The Uncovered Interest Rate Parity Assumption in a Hyperchaotic System

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Abstract—In this paper, the uncovered interest rate parity (UIP) assumption, using a well-known 3D chaotic dynamical system, which describes the variations of the interest rate, is studied. More specifically a 4D novel hyperchaotic financial system is proposed by introducing the exchange rate, due to the fact that the difference in interest rates between two countries is equal to the relative change in currency foreign exchange rates over the same period. The novel 4D financial dynamical system is investigated in two different cases with and without a base interest rate. The system's hyperchaotic behavior and a route to chaos through a crisis phenomenon and quasiperiodic behavior are observed through the simulation results.

Keywords—Econophysics, exchange rate, hyperchaotic finance system, interest rate.

I. Introduction

IN the last decades, there has been an increasing interest in nonlinear dynamical systems, which exhibit chaotic behavior, from researchers of many kinds of scientific fields [1]-[3]. Chaos theory has started, when Lorenz in 1963 discovered complex dynamics, while studying three nonlinear differential equations that led to turbulence in the weather system [4]. The main feature of chaotic systems is their great sensitivity to initial conditions, which means that a small change or perturbation may result in very different future behavior. Since the early 1980s, chaotic behavior has also been observed in economics [5]. It is well known that financial systems are complex nonlinear systems that interact with humans and contain many complicated factors [6].

Therefore, a new scientific field has emerged, which is called "Econophysics" and intends to study the dynamics of real economic systems. It is actually the science that uses models from physics to describe some economic phenomena that include uncertainty and nonlinear dynamics. Economic processes, which include systems with very large number of elements, such as financial or banking markets, stock markets, income and production show chaotic behavior and it is very difficult to provide effective forecasts [7]. When chaotic phenomena occur in a financial system, it means that the macroeconomic procedure has indefiniteness, which exists in it. Even though governments can enforce financial or monetary policies to intervene, the efficacy of the intervention may be very limited. So, it is absolutely necessary to study chaotic behaviors and phenomena in economic systems, especially during financial crises.

One of the variables that are very often studied in papers is the interest rate, which is the percentage of the amount of money lent, charged by the lender for the use of assets [8]. Another important variable is the exchange rate that is the rate at which one currency will be exchanged for another. It is also regarded as the value of one country's currency in relation to foreign currency. Exchange rates are determined in the foreign exchange market, which is open to a wide range of different types of buyers and sellers, and where currency trading is continuous [9], [10].

Furthermore, the assumption of uncovered interest parity (UIP) is an important building block for macroeconomic analysis of open economies [11]. It provides a simple relationship between the interest rate on an asset denominated in any one country's currency unit, the interest rate on a similar asset denominated in another country's currency, and the expected rate of change in the exchange rate between the two currencies. Actually, the theory of UIP postulates that market forces drive the forward exchange rate into equality with the expected future exchange rate.

Thus, in this work, a novel hyperchaotic finance system is introduced by adding the state variable of exchange rate to a well known third-order chaotic finance system. Hyperchaos was studied for the first time by Rössler in 1979 [12] and it means that the hyperchaotic system has much more complex behavior than a chaotic system. Usually, a hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents.

The rest of the paper is organized as follows. In the next section the basic features of the new financial system are presented. The simulation results of the proposed system and its analysis are presented in Section 3. Finally, Section 4 includes the conclusions of this work.

II. The System

In 2001, Ma and Chen reported a third-order dynamical model, describing a nonlinear finance system [13]. The model describes the time variations of three state variables, the interest rate x, the investment demand y, and the price index z. This nonlinear finance chaotic system is described by the following set of differential equations:

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$$\begin{cases} \dot{x} = z + (y - a)x \\ \dot{y} = 1 - by - x^2 \\ \dot{z} = -x - cz \end{cases} \tag{1}$$

Parameters α, b and c stand for: the saving amount, the cost per-investment and the elasticity of demand of commercial markets, respectively. All three parameters possess a positive value ($\alpha \geq 0$, $b \geq 0$, $c \geq 0$).

Also, the aforementioned finance system has been studied by other researchers in the same or in other mathematical forms [14]-[22], by extracting interesting results about their dynamical behavior.

From an economic point of view, it is found that the factors affecting the interest rate are related not only to investment demand and price index, but also to the exchange rates. According to uncovered interest rate parity theory [11] the difference in interest rates between two countries will equal to the relative change in currency foreign exchange rates over the same period.

So, by taking into mind the following equation

$$\dot{w} = x - x^* \tag{2}$$

where w is the exchange rate, x is the interest rate and x^* is the interest rate of a foreign country, system (1) is transformed to the following 4D order dynamical system.

$$\begin{cases} \dot{x} = z + (y - a)x \\ \dot{y} = 1 - by - x^2 - dw^2 \\ \dot{z} = -x - cz \\ \dot{w} = x - x^* \end{cases} \tag{3}$$

Furthermore, in the proposed system (3) a term $-dw^2$ ($d \geq 0$) has been added to the second equation due to the fact that the rate of change in investment demand is also depended on the currency foreign exchange rates. Due to the lack from literature of a specific relation between the rate of change in investment demand and the currency foreign exchange rates the $-dw^2$ function has been adopted in this work.

The equilibrium points of the novel finance system (3) are obtained by solving the following set of equations: $\dot{x} = 0,\ \dot{y} = 0,\ \dot{z} = 0,\ \dot{w} = 0$. So, the system has the two equilibrium points:

$(x, y, z, w)_1 = (x^*,\ \left((1+ad)/d\right),\ -x^*/d,\ \sqrt{\left(-b + d - abd - d\left(x^*\right)^2\right)/cd}$) and

$(x, y, z, w)_2 = (x^*,\ \left((1+ad)/d\right),\ -x^*/d,\ -\sqrt{\left(-b + d - abd - d\left(x^*\right)^2\right)/cd}$).

On the other hand, if parameter x^* is equal to zero, by considering it as a base interest rate into system (3), then, it is proved that the system has a set of infinite number of equilibria, as:

$(x, y, z, w) = (0,\ 1 - dk^2,\ 0,\ k)$, where $k \in R$

So, for $x^* = 0$, system (3) belongs to the new category of dynamical systems with hidden attractors [23].

Also, it is easy to see that system (3) is invariant under the change of coordinates $(x, y, z, w) \to (-x, y, -z, -w)$. Thus, it follows that the novel nonlinear finance chaotic system (3) has rotation symmetry about the y-axis and that any non-trivial trajectory must have a twin trajectory.

III. Simulation Results

In this section, the numerical simulation results of the novel finance system (3), in two cases with $x^* = 0$ and $x^* = 0.01$, by employing the fourth order Runge–Kutta algorithm, are presented. In this approach, some of the most well-known tools of nonlinear theory, such as the bifurcation diagram, the phase portrait, the Poincaré map and the Lyapunov exponents, have been used.

The bifurcation diagram is a very useful tool in nonlinear science, because it gives the change of system's dynamic behavior as a crucial variable increases (or decreases) with a small step. In more details, this work presents the bifurcation diagram of the variable y versus the parameter c, which is produced when the trajectory crosses the section plane $w = 0$ with $dw/dt < 0$. With this procedure a discrete number of points are produced for periodic behavior, while an infinite number of points are produced for a chaotic behavior. If we follow the aforementioned procedure for a specific value of c a Poincaré map is produced.

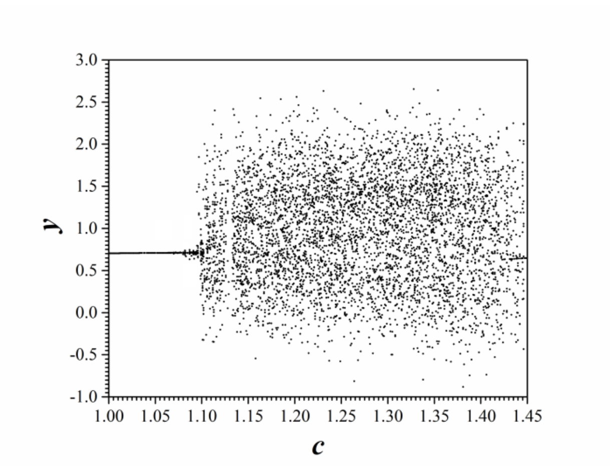
Another, useful tool, which is very common in the works related with nonlinear science, is the system's phase portrait, which is a geometric representation of the trajectories of a dynamical system in various phase planes. When the system is in a periodic state a closed curve is produced in the phase portrait, while when the system is in a chaotic state a more complex phase portrait is produced.

Furthermore, in mathematics the Lyapunov exponents of a dynamical system, which are also used in this work, are the quantities that characterize the rate of separation of infinitesimally close trajectories. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic, while when MLE is equal to zero the system is in a periodic state. Furthermore, if the system has two positive Lyapunov exponents the system is hyperchaotic. In this paper, the system's Lyapunov exponents have been calculated by employing the Wolf *et al*. algorithm [24].

A. First Case ($x^* = 0$)

In the first case, system's (3) behavior for $x^* = 0$, while the rest of parameters are: $a = 0.1$, $b = 0.3$, $d = 0.01$ and the initial conditions $(x_0, y_0, z_0, w_0) = (1, 0.5, 0.5, 0.1)$, is investigated. By taking the bifurcation diagram of y versus the parameter c (Fig.1(a)), as well as the respective spectrum of the three largest Lyapunov exponents (Fig.1(b)), an interesting dynamic behavior can be found.

As the parameter c increases the system passes from periodic to hyperchaotic behavior through a crisis [25]. The value of c for which this change occurred is equal to 1.099. Figure 2 depicts the phase portraits for a periodic ($c = 1.02$) state in two different planes, while in Fig.3 the phase portrait and the Poincaré map for hyperchaotic ($c = 1.30$) state are presented. The system's hyperchaotic behavior is confirmed by the calculation of two positive Lyapunov exponents in Fig.1(b).



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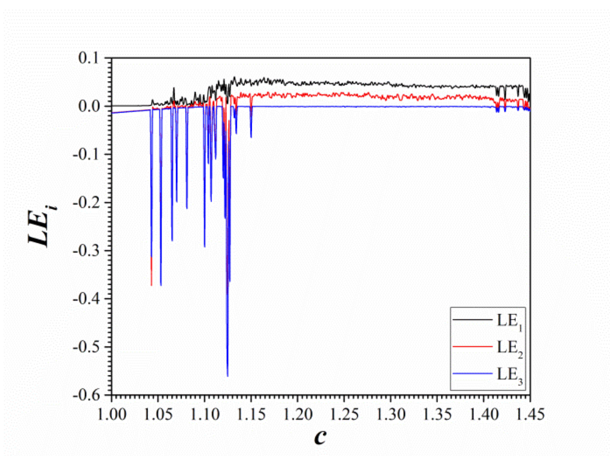

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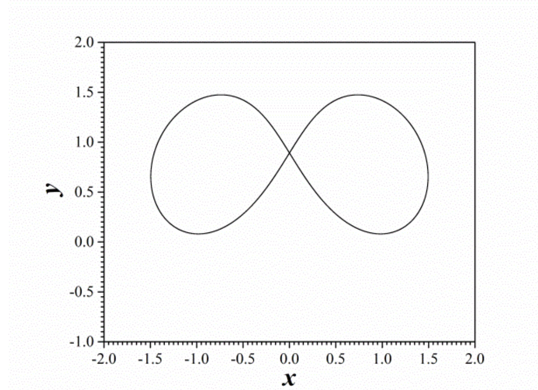
Fig. 1. (a) Bifurcation diagram and (b) the spectrum of the three largest Lyapunov exponents, of system (3), in the first case, when varying the value of the bifurcation parameter c from 1 to 1.45



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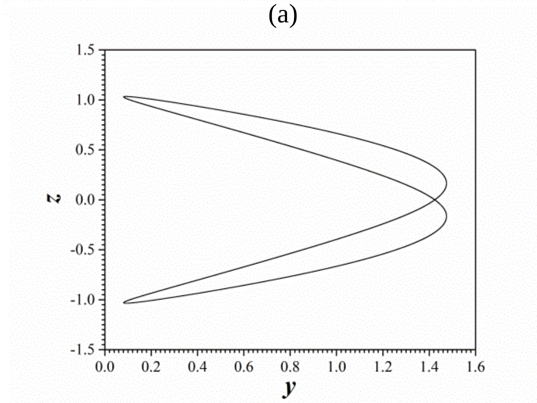

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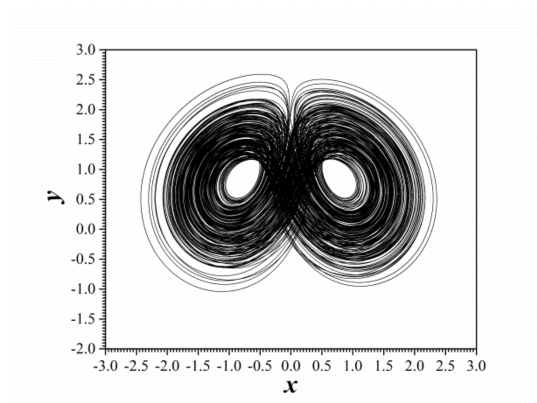
Fig. 2. Phase portraits, for $a = 0.1$, $b = 0.3$, $c = 1.02$, $d = 0.01$ and $x^* = 0$, in (a) $x – y$ plane and (b) $y – z$ plane



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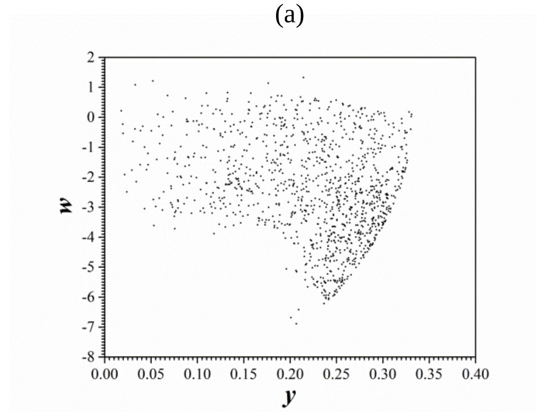

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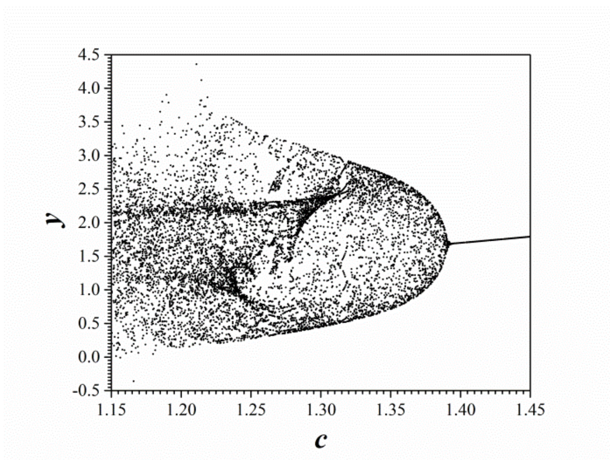
Fig. 3. (a) Phase portrait and (b) Poincaré map, for $a = 0.1$, $b = 0.3$, $c = 1.30$, $d = 0.01$ and $x^* = 0$

B. Second Case ($x^* = 0.01$)

In the second case system's (3) behavior for $x^* = 0.01$, while the rest of parameters are: a = 0.1, b = 0.1, d = 0.01 and the initial conditions (x_0, y_0, z_0, w_0) = (1, 0.5, 0.5, 0.1), is investigated. By taking the bifurcation diagram of y versus the parameter c (Fig.4(a)), as well as the respective spectrum of the three largest Lyapunov exponents (Fig.4(b)) an interesting dynamic behavior can be found.

As the parameter c decreases the system passes from periodic to chaotic behavior through a quasiperiodic region [25] and finally to hyperchaotic behavior. The value of c for which this change occurred is equal to 1.391. Figure 5 depicts the phase portraits for a periodic state (c = 1.40) in two different planes, while in Figs. 6 and 7 the phase portraits and Poincaré maps, for quasiperiodic (c = 1.34) and hyperchaotic (c = 1.16) state respectively, are presented. The system's quasiperiodic behavior is confirmed by the calculation of two zero Lyapunov exponents in Fig.4(b).



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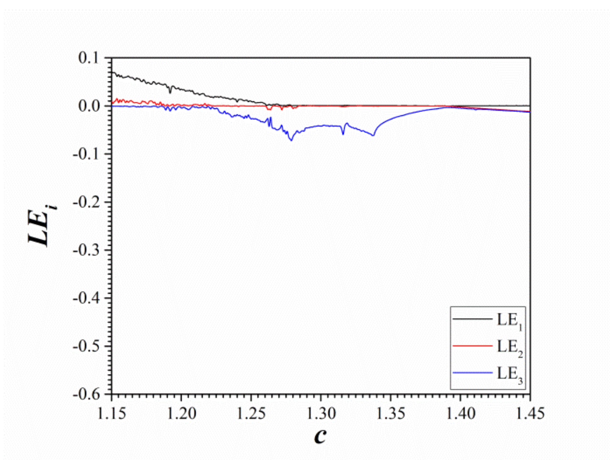

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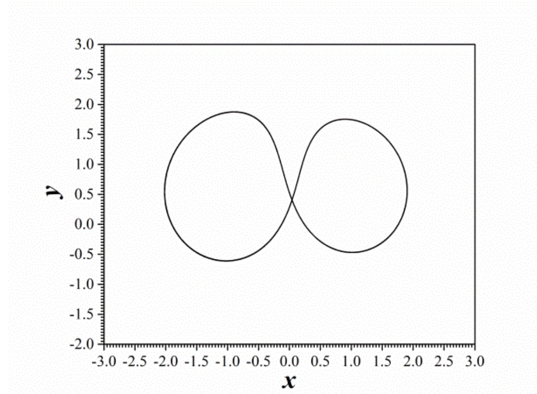
Fig. 4 (a) Bifurcation diagram and (b) the spectrum of the three largest Lyapunov exponents, of system (3), in the second case, when varying the value of the bifurcation parameter c from 1.15 to 1.45



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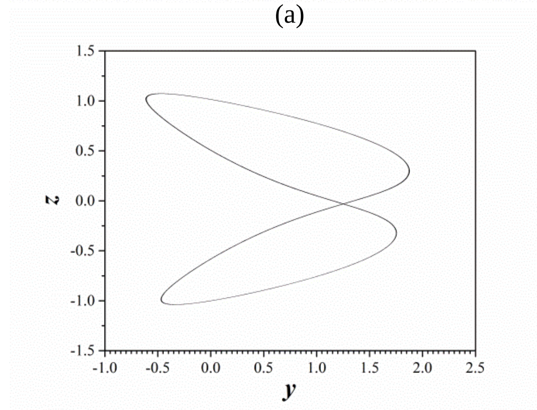

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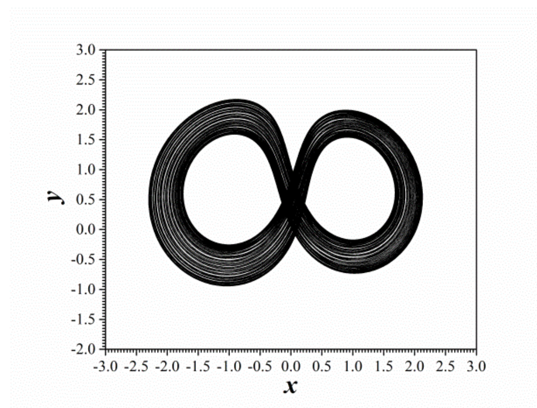
Fig. 5 (a) Phase portrait and (b) Poincaré map for a = 0.1, b = 0.1, c = 1.40, d = 0.01 and x^* = 0.01



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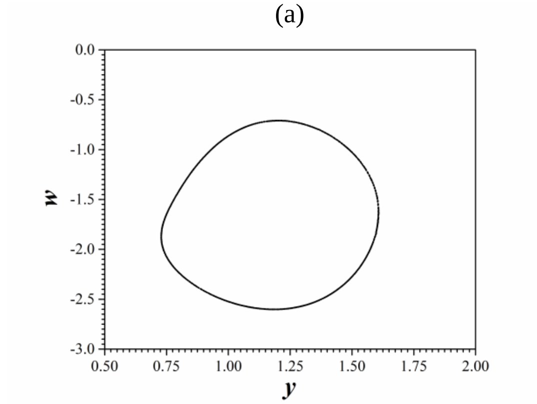

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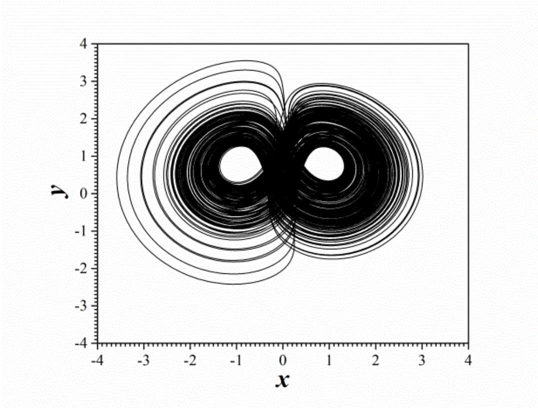
Fig. 6. (a) Phase portrait and (b) Poincaré map for a = 0.1, b = 0.1, c = 1.34, d = 0.01 and x^* = 0.01



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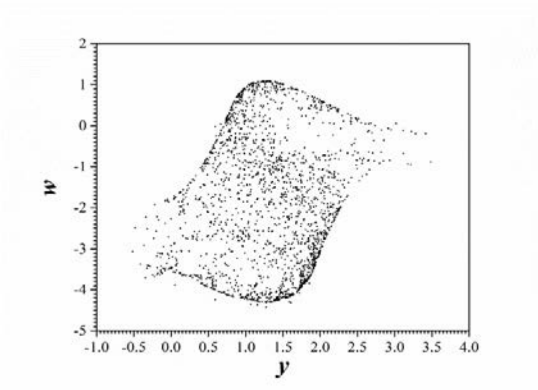

(b)

Fig. 7 (a) Phase portrait and (b) Poincaré map for a = 0.1, b = 0.1, c = 1.16, d = 0.01 and x^* = 0.01

IV. Conclusion

In this work, the uncovered interest rate parity assumption was adopted in a well-known 3D chaotic financial system. The new 4D financial model was studied, by using various tools of nonlinear theory, such as bifurcation diagram, Lyapunov exponents, phase portraits and Poincaré map, in two different cases with and without a base interest rate. In the first case, as parameter c, which denotes the elasticity of demand of commercial markets, increases, the system passes from periodic to hyperchaotic behavior through a crisis phenomenon. In the second case as parameter c decreases, the system passes from periodic to chaotic behavior through a quasiperiodic region and finally to hyperchaotic behavior. As a future work, the investigation of system's dynamical behavior for other values of parameters, has been planned.

References

[1] G. Chen and X. Dong, *From chaos to order: Methodologies, perspectives and applications*, Word Scientific, Singapore, 1998.

[2] I. R. Epstein and J. A. Pojman, *An introduction to nonlinear chemical dynamics: oscillations, waves, patterns, and chaos* (Topics in Physical Chemistry), Oxford University Press, Oxford, 1998.

[3] J. Walleczek, *Self-organized biological dynamics and nonlinear control: toward understanding complexity, chaos and emergent function in living systems*, Cambridge University Press, Cambridge, 2006.

[4] E. N. Lorenz, "Deterministic non periodic flow," *J. Atm. Sc*., vol. 20, pp. 130-141, 1963.

[5] M. J. Stutzer, "Chaotic dynamics and bifurcation in a macro model," *J. Econ. Dyn. Control*, vol. 3, pp. 353-376, 1980.

[6] R. A. Meyers, *Complex systems in finance and econometrics*, Springer, New York, 2009.

[7] G. Svoiu, *Econophysics: Background and applications in economics, finance, and sociophysics*, Academic Press, Oxford, 2013.

[8] https://www.investopedia.com/terms/i/interestrate.asp

[9] https://www.investopedia.com/terms/e/exchangerate.asp

[10] https://en.wikipedia.org/wiki/Exchange_rate

[11] P. Isard, "Uncovered Interest Parity," *International Monetary Fund Working Paper*, 2006.

[12] O. E. Rössler, "An equation for hyperchaos," *Phys. Lett. A*, vol. 71, pp. 155-157, 1979.

[13] J. H. Ma and Y. S. Chen, "Study for the bifurcation topological structure and the global complicated character of a kind of a nonlinear finance system (II)," *Appl. Math. Mech*. (English ed.), vol. 22, pp. 1375-1382, 2001.

[14] W. C. Chen, "Nonlinear dynamics and chaos in a fractional order financial system," *Chaos, Soliton Fract.*, vol. 36, pp. 1305-1314, 2008.

[15] Q. Gao and J. Ma, "Chaos and Hopf bifurcation of a finance system," *Nonlinear Dyn*., vol. 58, pp. 209-216, 2009.

[16] C. K. Volos, I. M. Kyprianidis, S. G. Stavrinides, I. N. Stouboulos, I. Magafas, and A. N. Anagnostopoulos, "Nonlinear financial dynamics from an engineer's point of view," *J. Eng. Sc. Techn. Rev*., vol. 4, pp. 281-285, 2011.

[17] H. Yu, G. Cai, and Y. Li, "Dynamic analysis and control of a new hyperchaotic finance system," *Nonlinear Dyn*., vol. 67, pp. 2171-2182, 2012.

[18] O. I. Tacha, C. K. Volos, I. N. Stouboulos, and I. M. Kyprianidis, "Analysis, adaptive control and circuit simulation of a novel finance system with dissaving," *Archives of Control Sciences*, vol. 26, pp. 95-115, 2016.

[19] S. Vaidyanathan, C. K. Volos, O. I. Tacha, I. M. Kyprianidis, I. N. Stouboulos, and V.-T. Pham, "Analysis, control and circuit simulation of a novel 3-D finance chaotic system," *Advances and Applications in Chaotic Systems, Series: Studies in Computational Intelligence (SCI)*, Springer, vol. 636, pp. 495-512, 2016.

[20] O. I. Tacha, C. K.Volos, I. M. Kyprianidis, I. N. Stouboulos, S. Vaidyanathan, and V.-T. Pham, "Analysis, adaptive control and circuit simulation of a novel nonlinear finance system," *Appl Math. Comput*., vol. 276, pp. 200-217, 2016.

[21] O. I. Tacha, J. M. Munoz-Pacheco E. Zambrano-Serrano, I. N. Stouboulos, and V.-T. Pham, "Determining the chaotic behavior in a fractional-order finance system with negative parameters," *Nonlinear Dyn*., vol. 94, pp. 1303–1317, 2018.

[22] J. M. Munoz-Pacheco, E. Zambrano-Serrano, C. Volos, O. I. Tacha, I. N. Stouboulos and V.-T. Pham, "A fractional order chaotic system with a 3D grid of variable attractors," *Chaos, Soliton Fract*., vol. 113, pp. 69-78, 2018.

[23] V.-T. Pham, S. Vaidyanathan, C. Volos, T. Kapitaniak, *Nonlinear dynamical systems with self-excited and hidden attractors*, Springer, Switzerland, 2018.

[24] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov exponents from a time series", *Physica D*, vol. 16, pp. 285-317, 1985.

[25] R. C. Hilborn, *Chaos and nonlinear dynamics: an introduction for scientists and engineers*, Oxford University Press, Oxford, 1994.