Frequency analysis of partially-filled rectangular water tank

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Abstract—This paper deals with experimental studies of sloshing of liquid in partially filled container subjected to external excitation horizontal harmonic motion. The theoretical background of fluid response on rectangular tank due to horizontal acceleration of tank bottom, impulsive and convective (sloshing) pressure and the fluid natural frequencies is presented in paper. The dynamic behavior of fluid filled rectangular container was monitored and was evaluated in realized experiment. The resulting peak slosh heights for various excitation frequencies and amplitudes in fluid filled rectangular tank are compared with the fluid natural frequencies.

Keywords— Dynamic, fluid, container, experiment.

I. INTRODUCTION

Ground-supported tanks are used to store a different kind of liquids. The motion of the container, full or partially filled with liquid, causes the hydrodynamic pressure and fluid flow up to sloshing of free surface and forms the basis for many complex problems. The free liquid surface may experience different types of motion including simple planar, non-planar, rotational, irregular beating, symmetric, asymmetric, quasi-periodic and chaotic, it is depended on the type of excitation and container shape. The amplitude of slosh depends on the frequency and amplitude of the tank motion, liquid-fill depth, liquid properties and tank geometry. The fluid resonance in the case of horizontal excitation occurs when the external forcing frequency is close to the natural frequency of the liquid. The liquid sloshing is a practical problem with regard to the reliability and safety structures, because an eventual damages of containers used for storage of hazardous liquids, e.g. petroleum, chemical and radioactive waste, are catastrophic, consequences are financial, and environmental loses [1-3].

The behaviour of fluid in a container was studied as first Poisson, then Rayleigh, Lamb, Westergaard, Hopkins, Jacobsen, Werner, Sundquist, Zangar. Housner in 1957 [4-6] presented a simplified analysis for the hydrodynamic pressure develop when the fluid container fixed to base is subjected to a horizontal acceleration.

The motion of the liquid inside the tank results in additional hydrodynamic pressure loading on the tank walls and tank bottom. Hopkins and Jacobsen gave the analytical and experimental observations of rigid tank. Graham and Rodriguez used spring-mass analogy. Housner recommended a simple procedure for estimating the dynamic fluid effect on rectangular tank. Epstein extended of Housner concept and gave the practical rule for design [7,8].

The response of the rigid tank could be split into two hydrodynamic components namely:
- “impulsive” component is due to rigid-body motion of the liquid, under dynamic loading the “rigid-impulsive” part of the liquid moves synchronously with the tank as an added mass and is subject to the same acceleration as the tank,
- “convective” component is due sloshing of the liquid at the free surface, the fluid oscillates and occurs the generation of pressures on the walls, base and roof of the tank.

In addition to causing forces and moments in the tank wall, the hydrodynamic pressures on the walls in conjunction with the pressures on the base result in a overturning moment on the tank. Based on the assumptions that
- the liquid is incompressible and inviscid,
- motion of liquid is irrotational and satisfies Laplace’s equation,
- structural and liquid motions remain linearly elastic.

Seismic design of liquid storage tanks requires knowledge of liquid sloshing frequencies [9-12]. The hydrodynamic pressure on the tank wall and bottom, caused seismic ground acceleration, depends on the tank geometry, height of liquid, properties of liquid and fluid-tank interaction. Pressures are related closely with nascent seismic forces. The knowledge of hydrodynamic pressures and forces acting on the solid domain of containers during an earthquake as well as frequency properties of tank–fluid systems are played fundamental role for a reliability design of earthquake-resistant structures/facilities – tanks [13-15].

II. ANALYSIS OF TANK-FLUID SYSTEM

The rectangular tank with rigid walls is exited by horizontal excitation. Due to earthquake the impulsive hydrodynamic pressure are generated in addition to hydrostatic pressure. The
The rectangular container with rigid walls exciting by horizontal acceleration \( \ddot{u}_x \) in the x-direction is considered, Figure 1a [16]. Due to the acceleration \( \ddot{u}_x \), the hydrodynamic pressures on the tank walls and tank bottom in addition to hydrostatic pressure are generated The tank has dimensions \( H, 2L \), Figure 1a, and tank wall unit thickness [17-20].

A. The Hydrodynamic Impulsive Pressure

The horizontal component of the fluid velocity \( \dot{u} \) is independent on the y-coordinate, therefore the behaviour of the fluid may be simulated as thin, massless, vertical membranes that are free to move in the x-direction. The distance between neighbouring membranes is \( dx \) (Figure 1b) [21-24].

\[
\dot{v} = (H - y) \frac{d\dot{u}}{dx}
\]

Due to the fluid incompressibility, the acceleration \( \ddot{v} \) is proportional to the velocity \( \dot{v} \) and the acceleration \( \ddot{u}_x \) also is proportional to the velocity \( \dot{u}_x \). The pressure of the fluid between two membranes is given by the standard hydrodynamic eq.

\[
\frac{\partial p}{\partial y} = -\rho \ddot{v},
\]

where \( \rho \) is density of the fluid [27,28].

The horizontal force on one membrane is given

\[
F = \int_a^b p \, dy.
\]

Therefore

\[
p = -\rho \int_0^1 (H - y) \frac{d\dot{u}}{dx} \, dy = -\rho H^2 \left( \frac{y}{H} - \frac{1}{2} \left( \frac{y}{H} \right)^2 \right) \frac{d\dot{u}}{dx}.
\]

and

\[
F = -\rho H^2 \left( \frac{y}{H} - \frac{1}{2} \left( \frac{y}{H} \right)^2 \right) \frac{d\dot{u}}{dx} \, dy = -\rho H^2 \left( \frac{y}{H} - \frac{1}{2} \left( \frac{y}{H} \right)^2 \right) \frac{d\dot{u}}{dx}.
\]

The kinetic energy of the fluid is

\[
W_k = \int_{x_0}^{x_f} \frac{1}{2} \rho (\dot{u}^2 + \dot{v}^2) \, dx \, dy.
\]

The potential energy of the fluid is zero, consequently the Hamilton’s principle defines that

\[
\delta \int_{x_0}^{x_f} \left( W_k - W_p \right) \, dt = 0.
\]

The equation of motion is

\[
\frac{d^2\dot{u}_x}{dx^2} - \frac{3}{H^2} \dot{u}_x = 0.
\]

The ground acceleration \( \ddot{u}_x \) produces an increasing of the pressure on one wall and a decreasing of the pressure on the other wall

\[
p_{in} = \rho \ddot{u}_x H \left( \frac{y}{H} - \frac{1}{2} \left( \frac{y}{H} \right)^2 \right) \sqrt{3} \tanh \sqrt{3} \frac{L}{H}.
\]

The \( \xi = z / H \) is dimensionless distance, where \( z = H - y \), then pressure is

\[
p_{in} \left( \frac{z}{H} \right) = \rho \ddot{u}_x H \sqrt{3} \left( 1 - \left( \frac{z}{H} \right)^2 \right) \tanh \sqrt{3} \frac{L}{H}.
\]

The hydrodynamic impulsive pressure on the wall of tank is as well given by

\[
p_{in} (\xi) = C_{in}(\xi) \rho H A_x(t),
\]

where \( A_x(t) \) represents the free-field ground motion, and \( \rho \) is mass density of the liquid.

The distribution of hydrodynamic pressures \( p_{in}(\xi) \) along the height of wall gives the function \( C_{in}(\xi) \)

\[
C_{in}(\xi) = \frac{\sqrt{3}}{2} \left( 1 - \xi^2 \right) \tanh \sqrt{3} \frac{L}{H}.
\]
B. The Hydrodynamic Convective Pressure

The effect of impulsive pressures of the fluid causes its oscillation. The fluid constrained between the two rigid membranes having possibility to freely rotate, Figure 2, is given

\[ \dot{u} = \frac{\dot{L} - x^2}{2} \frac{d\theta}{dy}, \]  
\[ \dot{v} = \dot{\theta} z. \]

The fluid pressure is given

\[ \frac{\partial p}{\partial x} = -\rho \ddot{u}. \]  

The equation of the motion of the fluid slice is

\[ \int_{\gamma} \frac{\partial^2}{\partial y} dxdy = -\rho \frac{(2L)}{12} \ddot{y} dy. \]  

The solution of eq. (21), respecting the boundary conditions appropriate to the problem, is for sinusoidal oscillations

\[ \theta = \theta_0 \frac{\sin \frac{5}{2} \frac{y}{L}}{\sin \frac{5}{2} \frac{H}{L}} \sin \omega t. \]

This defines the oscillation of the fluid. The maximum kinetic energy \( W_K \) and the maximum potential energy \( W_P \) are given

\[ W_K = \int_{0}^{\gamma} \frac{1}{2} \rho (u^2 + v^2) \omega^2 \sin^2 \omega t dxdy, \]  
\[ W_P = \int_{-\gamma}^{\gamma} \frac{1}{2} \rho g x^2 \sin \omega t dx. \]

That the natural frequency of vibration is determined

\[ \omega_n^2 = \frac{n^2 \pi^2 g}{L^2} \text{tanh} \left( \frac{n\pi H}{2L} \right). \]  

The difference is in the constants \( \pi/2 \) in eq. (26) and \( \sqrt{5/2} \) in eq. (27). These are values 0.570796 in eq. (26) and 1.570796 in eq. (27). The percentage difference of values is 0.658 %.

III. EXPERIMENT ANALYSIS

The rectangular tank made of glass was used in the experiment. The inner dimensions of the tank was 39.2 cm in length (L), 19.2 cm in height (H), and 24.2 cm in breadth (B), therefore is dimension of possible full fluid filling. The tank wall thickness was 3.8 mm and the tank bottom thickness 5 mm. The transparent grid with 1 cm raster was created on front of the largest side of the tank.

The rectangular tank was filled with water (H₂O) by using minimum quantities of potassium permanganate (2KMnO₅). The water acquired light violet colour to better showing and recording. The water depth or filling level (H) was 5 m, 10 cm and 15 cm. The tank motions were the horizontal harmonious motions

\[ x_j = A \sin \left( 2 \pi f \ t \right) \]  

where \( A \) are amplitude 0.5 cm and 1 cm of various frequencies \( f \).

The movement was performed using impulse hydraulic
pulsator, in direction of the longest tank bottom side, Figure 4. The movements were recorded with digital camera.

Fig. 4 the experiment workstation

The comparing of the first twenty calculated natural frequencies for tank fluid filling using eq. (26) was seen in Figure 5.

As seen in Figure 6, the values of first seven calculated natural frequencies for tank fluid filling using eq. (26) are different. The eight and more natural frequencies of the fluid filling of height 5 cm, 10 cm and 15 cm give the same frequency values.

In the case of a shallow filling of water, especially the filling height 5 cm, the phenomena of splashes appear more pronounced at the surface than at higher heights of fluid filling. The monitored wave heights were varied in depending of excited frequency and the amplitude. The Figures 6-8 are documented height if water waves in depending of frequencies, for 5 cm (in Figure 7), 10 cm (in Figure 8) and 15 cm water filling (in Figure 9). The movement of fluid free surface and peak wave heights confirmed the correctness of the eq. (26) and eq. (27) for the calculation of the first natural frequency in all cases of water filling using in experiment.

The peak waves of fluid, for 5 cm fluid filling, were shown in Figure 7 in depending on frequencies. The peak waves of fluid excited with amplitude 0.5 cm may be seen with black “+” and amplitude 1.0 cm with red “x”. Using eq. (26), the first frequency of the glass rectangular fluid filled tank is \(f_1 = 0.875\) Hz and second frequency is \(f_2 = 1.238\) Hz. The first frequency and the second frequency were shown by vertical dash violet lines in Figure 7. It is seen that the peak waves of fluid in the glass rectangular tank culminated close the first frequency.

The peak waves of fluid for 15 cm fluid filling were presented in Figure 9 in depending on frequencies. The peak waves of fluid excited with amplitude 0.5 cm can be seen with black “+” and amplitude 1.0 cm with red “x”. The first frequency of the glass rectangular fluid filled tank is
The second sloshing mode was not seen in case of 5 cm water filling, with exciting amplitude 0.5 cm, the free surface of water was calm. The second natural frequency was not observable. Obviously, it is due to the interference of waves on the surface. The other higher shapes could not be watched. The shapes of free surface and fluid waves were shown in Figures 4, 10, 11 and 12.
The next second and higher modes (shapes) were not monitored in case of water filling 5 cm with exciting amplitude 1.0 cm, 10 cm and 15 cm, because the water splashes were observed very strong but the fluid gave marked splashes close the first mode of the liquid.

Figures 13-15 show:
- water splash out from rectangular tank in case of 10 cm water filling, Figure 13a), 13b) and 13c),
- waves that could be observed and used for data evaluation, Figure 14a) and 14b) in case of 10 cm fluid filling,
- fluid splash out from rectangular tank in case of 15 cm water filling, Figures 15a) and 15b).

The outpourings of water out from rectangular tank may be seen in Figures 13a) and 13c), the light violet colour water puddle close of tank bottom.
IV. CONCLUSIONS

The dynamic behaviour of fluid in rectangular tank due to horizontal harmonic motion of tank bottom was analysed in this paper. The rectangular tank made of glass was used in the experiment and was filled with water to the filling level 5 cm, 10 cm and 15 cm. The tank motions were realized the horizontal harmonious motions. The first twenty frequencies were calculated and compared for three levels of water filling.

The observations of liquid behaviour and waves heights were summarized from the experiment. The peak waves of fluid in of the glass rectangular tank were culminated close the first calculated natural frequency. The dynamic behaviours of fluid in rectangular tank due to horizontal harmonic motions of tank bottom were confirmed to the 1\textsuperscript{st} mode of natural frequencies during to realization of the experiment.

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REFERENCES


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