 \ \ \forall \in L_0^2(\Omega) \quad (88)$$

Proof: Let $q \in L_0^2(\Omega)$,we have from [2]. There exist a constant positive k_0 such as

$$\sup_{v \in H_0^1(\Omega)} \frac{b(v,q)}{| v |_{1,\Omega}} \geq k_0 \parallel q \parallel_{0,\Omega}$$

Since $H_0^1(\Omega) \subset V(\Omega)$, and

$$| v |_{1,\Omega} = \parallel v \parallel_{1,\Omega} \ \forall v \in H_0^1(\Omega)$$

$$\sup_{v\in V(\Omega)} \frac{b(v,q)}{\| v \|_{1,\Omega}} \geq \sup_{v\in H_0^1(\Omega)} \frac{b(v,q)}{\| v \|_{1,\Omega}} \geq k_0 \| q \|_{0,\Omega}$$

The bilinear form d satisfy the weak coerciveness such as. There exist a constant (negative) ε such as

Proposition 9:

$$d(\chi,\chi) \geq -\varepsilon \| \chi \|_{0,\Omega}^2 \ \ \forall\chi \in L_0^2(\Omega) \tag{89}$$

Proof: The weak coerciveness of the form bilinear d is always satisfied for any $\varepsilon \geq -(\mu+\lambda)$

Theorem 10: The generalized variational problem (17) has a unique solution $(u,\psi) \in V(\Omega)\times L_0^2(\Omega)$ for any $L \in (V(\Omega))^{'}$ as long as

$$\xi = \frac{\delta^2 \ \| \ b_\Gamma \ \|\| \ b_\Gamma \ - \ b \ \|}{\delta \ \| \ a \ \|^{-2} \ \vartheta^2 - \varepsilon} < 1 \tag{90}$$

With $\delta \| a \|^{-2} \vartheta^2 > \varepsilon$

Further, the solution has the stability estimates

$$\| u \|_{1,\Omega} \leq \frac{1}{1-\xi} \ \| u_0 \|_{1,\Omega} \tag{91}$$

$$\| \chi \|_{0,\Omega} \leq \| \chi_0 \ \|_{0,\Omega} + \frac{\xi \ \delta}{\| \ b_\Gamma \ \| \ (1 \ - \ \xi)} \ \| u_0 \|_{1,\Omega} \tag{92}$$

Where (u_0,χ_0) the bounded solution of (34) with $G=0$ such as

$$\| \chi_0 \|_{0,\Omega} \leq \frac{\delta^{-1} \ \| \ b_\Gamma \ \| \ \| \ L \ \|}{\delta \ \| \ a \ \|^{-2} \ \vartheta^2 \ - \ \varepsilon} \tag{93}$$

$$\| u_0 \|_{0,\Omega} \leq \ \delta^{-1} \ (\| \ L \ \| \ + \ \| \ B_\Gamma \ \| \ \| \ \chi_0 \ \|_{0,\Omega}) \tag{94}$$

Proof: We just apply the propositions 3, 8, 7, 9 previous and by the theorem 1 taking $G=0$

since $\varepsilon \geq -(\mu+\lambda)$ the condition (90) will occur if

$$\| b_\Gamma \|\| b - b_\Gamma \| < \frac{\delta \| a \|^{-2} \vartheta^2 + \mu + \lambda}{\delta^2}$$

IV. Mixed Finite Element

The term mixed method was first used in the 1960's to describe finite element methods in which both stress and displacement fields are approximated as primary variables.

In numerical analysis, the mixed finite element method, also known as the hybrid finite element method, is a type of finite element method in which extra independent variables are introduced as nodal variables during the discretization of a partial differential equation problem. The extra independent variables are constrained by using Lagrange multipliers. To be distinguished from the mixed finite element method, usual finite element methods that do not introduce such extra independent variables are also called irreducible finite element methods. The mixed finite element method is efficient for some problems that would be numerically ill-posed if discretized by using the irreducible finite element method, one example of such problems is to compute the stress and strain fields in an almost incompressible elastic body [13].

To apply the method of mixed finite element $P1-bubble/P1$ for the variational problem (17). We need some mathematical tools, then we us the approximation of the standard Galerkin method, for more explication we can see in the articles and books [1], [6]–[9], [15], [19]. Let's consider a uniform triangulation T_h of the rectangular domain Ω, where $h>0$ is the maximum diameter of all elements $K \in T_h$, and T_h consists of triangles in two dimensions. We assume that we have a sequence of triangulations $(T_h)_h h \to 0$. Let λ_1^K, λ_2^K, λ_2^K be the barycenter coordinates with respect to a triangle K . μ^K is the bubble function associated with the triangle K defined by $\mu^K = \lambda_1^K\lambda_2^K\lambda_3^K$ in K and equal to 0 elsewhere

We define the discrete domain

$\Omega_h = \bigcup_{k=1}^n T_k$, and Ω_h is closed if Ω is polygon, then $\Omega_h = \overline{\Omega}$ and $\Gamma_h = \partial\Omega_h = \partial\Omega = \Gamma$ $\mathbb{P}_1(K)$ is the space of polynomials defined on the triangle K of the degree lower or equal to 1. The functions of $V_h \times M_h$ are not globally affine in all Ω, but only affine by piece. On the other hand, they are generally continuous. The functions of the space are completely determined by their values in each of the mesh vertices.For the solution of an elasticity problem, the displacement/div-displacement (u/ψ) finite element discretization are effective in [3].

Let V_h be the finite element displacement interpolation space and M_h be the finite element div-displacement interpolation space (corresponding to the spaces $V(\Omega)$ and $M = L_0^2(\Omega)$ of the continuous problem. The functions of the space V_h are completely determined by their values in each of the mesh vertices. Moreover the dimension of the space V_h is $N-n_s$, with N is the overall number of vertices and n_s the number of vertices on the boundaries. Then the mixed finite elements problem is like. We define the approached spaces as follow

For all $(u_h,\psi_h) \in V_h\times M_h \subset V\times M$, to facilitate writing we note the restriction of u_h and ψ_h on K by u_h and ψ_h respectively, then we have

$$u_h = \sum_{\mathbf{i=1}}^{\mathbf{3}} \alpha_{\mathbf{i}}^{\mathbf{K}}\lambda_{\mathbf{i}}^{\mathbf{K}} + \beta^{\mathbf{K}}\mu^{\mathbf{K}}(\mathbf{x}) \ , \ \alpha_{\mathbf{i}}^{\mathbf{K}}, \beta^{\mathbf{K}} \in \mathbb{R}^{\mathbf{2}} \tag{95}$$

$$\psi_h = \sum_{\mathbf{i=1}}^{\mathbf{3}} \theta_{\mathbf{i}}^{\mathbf{K}}\lambda_{\mathbf{i}}^{\mathbf{K}} \ , \ \theta_{\mathbf{i}}^{\mathbf{K}} \in \mathbb{R} \ , \ \forall\mathbf{K} \in \mathrm{T}_{\mathbf{h}}. \tag{96}$$

(97)

Let's seek $(u_h,\psi_h) \in V_h\times M_h$

$$\begin{cases} a(u_h,v_h) + b_\Gamma(v_h,\psi_h) = L_h(v_h), \\ b(u_h,q_h) - d_h(\psi_h,q_h) = 0 \end{cases} \tag{98}$$

$\forall v_h \in V_h$, $\forall q_h \in M_h$, where

$$a(u_h,v_h) = \int_K \mu\nabla u_h : \nabla v_h dK \tag{99}$$

$$+ \int_{\Gamma_h} \mathbf{B^{-1}Au_h.v_h d\Gamma_h}, \tag{100}$$

$$b(v_h,q_h) = \int_K (\lambda+\mu) q_h \nabla.v_h dK, \tag{101}$$

$$b_\Gamma(v_h,q_h) = b(v_h,q_h) + \int_{\Gamma_h} \mu q_h n_K.v_h d\Gamma_h, \tag{102}$$

$$d(\psi_h,q_h) = \int_K (\lambda+\mu)\psi_h q_h dK, \tag{103}$$

$$L(v_h) = \int_K f.v_h dK + \int_{\Gamma_h} \mathbf{B^{-1} g_h . v_h d\Gamma_h}, \quad (104)$$

With $\Gamma_h = \Gamma \bigcap \partial K$ and n_K the normal on K .
The existence and uniqueness of the solution of the mixed formulation (98) is shown by using the continuity of the bilinear forms a on $V_h \times V_h$, b_Γ on $V_h \times M_h$, b on $V_h \times M_h$ and d on $M_h \times M_h$ is clear by using the korn's inequality. On the other hand the coercivity of the bilinear form a on V_h and d on M_h is hold by using theirs coercivity on $V(\Omega)$ and $M(\Omega)$ respectively since $V_h \subset V(\Omega)$. We can see the uniform $\inf - \sup$ condition uniformly of the bilinear form b in [5] with respect to the mesh-size. We will prove that the uniform $\inf - \sup$ condition for the bilinear form b_Γ on $V_h \times M_h$, it means that we have to prove the existence of a constant $\vartheta > 0$ independent of the mesh-size as the following theorem clarifies

Theorem 11: there exist $\vartheta > 0$ such as

$$\begin{array}{c} \forall q_h \in M_h \; \exists u_h \in V_h, u_h \neq 0, \\ b_\Gamma(u_h, q_h) \geq \vartheta \parallel u_h \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega} \end{array} \quad (105)$$

This theorem guarantee the verification of the condition $\inf - \sup$ of the bilinear form b_Γ. It should be noted that for all $u_h \in V_h$

$$b_\Gamma(u_h, q_h) = b(u_h, q_h) + \int_{\Gamma_h} \mu \, q_h \, n_K.v_h \, d\Gamma_h \quad (106)$$

Proof: first we prove that the bilinear form b verifies (105) of theorem (11). It is assumed that the triangulation $\mathtt{T}_h$ is uniformly regular. Let $q_h \in M_h$ be fixed
$M_h \subset M$ and that the bilinear form b satisfies the $\inf - \sup$ condition in $V \times M$, so that there exist $u \in V$ and $b(u, q_h) \geq \beta \parallel u \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega}$. With $\beta > 0$ independent of q_h, but u depends of q_h. For this u we have just show that $u_h \in V_h$ clarified below

$$b(u_h, q_h) = b(u, q_h), \quad (107)$$

$$\parallel u_h \parallel_{1,\Omega} \leq c \parallel u \parallel_{1,\Omega}, \quad (108)$$

The relations (107) and (108) are the subject of the lemma (12) that comes afterwards. Where $c > 0$ is independent of q_h and h, indeed if (107) are checked so

$$b(u_h, q_h) = b(u, q_h) \geq \quad \beta \parallel u \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega} \quad (109)$$

$$\geq \quad \frac{\beta}{c} \parallel u_h \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega}, \quad (110)$$

just take $\beta' = \frac{\beta}{c}$. Otherwise, we can easily check that

$$\int_\Gamma \mid \mu q n.u \mid d\Gamma \leq c' \mid \mu \mid \parallel u \parallel_{1,\Omega} \parallel q \parallel_{0,\Omega}, \quad (111)$$

Then, $V_h \subset V$ and $M_h \subset M$ we obtain

$$\int_\Gamma \mid \mu q_h n.u_h \mid d\Gamma \leq c' \mid \mu \mid \parallel u_h \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega} \quad (112)$$

Combining (112) and (109) Moreover, we assume that $\mu q_h n.u_h$ adopts a negative sign, then we have

$$b_\Gamma(u_h, q_h) \geq (\beta' - c')\mu \parallel q_h \parallel_{0,\Omega} \parallel u_h \parallel_{1,\Omega} \quad (113)$$

If we suppose $\mu < \frac{\beta'}{c'}$, also we get $\vartheta_0 = (\beta' - c')\mu$
So whatever the sign of $\mu q_h n.u_h$ we conclude that

$$b_\Gamma(u_h, q_h) \geq \min(\vartheta_0, \beta') \parallel u_h \parallel_{1,\Omega} \parallel q_h \parallel_{0,\Omega} \quad (114)$$

just take $\vartheta = \min(\vartheta_0, \beta')$, it means that we answer the theorem (11)

Lemma 12: there exist $u_h \in V_h$, $c > 0$, and we suppose that u is fixed, such as

$$b(u_h, q_h) = b(u, q_h) \; \forall q_h \in M_h \quad (115)$$

$$\parallel u_h \parallel_{1,\Omega} \leq c \parallel u \parallel_{1,\Omega} \quad (116)$$

Proof: First we define the linear operator $R_h \in \mathrm{L(V, V_h)}$, which verifies
$\forall v_h \in V_h, \; \exists ! \; v \in V$, so that $R_h v = v_h$. This operator is a projector of V on V_h, it is well defined. Indeed Lax Milgram ensures the unique existence of the variational problem (117) Find v in V so as

$$\int_\Omega \nabla(R_h v - v).\nabla v_h dx = 0 \; , \; \forall v_h \in V_h \quad (117)$$

If we take $v_h = R_h v$ in (117), we obtain

$$\parallel \nabla R_h v \parallel_{0,\Omega}^2 = \int_\Omega \nabla v.\nabla v_h dx \quad (118)$$

We use the inequality of Hölder, then we get

$$\parallel \nabla R_h v \parallel_{0,\Omega}^2 \leq \parallel \nabla R_h v \parallel_{0,\Omega} \parallel \nabla v \parallel_{0,\Omega} \quad (119)$$

We conclude that

$$\parallel \nabla R_h v \parallel_{0,\Omega} \leq \parallel \nabla v \parallel_{0,\Omega} \quad (120)$$

we find that R_h is continuous in the sense of $H_0^1(\Omega) \cap V$, and by the density of $H_0^1(\Omega)$ in $H^1(\Omega)$, it means that $\overline{H_0^1(\Omega) \cap V}^{\|\cdot\|_{1,\Omega}} \subset H^1(\Omega)$, so we have its continuity in the space V then, we suppose that the operator R_h checks the next properties of estimation, that is to say, there exist $c > 0$ independent of h and v

$$\parallel (R_h v - v) \parallel_{0,\Omega} \leq \; c \, h \parallel \nabla v \parallel_{0,\Omega} \quad (121)$$

The above findings are workable for all $v \in V$.
Now, let prove that the relation (115) in lemma (12), just show

$$\int_{\Omega_h} \nabla.u_h q_h dx = \int_{\Omega_h} \nabla.u q_h dx \; \forall K \in \mathtt{T}_h \quad (122)$$

By applying the Green formula, we find that (122) is equivalent to

$$\int_{\Omega_h} \nabla q_h.u_h dx = \int_{\Omega_h} \nabla q_h.u dx \quad (123)$$

But ∇q_h is constant when $q_h \in \mathbb{P}_1(K)$. So, just prove that there exist $u_h \in V_h$, such as

$$\int_K u_h dx = \int_K u dx \; \forall K \in \mathtt{T}_h \quad (124)$$

Indeed, by the definition of the space V_h , for every function u_h of V_h is determined by the relation

$$u_h = \sum_{i=1}^{3} u_h(a_i^K)\lambda_i(x) + \beta^K \mu^K(x) \; \forall x \in K \quad (125)$$

We go to the integral on K, we have

$$\int_K u_h dx = \sum_{i=1}^{3} u_h(a_i^K) \int_K \lambda_i(x)dx \qquad (126)$$
$$+ \quad \beta^K \int_K \mu^K(x)dx \qquad (127)$$
$$= \sum_{i=1}^{3} R_h u(a_i^K) \int_K \lambda_i(x)dx \qquad (128)$$
$$+ \quad \beta^K \int_K \mu^K(x)dx \qquad (129)$$
$$= \int_K R_h u dx = \int_K u dx \qquad (130)$$

We have chosen $u_h \in V_h$ such as, $u_h(a_i^K) = R_h u(a_i^K)$
For all a_i^K top of K, who verifies (115) of lemma (12).
Now let prove the relation (116) of lemma (12) we have

$$u_{h|_K} = \sum_{i=1}^{3} R_h u(a_i^K)\lambda_i + \beta^K \mu^K, \qquad (131)$$

with a simple writing $u_{h|_K} = R_h u + \beta^K \mu^K$
And
$\| u_h \|^2_{1,\Omega} = \sum_{K\in \mathbf{T_h}} \| u_h \|^2_{1,K}$, then

$$\| u_h \|^2_{1,\Omega} \leq \sum_{K\in \mathbf{T_h}} (\| R_h u \|_{1,K} + \| \beta^K \| \| \mu^K \|_{1,K})^2 \qquad (132)$$
$$\leq 2 \sum_{K\in \mathbf{T_h}} (\| R_h u \|^2_{1,K} + \| \beta^K \|^2 \| \mu^K \|^2_{1,K}) \qquad (133)$$
$$\leq 2 \| R_h u \|^2_{1,\Omega} + 2 \sum_{K\in \mathbf{T_h}} \| \beta^K \|^2 \| \mu^K \|^2_{1,K} \qquad (134)$$

By the continuity of the operator R_h

$$\| u_h \|^2_{1,\Omega} \leq c \ \| u \|^2_{1,\Omega} + 2 \sum_{K\in \mathbf{T_h}} \| \beta^K \|^2 \| \mu^K \|^2_{1,K}, \qquad (135)$$

By using the relation (115) of lemma (12)
And
$\int_K u_h dx = \int_K R_h u dx + \beta^K \int_K \mu^K dx$
We find that

$$\beta^K = \frac{\int_K (u - R_h u)dx}{\int_K \mu^K dx} \qquad (136)$$

Using Cauchy-Schwarz, it gives the existence of $c_0 > 0$ independent of h and K, so as

$$\| \beta^K \|^2 \leq c_0 \frac{\| u - R_h u \|^2_{0,K}}{h^2} \qquad (137)$$

The function μ^K is bounded in the sense of the norm $\|, \|_{1,K}$, and by the relation (121) we get the relation (116) of lemma (12) .
Finally, it was also shown that the mini-element $P1-bubble \setminus P1$ satisfies the condition $\inf-\sup$ discreet

Theorem 13: There exist strictly positive constant ξ, for all $u_h \in V_h$ we have

$$a(u_h, u_h) \geq \xi \| u_h \|^2_{1,\Omega} \qquad (138)$$

The formula (138) indicates that the bilinear form a is coercive on the space V_h.
Proof: we define the bilinear form a_0 on V_h

$$a_0(u_h, v_h) = \int_\Omega \mu \nabla u_h : \nabla v_h d\Omega \qquad (139)$$

With (139), the bilinear form can a be written in the form $a(u_h, v_h) = a_0(u_h, v_h) + \int_\Gamma \mathbf{B^{-1}Au_h.v_h d\Gamma}$
To show that a_0 verifies this property, it suffices to show that it is definite positive, since the space V_h is of finite dimension.
if $a(u_h, v_h)$ is zero, then u_h is constant on each triangle. Since there is continuity in the middle of each edge, then u_h is globally constant. As it vanishes on the midst of the edge contained in the boundary of Ω. So u_h is identically zero. Then we can write

$$a_0(u_h, u_h) \geq \xi_0 \| u_h \|^2_{1,\Omega}, \qquad (140)$$

By using (9), it is easy to show that there exist a strictly positive constants ν, and for all $u_h \in V_h$ we have

$$\int_\Gamma | \mathbf{B^{-1}Au_h.u_h} | \mathbf{d\Gamma} \leq \nu \| \mathbf{u_h} \|^{\mathbf{2}}_{\mathbf{1,\Omega}} \qquad (141)$$

If $\mathbf{B^{-1}Au_h.u_h < 0}$, with (140) and (141), we obtain

$$a(u_h, u_h) \geq (\xi_0 - \nu) \| u_h \|^2_{1,\Omega}, \qquad (142)$$

since $\xi_0 - \nu > 0$.
if not $\mathbf{B^{-1}Au_h.u_h > 0}$, we have

$$a(u_h, u_h) \geq a_0(u_h, u_h) \qquad (143)$$
$$\geq \xi_0 \| u_h \|^2_{1,\Omega} \qquad (144)$$

Finally, we Combine (144) and (142), we result that $a(u_h, u_h) \geq (\xi_0 - \nu) \| u_h \|^2_{1,\Omega}$for all $u_h \in V_h$, since $\xi_0 - \nu > 0$

V. ALGEBRIC PROBLEM

In this section we introduce the Matrices A, B_Γ, B, D, L related to the descrete bilinear forms a_h,$b_{\Gamma h}$, b_h, d_h, L_h respectively in the following way and we can express the bilinear forms according to the operators as well defined here

$$\begin{cases} a_h(u_h, v_h) = (Au_h, v_h), \\ b_{\Gamma h}(v_h, q_h) = (B_\Gamma v_h, q_h), \\ b_h(u_h, q_h) = (Bv_h, q_h), \\ d_h(\psi_h, q_h) = (D\psi_h, q_h), \\ L_h(v_h) = Lv_h, \end{cases} \qquad (145)$$

$\forall (u_h, v_h) \in V_h(\Omega) \times V_h(\Omega)$,
$\forall (\psi_h, q_h) \in M_h(\Omega) \times M_h(\Omega)$
With(145), we find that the discrete formulation (98) can be expressed as a system of operator equations

$$\begin{cases} A\ u_h + B_\Gamma^t\ \psi_h = L, \\ B\ u_h - D\ \psi_h = 0, \end{cases} \qquad (146)$$

We find that the discrete formulation can be expressed as a system of linear equations as well

$$\begin{pmatrix} A & B_\Gamma^t \\ B & -D \end{pmatrix}\begin{pmatrix} u_h \\ \psi_h \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix} \tag{147}$$

with $u_h = (u_x, u_y)^t$, we can express the algebric system (146) as follows

$$\begin{pmatrix} A_x & 0 & B_{\Gamma,x}^t \\ 0 & A_y & B_{\Gamma,y}^t \\ B_x & B_y & -D \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ \psi \end{pmatrix} = \begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} \tag{148}$$

Let $\{\varphi_1;\varphi_2....;\varphi_n\}$ be the finite element basis formed of scalar functions φ_i, $i = 1...n$.In practice the two components of (u_h^x, u_h^y) of u_h are always appreciated by one space finite element.Let N be the number of nodes in the finite element mesh, and $n = N - n_s$ with ns the number of vertices on the boundaries.The basis of the space V_h

$$\mathbf{B_{V_h}} = \{\phi_\mathbf{1} = (\varphi_\mathbf{1}, \mathbf{0})...\phi_\mathbf{n} \tag{149}$$

$$= (\varphi_\mathbf{n}, \mathbf{0}), \phi_\mathbf{n+1} = (\mathbf{0}, \varphi_\mathbf{1})...\phi_\mathbf{2n} = (\mathbf{0}, \varphi_\mathbf{n})\}, \tag{150}$$

Then, $u_h = (u_h^x, u_h^y) \in V_h$ can be gives by the relation

$$u_h = u_1^x\phi_1 + ... + u_n^x\phi_n + u_1^y\phi_{n+1} + ... + u_n^y\phi_{2n}, \tag{151}$$

For a given triangle K, the displacement field u_h and the divergence ψ_h are approximated by linear combinations of the basis functions in the form

$$\mathbf{u_h^x} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \mathbf{u_i^x}\varphi_\mathbf{i} + \mathbf{u_b}\varphi_\mathbf{b}, \tag{152}$$

$$\mathbf{u_h^y} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \mathbf{u_i^y}\varphi_\mathbf{i} + \mathbf{u_b}\varphi_\mathbf{b}, \tag{153}$$

$$\psi_\mathbf{h} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \psi_\mathbf{i}\varphi_\mathbf{i} \tag{154}$$

The linear system (148), attached to the discrete system (146) is evaluated over each triangle K to obtain the element of the local matrices and the global matrices are denoted by uppercase letters, and are given by direct-summing.
Assuming that $(\mathbf{B^{-1}A})_\mathbf{ij} = \alpha_\mathbf{ij}$, and $(\mathbf{B^{-1}})_\mathbf{ij} = \beta_\mathbf{ij}$ for $i, j = 1, 2$
The matrices elements over the domain Ω are given by

$$a_{ij}^0 = \int_K \mu\nabla\varphi_i\nabla\varphi_j dK \tag{155}$$

$$A_0 = \sum_{K\in T_h} a_{ij}^0, \tag{156}$$

$$a_{ij}^x = a_{ij}^0 + \int_{E\cap K\subset\Gamma_h} \alpha_{11}\varphi_i\varphi_j d\Gamma_h, \tag{157}$$

$$A_x = \sum_{K\in T_h} a_{ij}^x, \tag{158}$$

$$a_{ij}^y = a_{ij}^0 + \int_{E\cap K\subset\Gamma_h} \alpha_{22}\varphi_i\varphi_j d\Gamma_h, \tag{159}$$

$$A_y = \sum_{K\in T_h} a_{ij}^y, \tag{160}$$

$$b_{ij}^x = \int_K (\lambda+\mu)\frac{\partial\varphi_i}{\partial x}\varphi_j dK \tag{161}$$

$$B^x = \sum_{K\in T_h} b_{ij}^x, \tag{162}$$

$$b_{ij}^y = \int_K (\lambda+\mu)\frac{\partial\varphi_i}{\partial y}\varphi_j dK \tag{163}$$

$$B^y = \sum_{K\in T_h} b_{ij}^y, \tag{164}$$

$$b_{\Gamma_{ij}}^x = b_{ij}^x + \int_{E\cap K\subset\Gamma_h} \mu\,\varphi_i n_{ij}\varphi_j\, dE \tag{165}$$

$$B_\Gamma^x = \sum_{K\in T_h} b_{\Gamma_{ij}}^x, \tag{166}$$

$$b_{\Gamma_{ij}}^y = b_{ij}^y + \int_{E\cap K\subset\Gamma_h} \mu\,\varphi_i n_{ij}\varphi_j\, dE \tag{167}$$

$$B_\Gamma^y = \sum_{K\in T_h} b_{\Gamma_{ij}}^y, \tag{168}$$

$$d_{ij} = \int_K (\lambda+\mu)\,\varphi_i\,\varphi_j dK \tag{169}$$

$$D = \sum_{K\in T_h} d_{ij}, \tag{170}$$

$$l_i^{0x} = \int_K f_1\varphi_i dK \tag{171}$$

$$l_i^x = l_i^{0x} + \int_{E\cap K\subset\Gamma_{\Gamma h}} (\beta_{11}+\beta_{21})g_1\varphi_i dE \tag{172}$$

$$L^x = \sum_{K\in T_h} l_i^x, \tag{173}$$

$$l_i^{0y} = \int_K f_2\varphi_i dK \tag{174}$$

$$l_i^y = l_i^{0y} + \int_{E\cap K\subset\Gamma_{\Gamma h}} (\beta_{12}+\beta_{22})g_2\varphi_i \tag{175}$$

$$L^y = \sum_{K\in T_h} l_i^y dE, \tag{176}$$

knowing that $f = (f_1, f_2)^t, g = (g_1, g_2)^t$, $n_{ij} = 0, 1$ or -1

Element Matrices

In this paragraph, we calculate elementary matrices in (159), in order to carry them out.
The calculations which have already been made in the article [11] will be based.
For a triangle K let $(x_i; y_i)_{i=1,2,3}$ be the vertices and the basis functions are defined by

$$\varphi_1(x,y) = 1 - x - y\ ,\ \varphi_2(x,y) = x \tag{177}$$

$$\varphi_3(x,y) = y\ ,\ \varphi_b = 27\varphi_1\varphi_2\varphi_3, \tag{178}$$

We need the following notations

$$x_{ij} = x_i - x_j\ ,\ y_{ij} = y_i - y_j \quad i,j = 1,2,3 \tag{179}$$

And

$$x^{(K)} = \begin{pmatrix} x_{32} \\ x_{13} \\ x_{21} \end{pmatrix} = \begin{pmatrix} x_1^{(K)} \\ x_2^{(K)} \\ x_3^{(K)} \end{pmatrix} \tag{180}$$

$$y^{(K)} = \begin{pmatrix} y_{32} \\ y_{13} \\ y_{21} \end{pmatrix} = \begin{pmatrix} y_1^{(K)} \\ y_2^{(K)} \\ y_3^{(K)} \end{pmatrix} \tag{181}$$

The nonbubble part of the matrix A is written like

$$A_0^K = (a_{ij}^0)_{i,j=1,2,3} = \frac{\mu}{4 \mid K \mid}(y^{(K)}y^{(K)_t} + x^{(K)}x^{(K)_t}), \tag{182}$$

With $\mid K \mid = \frac{x_3^{(K)}y_1^{(K)} - x_2^{(K)}y_3^{(K)}}{2}$. The bubble part of A_0 are $A_{b.} = (a_{bj})_{j=1,2,3} = \mathbf{0}_{31}$, with $\mathbf{0}_{31}$ is a Zero column vector with 3 elements, and for the diagonal

$$a_{bb} = \frac{81\mu}{10} \mid K \mid (\mid \nabla\varphi_1 \mid^2 \tag{183}$$

$$+ \mid \nabla\varphi_2 \mid^2 + \nabla\varphi_1.\nabla\varphi_2) \tag{184}$$

$$= \frac{81\mu}{40}\frac{1}{\mid K \mid}((x_1^{(K)})^2 + (x_2^{(K)})^2 \tag{185}$$

$$+ (y_1^{(K)})^2 + (y_2^{(K)})^2 + x_1^{(K)}x_2^{(K)} + y_1^{(K)}y_2^{(K)}) \tag{186}$$

Since the stiffness matrix A_0 is symmetric, then we have

$$A_0 = \begin{pmatrix} A_0^K & A_{b.} \\ A_{b.}^t & a_{bb} \end{pmatrix} = \begin{pmatrix} A_0^K & 0 \\ 0 & a_{bb} \end{pmatrix} \tag{187}$$

the the mass matrix D is given by for $i, j = 1, 2, 3$

$$D = \begin{cases} \frac{\mu + \lambda}{6} \mid K \mid, \ i = j \\ \frac{\mu + \lambda}{12} \mid K \mid \quad i \neq j \end{cases} \tag{188}$$

Now we will implement the divergence matrices, the element matrices of nonbubble part of B_x and B_y are given by

$$B_x^K = \frac{1}{6}(\mu + \lambda)\begin{pmatrix} y^{(K)_t} \\ y^{(K)_t} \\ y^{(K)_t} \end{pmatrix} \tag{189}$$

$$B_y^K = \frac{1}{6}(\mu + \lambda)\begin{pmatrix} x^{(K)_t} \\ x^{(K)_t} \\ x^{(K)_t} \end{pmatrix} \tag{190}$$

the bubble part are given like

$$B_{xb} = (\mu + \lambda)\frac{9}{40}y^{(K)_t} \tag{191}$$

$$B_{yb} = (\mu + \lambda)\frac{9}{40}x^{(K)_t} \tag{192}$$

Finally we find that

$$B_x = \begin{pmatrix} B_x^K \\ B_{xb} \end{pmatrix} \tag{193}$$

$$B_y = \begin{pmatrix} B_y^K \\ B_{yb} \end{pmatrix} \tag{194}$$

Element right-hand side

The nonbubble part of the external forces is given by

$$l_K^{0x} = \frac{\mid K \mid}{3} f_{1K} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{195}$$

$$l_K^{0y} = \frac{\mid K \mid}{3} f_{2K} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{196}$$

where

$$f_{iK} = \frac{(f_i(x_1) + f_i(x_2) + f_i(x_3))}{3} \ , \ \ i = 1, 2 \tag{197}$$

The bubble part of the right-hand side are

$$l_{xb} = \frac{9}{20} \mid K \mid f_{1K} \tag{198}$$

$$l_{yb} = \frac{9}{20} \mid K \mid f_{2K} \tag{199}$$

Finally we have

$$l_x = \begin{pmatrix} l_K^{0x} \\ l_{xb} \end{pmatrix} \tag{200}$$

$$l_y = \begin{pmatrix} l_K^{0y} \\ l_{yb} \end{pmatrix} \tag{201}$$

Now we will build the matrix of 11×11 corresponding to the system (148)

$$\begin{pmatrix} A_x^k & 0 & 0 & 0 & B_{\Gamma,x}^t \\ 0 & a_{bb} & 0 & 0 & B_{b\Gamma,x}^t \\ 0 & 0 & A_y^k & 0 & B_{\Gamma,y}^t \\ 0 & 0 & 0 & a_{bb} & B_{b\Gamma,y}^t \\ B_x^K & B_{bx} & B_y^K & B_{by} & -D \end{pmatrix} \begin{pmatrix} u_x \\ u_{bx} \\ u_y \\ u_{by} \\ \psi \end{pmatrix} = \begin{pmatrix} l_K^x \\ l_{bx} \\ l_K^x \\ l_{by} \\ 0 \end{pmatrix} \tag{202}$$

We will classify the system (202) in such a way that we can describe the bubble part of the unknown u as a function of the nonbubble part of u and ψ, in order to be able to eliminate it thereafter

$$\begin{pmatrix} A_x^k & 0 & 0 & 0 & B_{\Gamma,x}^t \\ 0 & A_y^k & 0 & 0 & B_{\Gamma,y}^t \\ 0 & 0 & a_{bb} & 0 & B_{b\Gamma,x}^t \\ 0 & 0 & 0 & a_{bb} & B_{b\Gamma,y}^t \\ B_x^K & B_y^K & B_{bx} & B_{by} & -D \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_{bx} \\ u_{by} \\ \psi \end{pmatrix} = \begin{pmatrix} l_x^K \\ l_y^K \\ l_{bx} \\ l_{by} \\ 0 \end{pmatrix} \tag{203}$$

From line 3 and 4 of the system (203), we deduces

$$\begin{aligned} u_{bx} &= a_{bb}^{-1}(l_{bx} - B_{b\Gamma,x}^t\psi) \\ u_{by} &= a_{bb}^{-1}(l_{by} - B_{b\Gamma,y}^t\psi) \end{aligned} \tag{204}$$

with the elimination of u_{bx} and u_{by} we find the new linear system with the Matrix in $\mathbb{U} = \begin{pmatrix} u_x \\ u_y \\ \psi \end{pmatrix}$

$$\mathbb{A} = \begin{pmatrix} A_x^k & 0 & B_{\Gamma,x}^t \\ 0 & A_y^k & B_{\Gamma,y}^t \\ B_x^K & B_y^K & -B_{bx}B_{b\Gamma,x}^t a_{bb}^{-1} - B_{by}B_{b\Gamma,y}^t a_{bb}^{-1} - D \end{pmatrix} \tag{205}$$

and the right hand

$$\mathbb{F} = \begin{pmatrix} l_x^K \\ l_y^K \\ -B_{bx} l_{bx} a_{bb}^{-1} - B_{by} l_{by} a_{bb}^{-1} \end{pmatrix} \qquad (206)$$

VI. Numerical Results

This section is the fruit of the study that is done in the previous sections in this paper, with its two sides: mathematics and programming. So there is some numerical results of calculations with mixed finite element $P1-bublle/P1$ method and the ordinary finite element method that will be presented later.

Matlab is an interactive environment and programming language for scientific computation. It is nowadays a widely used tool in education, engineering and research and becomes a standard tool in many areas. But Matlab is a matrix language and its distinguishing features is the use of matrices as the main data type. For best performance in large scale problems, one should take advantage of this by using vector and matrix operations.

We propose a vectorized Matlab implementation of the $P1-bubble/P1$ finite element (Mini element) for the generalized elasticity problem. Vectorization means that our code operates on array and does not use for loops for the assembling operations. Our implementation needs only Matlab basic distribution functions and can be easily modified and refined [11].

A two-dimensional problem with the elastic square body with a hole (see. Figs 1, 2), $\Omega = [0,2] \times [0,2] \setminus B(0,1)$, $(E = 2900 \; and \; nu = 0.25)$, is stretched at the top $(y = 2)$ with a surface load $g = n$, where n denotes the outer normal to $\partial\Omega$, the rest of the boundary is traction free [20]. By reason of symmetry since the domain is homogeneous,one will present the quarter of the domain that was discretized. The Dirichlet conditions are $h = 0$ on $[1,2] \times 0$ and $0 \times [1,2]$ the forces charges are taking as $f = (0, -(\mu + \lambda))^t$ for all nodes, with this f we propose the exact solution as like as $u(x,y) = (xy, xy + x)$.

The first objective of this numerical experiment is to test the stability of the divergence of the field of displacement u_h numerical solution,in fact we will calculate the error $e_\infty = \max_{i,j}(| \; divu(x_i, y_j) - \psi_{i,j} \; |)$ for three meshes,and we will see how this error behaves when the mesh size grows,That is when $h \rightarrow 0$. So from the tabular the errore_∞ go to zero when h is too small. The example demonstrates how the mixed finite element P1 bubble-P1 method is more efficient then the ordinary one. Because it allows us to calculate the displacements and their divergences simultaneously,and that it guarantees the stability of these divergences on each node.

Secondly, we calculate $\| \; u - u_h \; \|_{1,\Omega}$ for each method, then we obtain the two slopes by using the linear regression and the table 1 resume all the numerical results.

This example shows how to perform simple linear regression using the errors data set. The example also shows you how to calculate the coefficient of determination to evaluate the regressions.

we present two figures 9, and 10 of the linear regression for each methods

Fig. 1: reusable coolers

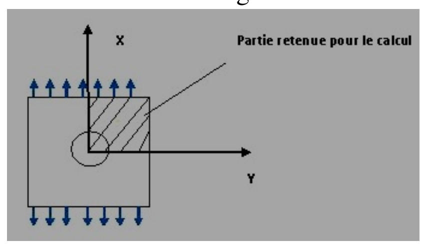

Fig. 2: square with hole

Below, we find the tabular 1 summarizes all the errors that have been calculated, and the figures 3,4,5,6,7,8, which represent the displacements of the membrane with hole defined above for three steps $h = 0.3, 0.2, 0.1$ for both method MFEM and abaqus software. The blue color that appears at the edges $[1,2] \times \{0\}$ and $\{0\} \times [1,2]$ means that displacements are almost zero, so it reflects the boundary conditions of Dirichlet $u = 0$. Then we observe the great displacements are those which exits to the edge $[1,2] \times \{2\}$ where there is a traction. We observe that there is a similarity of results between our method and abaqus, it means that this method of MFEM is reliable and provided a good solution. The figures 9, and 10 represent the linear correlation between $log(\| \; u - u_h \; \|_{1,\Omega})$ and $log(h)$, with which the speed of convergence has been calculated.

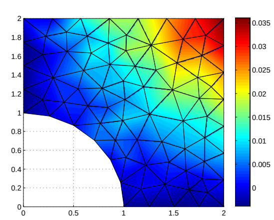

Fig. 3: Deformed mesh for membrane with hole,case h=0.3 with MFEM

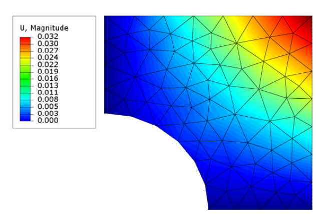

Fig. 4: Deformed mesh for membrane with hole,case h=0.3 with abaqus system

VII. Conclusion and discussion

In this paper, we have proposed a mixed finite element $P1$ bubble - $P1$ method for solving the system of Navier Lamé.

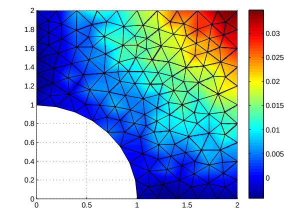

Fig. 5: Deformed mesh for membrane with hole,case h=0.2 with MFEM

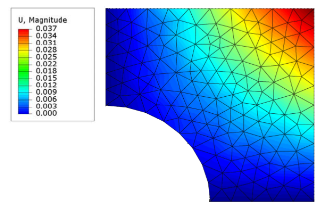

Fig. 6: Deformed mesh for membrane with hole,case h=0.2 with abaqus system

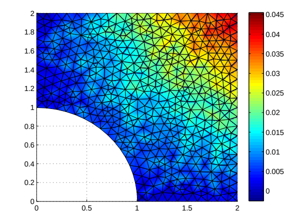

Fig. 7: Deformed mesh for membrane with hole,case h=0.1 with MFEM

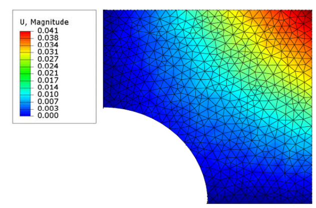

Fig. 8: Deformed mesh for membrane with hole,case h=0.1 with abaqus system

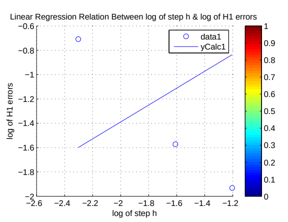

Fig. 9: the slope $\alpha = 0.694$ with FEM

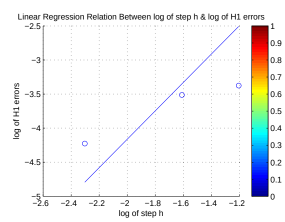

Fig. 10: the slope $\alpha = 2.082$ with MFEM

TABLE I: The table showing the different errors

Number of nodes np	80	164	589
Number of elements nt	130	287	1099
step h	0.3	0.2	0.1
e_∞	0.2252	0.2004	0.1699
$\| u - u_h \|_{1,\Omega}$ with MFEM	0.0342	0.0298	0.0146
$\| \psi - \psi_h \|_{0,\Omega}$ with MFEM	0.0567	0.0346	0.0264
$\| u - u_h \|_{1,\Omega}$ with FEM	0.4924	0.2075	0.1448
MFEM slope $\alpha = 2.082$			
FEM slope $\alpha = 0.694$			

A number of reasons have been put to prefer mixed methods over displacement or equilibrium methods in some situations. First of all, equilibrium methods are rarely used in practical computation due to the difficulty of creating finite element spaces incorporating the necessary constraints (the conditions of static admissability and, in particular, the equilibrium condition in the case of elasticity). As remarked above, for the elasticity problem, in which the a form is coercive, stability can always be achieved by adequate enrichment of the displacement space. There are a number of ways to enrich the space. For our example, the unstable pair (linear displacement, linear divergence) element may be stabilized by the addition of a single internal displacement degree of freedom via a bubble [5].

It can be observed from our numerical experiments with the calculation of the slops for each methods, we find that the slope with $P1$ bubble- $P1$ method is more superior then the slope with the classical method. This numerical result means that the numerical solution u_{app} obtained by the mixed finite element $P1$ bubble- $P1$ method converge very speedy to the exact solution then the other solution obtained by the classical method.

We have demonstrated that, for solving the elasticity problem in Matlab with the mini-element $P1$ bubble -$P1$ is much more efficient than a standard implementation with ordinary finite element.

Moreover, the advantage of this problem with $\mathbf{C_{A,B}}$ boundary condition is the program level Matlab, it's enough to make a single program Matlab and can be reduced to ordinary problems as Dirichlet and Neumann.

Further work is underway to derive with the mini-element, for solving 3D elasticity problems.

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