Mathematical Modeling of the Rotating Stratified Fluid in a Vicinity of the Bottom of the Ocean

A. Giniatoulline

Abstract—We obtain the explicit form of the solutions for an initial boundary value problem in a finite layer, which describes the dynamics of the Ocean in case of rotating stratified viscous flows.
We study the asymptotical properties of the solutions. For large values of *t*, we obtain uniform asymptotical decompositions, as well as decompositions with respect of the small parameter 1/*t* on an arbitrary compact set in the considered layer of the Ocean. For inviscid fluid, we find the spectrum of normal inner waves and establish its structure. We construct a Weyl sequence for the essential spectrum, which is an explicit representation of non-uniqueness of the solution. The localization of the essential spectrum may be used for bifurcation points where small nonlinear flows arise**.** The results may be applied in mathematical modeling of fluid dynamics of the Atmosphere and the Ocean, particularly, in the construction of stable numerical algorithms for the solutions of the studied models.

Keywords—Computational fluid dynamics, Fourier series and Fourier transform, spectrum of inner vibrations, stratified fluid, turbulence and multiphase flows.

I. Introduction, Construstion of Weak and Strong Solutions, their Existence and Uniqueness

We consider the following system of differential equations in partial derivatives

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \omega v_2 - \nu\Delta v_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial v_2}{\partial t} + \omega v_1 - \nu\Delta v_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial p}{\partial x_3} - N v_4 = 0 \\ \operatorname{div}\vec{v} = 0 \\ \dfrac{\partial v_4}{\partial t} + N v_3 = 0 \qquad x \in \Omega, \quad t > 0 \end{cases} \tag{1}$$

in the domain

$$Q = \Omega \times \{t > 0\},\ \Omega = \{x = (x', x_3) = (x_1, x_2, x_3),\ x' \in R^2,\ 0 < x_3 < h\}.$$

Here $\vec{v} = (v_1, v_2, v_3)$ is the velocity field of the fluid, $p(x,t)$ is the dynamic pressure, $v_4(x,t)$ is the dynamic density of the fluid, ω is the Coriolis parameter which corresponds to the rotation of the Earth over the vertical axis, and N is a positive constant stratification parameter. For the kinematic viscosity coefficient ν, we assume $\nu > 0$.

The considered equations are deduced in [1]. The study of mathematical properties of different systems of fluid dynamics of rotating fluid was started in [2]-[4]. Various problems involving the spectrum of normal vibrations for stratified and rotating fluid were considered in [5]-[9]. For non-linear model considered in bounded domains, the solution of similar systems was studied in [10]. The system is deduced for the cases when the horizontal dimensions are considerably larger than vertical dimensions, ([11]) and describes the motion of the Ocean flows near the bottom for the cases of rotating Earth and exponentially decreasing initial distribution of density due to the gravitational force.

We will consider the initial conditions

$$v_i\big|_{t=0} = v_i^0(x),\ i = 1,2,4 \tag{2}$$

and boundary value conditions

$$\left.\frac{\partial v_i}{\partial x_3}\right|_{\substack{x_3=0\\x_3=h}} = 0,\ i = 1,2;\ v_i\big|_{\substack{x_3=0\\x_3=h}} = 0,\ i = 3,4. \tag{3}$$

We use the Laplace transform with respect to *t*, the Fourier transform with respect to x' and finite integral transforms with respect to x_3. We apply the Cosine-Fourier transform to the first, the second and the fourth equations of (1), and the Sine-Fourier transform to the rest of the equations. For that purpose, we multiply the first, the second and the fourth equations by $\cos\lambda_n x_3$, the rest of the equations we multiply by $\sin\lambda_n x_3$, and integrate with respect to x_3 on the interval $0 < x_3 < h$. Let us introduce the following notations:

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A. Giniatoulline is with the Los Andes University, Bogotá D.C., 110111, Carrera 1 Este No. 18A-10, Colombia, South America (phone: 571-339-4949; fax: 571-332-4330; e-mail: aginiato@uniandes.edu.co).

$$\lambda_n = \pi n / h,$$

$$(\hat{v}_i, \hat{p})(x', n, t) = \int_0^h (v_i, p)(x', x_3, t)\cos\lambda_n x_3 dx_3, \ i = 1,2,$$

$$(\hat{v}_3, \hat{v}_4)(x', n, t) = \int_0^h (v_3, v_4, v_5)(x', x_3, t)\sin\lambda_n x_3 dx_3,$$

$$(\hat{v}_i, \hat{v}_4)(x', n, t)\big|_{t=0} = (\hat{v}_i^0, \hat{v}_4^0, \hat{v}_5^0)(x', n,) \ , \ i = 1,2.$$

We assume that the initial conditions are sufficiently smooth and rapidly decreasing functions for $|x'| \to \infty$, which allows us to apply the Fourier transform in x' and Laplace transform in t.

Additionally, we introduce the notations

$$\Psi_0(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\Lambda^n}\sin^2\left(\frac{\Lambda t}{2\lambda_n}\right),$$

$$\Psi_1(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\lambda_n \Lambda}\sin\left(\frac{\Lambda t}{\lambda_n}\right),$$

$$\Psi_2(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\lambda_n^2}\cos\left(\frac{\Lambda t}{\lambda_n}\right),$$

$$\Lambda = \sqrt{\omega^2 \lambda_n^2 + \nu|\xi'|^2}.$$

For the following, we assume $v_i^0 \in W_1^4(\Omega)$, $i = 1,2,4$,

$$\int_0^h \left[\frac{\partial v_1^0}{\partial x_1} + \frac{\partial v_2^0}{\partial x_2}\right] dx_3 = 0 ,$$

We also suppose that the condition of consistency of the initial data and boundary values is fulfilled.

It is proved in [12] that the Fourier coefficients of the solution are expressed as follows

$$\hat{v}_k(x', n, t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left\{\tilde{v}_k^0 e^{Ht} - \left(\nu\xi_k^2 + \omega^2\lambda_n^2\right)\tilde{v}_k^0\Psi_0 - \right.$$
$$-(-1)^k\left[\lambda_n^2\omega\Psi_1 + (-1)^k\xi_1\xi_2\nu\Psi_0\right]\tilde{v}_{3-k}^0 +$$
$$\left. +\lambda_n\left[i\xi_k\Psi_1 - (-1)^k i\xi_{3-k}\omega\Psi_0\right]\tilde{U}_3^0\right\} d\xi' \ k = 1,2,$$

$$\hat{v}_3(x', n, t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\begin{bmatrix}\lambda_n\left(\omega\tilde{U}_2^0\Psi_1 - \tilde{U}_1^0\Psi_2\right) + \\ +|\xi'|^2\tilde{U}_3^0\Psi_1\end{bmatrix} d\xi',$$

$$\hat{p}(x', n, t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left[\nu\left(\omega\tilde{U}_2^0\Psi_0 - \tilde{U}_1^0\Psi_1\right) - \right.$$
$$\left. -\lambda_n\left(\omega^2\Psi_0 + \Psi_2\right)\tilde{U}_3^0\right] d\xi',$$

$$\hat{v}_4(x', n, t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left[\tilde{v}_4^0 e^{Ht} + |\xi'|^2\Psi_0\left(-\nu\tilde{v}_4^0\right) + \right.$$
$$\left. +N\lambda_n\left(\tilde{U}_1^0\Psi_1 - \omega\tilde{U}_2^0\Psi_0\right)\ \right] d\xi',$$

where

$$\tilde{U}_1^0(\xi', n) = i\xi_1\tilde{v}_1^0 + i\xi_2\tilde{v}_2^0 \ , \ \tilde{U}_2^0(\xi', n) = i\xi_1\tilde{v}_2^0 - i\xi_2\tilde{v}_1^0,.$$
$$\tilde{U}_3^0(\xi', n) = N\tilde{v}_4^0 \ ,$$
$$H = -\nu\left(|\xi'|^2 + \lambda_n^2\right).$$

In this way, the solution of the problem (1)-(3) can be represented as follows ([12]):

$$(v_i, p)(x, t) = \frac{1}{h}(\hat{v}_i, \hat{p})(x', 0, t) + \frac{2}{h}\sum_{n=1}^{\infty}(\hat{v}_i, \hat{p})(x', n, t)\cos(\lambda_n x_3),$$
$$i = 1,2,$$

$$(v_3, v_4)(x, t) = \frac{2}{h}\sum_{n=1}^{\infty}(\hat{v}_3, \hat{v}_4, \hat{v}_5)(x', n, t)\sin(\lambda_n x_3). \tag{4}$$

We denote $Q_\tau = \Omega \times \{0 < t < \tau\}$,

$$\vec{U}^0(x', x_3) = (v_1^0, v_2^0, v_4^0)(x', x_3) \ , \ \|f\|_k = \|f\|_{W_1^k(\Omega)} ,$$

$\hat{V}(Q_\tau) = \{v_i \in C([0,\tau], L_2(\Omega)) \cap L_2((0,\tau), W_2^1(\Omega)), \ i = 1,2,$

$v_3 \in L_2\left((0,\tau), \overset{0}{W}{}_2^1(\Omega)\right)$, $\mathrm{div}\vec{v} = 0$,

$v_i \in C([0,\tau], L_2(\Omega)) \cap L_2\left((0,\tau), \overset{0}{W}{}_2^1(\Omega)\right), \ i = 4\}$,

$$V(Q_\tau) = \{(\vec{v}, v_4, v_5) \in \hat{V}(Q_\tau) : D_t v_i \in L_2(Q_\tau) \ , \ i = 1,2,4\}.$$

Let us define a *strong solution* of the problem (1)-(3) as a system of the functions $\{\vec{v}, p, v_4\}$ such that

$$v_i \in C_{x,t}^{2,1}(Q) \cap C_{x,t}^{1,0}(\bar{Q}) \ , \ i = 1,2, \ p \in C_{x,t}^{1,0}(Q),$$
$$v_3 \in C_{x,t}^{1,0}(Q) \cap C(\bar{Q}) \ , \ v_i \in C_{x,t}^{2,1}(Q) \cap C(\bar{Q}) \ , \ i = 4$$

satisfy (1) and the conditions (2), (3).

We define a *weak solution* of the problem (1)-(3) as a system of the functions $\{\vec{v}, v_4, v_5\} \in V(Q_\tau)$ which satisfy the condition (2) and the integral identity

$$\int_{Q_\tau}\{\sum_{i=1}^2 \frac{\partial v_i}{\partial t}\Phi_i + \frac{\partial v_4}{\partial t}\Phi_4 + \nu\sum_{i=1}^2\sum_{j=1}^3 \frac{\partial v_i}{\partial x_j}\frac{\partial \Phi_i}{\partial x_j} + \nu\sum_{i=1}^2\left(\frac{\partial v_4}{\partial x_i}\frac{\partial \Phi_4}{\partial x_i}\right) + \nu\left(\frac{\partial v_4}{\partial x_3}\frac{\partial \Phi_4}{\partial x_3}\right) +$$
$$+\omega(v_1\Phi_2 - v_2\Phi_1) + N(v_3\Phi_4 - v_4\Phi_3)\} dx dt = 0$$

for all $t \in [0,\tau]$ and for every vector function

$$\vec{\Phi}(x, t) = (\Phi_i)_{i=1}^4 \in \hat{V}(Q_\tau).$$

In [12] it is proved that the relations (4) define both weak and strong solutions, and that the strong solution is unique in the class of functions $V(Q_\tau)$.

Our aim now is to study the velocity of the asymptotical decay of the solution for large values of t, and obtain uniform estimates of decreasing, as well as asymptotical decomposition on an arbitrary compact set in the considered layer of the Ocean.

II. Problem Solution

Let us establish first some helpful auxiliary statements.

Lemma 1 For the initial conditions v_1^2, v_2^0, the following relation is valid:

$$\int_\Omega v_i^0(x)dx = 0\ ,\ i = 1,2\ .$$

Proof. We apply transform $\left(F_{x'\to\xi'}\left[\varphi(x')\right] = \breve{\varphi}(\xi')\right)$ to the relation $\int_0^h \left[\frac{\partial v_1^2}{\partial x_1} + \frac{\partial v_2^0}{\partial x_2}\right]dx_3 = 0$ and thus obtain $\xi_1 \int_0^h \breve{v}_1^0(\xi', x_3)dx_3 + \xi_2 \int_0^h \breve{v}_2^0(\xi', x_3)dx_3 = 0$ for all $\xi' \in R^2$.

Suppose $\int_0^h \breve{v}_1^0(0, x_3)dx_3 \neq 0$. Then, there exist two numbers ρ_1, ρ_2 such that for all $\xi' : |\xi'| < \rho_1$, the inequality holds:

$$\left|\int_0^h \breve{v}_1^0(\xi', x_3)dx_3\right| > \rho_2\ .$$

In this way, for all ξ' from the circle $|\xi'| < \rho_1$, we have $|\xi_1|\rho_2 < |\xi_1|\left|\int_0^h \breve{v}_1^0(\xi', x_3)dx_3\right| \le |\xi_2|\left\|v_2^0\right\|$.

On the other hand, the inequality $|\xi_1|\rho_2 \le |\xi_2|\left\{\left\|v_1^0\right\| + \left\|v_2^0\right\|\right\}$ is false for those points of the circle $|\xi'| < \rho_1$ which belong to the cone $\xi_1^2\rho_2^2 \le \xi_2^2\left\{\left\|v_1^0\right\| + \left\|v_2^0\right\|\right\}^2$.

Therefore, $\int_0^h \breve{v}_1^0(0, x_3)dx_3 = 0$.

For v_2^0, the proof is analogous and thus the Proposition is proved.

Now, let us consider the integrals of the type

$$I_{k,j}^s(t) = \int_0^\infty e^{-\alpha\beta^2 t} \frac{\beta^{2k+1}}{\left(\gamma^2 + \beta^2\right)^{s/2}} \cos\left(\frac{\pi}{2} j - \sigma t\sqrt{\gamma^2 + \beta^2}\right)d\beta,$$

$0 \le k, j, s \le 1$, $\alpha > 0, \sigma > 0, \gamma > 0$.

Lemma 2 For $t \to \infty$, the following asymptotical decompositions are valid:

$$I_{k,j}^s(t) = A_{k,j}^s \frac{\cos(\sigma\gamma t)}{t^{k+1}} + B_{k,j}^s \frac{\sin(\sigma\gamma t)}{t^{k+1}} + O\left(t^{-(k+2)}\right),$$

where the constants $A_{k,j}^s$, $B_{k,j}^s$ do not depend on t.

Proof. We consider first the case $s = k = 0$. After integrating by parts in the integral $I_{0,0}^0(t)$, we obtain

$$I_{0,0}^0(t) = \frac{\cos(\sigma\gamma t)}{2\alpha t} - \frac{\sigma}{2\alpha}\int_0^\infty e^{-\alpha\beta^2 t} \frac{\beta}{\left(\gamma^2 + \beta^2\right)^{1/2}} \sin\left(\sigma t\sqrt{\gamma^2 + \beta^2}\right)d\beta. \quad (5)$$

We use the representation of $\left(\gamma^2 + \beta^2\right)^{-1/2}$ as Taylor series with the residual term in integral form

$$\left(\gamma^2 + \beta^2\right)^{-1/2} = \frac{1}{\gamma} - \int_0^\beta \frac{\xi d\xi}{\left(\gamma^2 + \beta^2\right)^{3/2}} = \frac{1}{\gamma} + O\left(\beta^2\right), \quad (6)$$

and thus obtain the relation (5) for $t \ge t_0 > 0$ as follows:

$$I_{0,0}^0(t) = \frac{\cos(\sigma\gamma t)}{2\alpha t} - \frac{\sigma}{2\alpha\gamma} I_{0,1}^0(t) + O\left(t^{-2}\right). \quad (7)$$

Now, we integrate by parts in the representation of the integral $I_{0,1}^0(t)$ and proceed as we did for (7):

$$I_{0,1}^0(t) = \frac{\sin(\sigma\gamma t)}{2\alpha t} + \frac{\sigma}{2\alpha\gamma} I_{0,0}^0(t) + O\left(t^{-2}\right)\ . \quad (8)$$

In this way, from (7), (8) we have

$$I_{0,0}^0(t) = \frac{\cos(\sigma\gamma t)}{2\alpha t} - \frac{\sigma\sin(\sigma\gamma t)}{4\alpha^2\gamma t} - \frac{\sigma^2}{4\alpha^2\gamma^2} I_{0,0}^0(t) + O\left(t^{-2}\right),$$

and, consequently, we obtain the asymptotical decomposition for $I_{0,0}^0(t)$, where

$$A_{0,0}^0 = 2\alpha\gamma^2 / \left(4\alpha^2\gamma^2 + \sigma^2\right)$$
$$B_{0,0}^0 = -\sigma\gamma / \left(4\alpha^2\gamma^2 + \sigma^2\right).$$

The decomposition for $I_{0,1}^0(t)$ can be obtained analogously.

Now, let $s = 0$, $k = 1$. We integrate by parts in the integrals $I_{1,j}^0(t)$, $j = 0,1$ and proceed in the similar way as we did for the case $k = 0$:

$$I_{1,j}^0(t) = (-1)^{1+j} I_{1,1-j}^0(t) + \frac{1}{\alpha t} I_{0,j}^0(t) + O\left(t^{-2}\right)\ , j = 0,1.$$

Using the last relation, as well as the obtained decompositions for $I_{0,j}^0(t), j = 0,1$, we obtain the asymptotical decompositions for the integrals $I_{1,j}^0(t), j = 0,1$.

Finally, let $s = 1$. From (6) we have

$$I_{k,j}^1(t) = \frac{1}{\gamma} I_{k,j}^0(t) + O\left(t^{-(k+2)}\right)\ , 0 \le j,k \le 1,$$

and thus we obtain the asymptotical decompositions for the integrals $I_{k,j}^1(t)$, $0 \le j,k \le 1$, which concludes the proof.

Now, let us study the integrals of the type

$$E_{k,j}^{s,l}(x',t)=\frac{1}{(2\pi)^2}\cdot$$
$$\cdot\int_{R^2} e^{i(x',\xi')}e^{-\alpha|\xi'|^2 t}\frac{(i\xi_k)^{1-l}|\xi'|^{2l}}{(\gamma^2+|\xi'|^2)^{5/2}}\cos\left(\frac{\pi}{2}j-\sigma t\sqrt{\gamma^2+|\xi'|^2}\right)d\xi',$$
$k=1,2$; $0\le j,l,s\le 1$; $\alpha>0,\gamma>0,\sigma>0.$

Lemma 3 For $t\to\infty$, on an arbitrary compact set $K\subset R^2$, the following asymptotical decompositions are valid:
$$E_{k,j}^{s,l}(t)=C_{k,j}^{s,l}(x')\frac{\cos(\sigma\gamma t)}{t^2}+D_{k,j}^{s,l}(x')\frac{\sin(\sigma\gamma t)}{t^2}+O\left(\frac{1+|x'|^3}{t^3}\right),$$
where the coefficients $C_{k,j}^{s,l}$, $D_{k,j}^{s,l}$ do not depend on t.

Proof. Let $l=1$. We use the integral representation of the Bessel function $J_0(z)$
$$J_0(z)=1-\frac{z}{\pi}\int_0^{\pi}\left(\int_0^1\sin(z\gamma\sin\alpha)d\gamma\right)\sin\alpha d\alpha\ ,$$
as well as the following Bochner formula for Fourier integrals ([13])
$$\int_{R^2} e^{i(x',\xi')}f(|\xi'|)d\xi'=2\pi\int_0^{\infty}\beta J_0(|x'|\beta)f(\beta)d\beta . \qquad (9)$$
Therefore, we can represent the integral $E_{k,j}^{s,1}(x',t)$ in the form:
$$E_{k,j}^{s,1}(x',t)=\frac{1}{(2\pi)}\cdot$$
$$\cdot\int_0^{\infty} e^{-\alpha\beta^2 t}\frac{\beta^3}{(\gamma^2+\beta^2)^{5/2}}\cos\left(\frac{\pi}{2}j-\sigma t\sqrt{\gamma^2+\beta^2}\right)d\beta+$$
$$+G_{s,j}(x',t),$$
$$G_{s,j}(x',t)\le \mathrm{Const}\left(1+|x'|^2\right)t^{-3}\ ,\ 0\le j,s\le 1.$$
We obtain the asymptotical decomposition of the last integral directly from the Lemma 2. The coefficients $C_{k,j}^{s,1}$, $D_{k,j}^{s,1}$ are equal to the corresponding coefficients $A_{k,j}^{s}$, $B_{k,j}^{s}$, divided by 2π.

The case $l=0$ can be considered analogously, with the use of the Lemma 2, the integral representation of the Bessel function $J_1(z)=\frac{z}{2}-\frac{z^2}{\pi}\int_0^{\pi}\left(\int_0^1\sin(z\gamma\sin\alpha)d\gamma\right)\cos\alpha d\alpha$,

and the following corollary of the Bochner formula ([13]):
$$\int_{R^2} e^{i(x',\xi')}i\xi_j f(|\xi'|)d\xi'=$$
$$=-2\pi\frac{x_j}{|x'|}\int_0^{\infty}\beta^2 J_1(|x'|\beta)f(\beta)d\beta\ ,\ j=1,2. \qquad (10)$$
In this way, the Lemma is proved.

Now we can state our first main result.

Theorem 1 Let the initial data satisfy $|x'|v_i^0\in L_1(\Omega),\ i=1,2$. Then, for $t\ge t_0>0$, the solution of the problem (1)-(3) satisfies the following estimates uniformly with respect to $x\in\Omega$
$$|v_i(x,t)|\le C_i t^{-3/2}, i=1,2\ ;\ |v_3(x,t)|\le C_3 t^{-3/2}e^{-\nu\lambda_1^2 t},$$
$$|p(x,t)|\le C_4 t^{-1}\ ,\ |v_4(x,t)|\le C_5 t^{-1}e^{-\nu\lambda_1^2 t},\ \lambda_1=(\pi/h),$$
and the constants depend on the norms of the initial data.

If, additionally, $|x'|v_4^0\in L_1(\Omega)$ and for some fixed integer $k\ge 1$ the condition holds: $|x'|^{2k+2}v_i^0\in L_1(\Omega), i=1,2$;

then, for $t\to\infty$, the solution of the problem (1)-(3) has the following asymptotical decomposition on an arbitrary compact set $K\subset\Omega$:
$$v_i(x,t)=(\nu t)^{-2}\sum_{n=0}^{k-1}M_n^i(x')(\nu t)^{-n}+G_i(x,t), i=1,2$$
$$v_3(x,t)=(\nu t)^{-2}\sin(\lambda_1 x_3)e^{-\nu\lambda_1^2 t}\sum_{j=0}^{1}M_j^3(x')\cos\left(\frac{\pi}{2}j-\omega t\right)+G_3(x,t)$$
$$p(x,t)=(\nu t)^{-1}\sum_{n=0}^{k-1}M_n^4(x')(\nu t)^{-n}+G_4(x,t)$$
$$v_4(x,t)=(\nu t)^{-1}\sin(\lambda_1 x_3)e^{-\nu\lambda_1^2 t}+G_5(x,t),$$
where the residual terms satisfy the estimates for $t\ge t_0>0$:
$$|G_i(x,t)|\le C^i\left(1+|x'|\right)^{2k+2-\delta_{4,i}}(\nu t)^{-k-2+\delta_{4,i}}, i=1,2,4;$$
$$|G_j(x,t)|\le C^j\left(1+|x'|\right)^3 t^{-\frac{3}{2}(1+\delta_{3,j})}e^{-\nu\lambda_1^2 t}, j=3,5,$$
the constants C^i, C^j depend on the norms of the initial data and the coefficients of the decomposition are continuous with respect to x and are expressed only in terms of the initial data.

Proof. Let us study the component $v_1(x,t)$ from (4). Using the Taylor formula with the residual term in integral form, we decompose the Fourier coefficient $\tilde v_1^0(\xi',0)$ in a vicinity of the point $\xi'=0$:
$$\tilde v_1^0(\xi',0)=\tilde v_1^0(0,0)-$$
$$-\sum_{j=1}^{2}i\xi_j\left\{\int_0^h\int_0^1\left[\int_{R^2}x_j e^{-i(\vartheta\xi',x')}v_1^0(x',x_3)dx'\right]d\vartheta dx_3\right\}. \quad (11)$$
By virtue of Lemma 1 we have $\tilde v_1^0(0,0)=0$. Therefore, from (11) we obtain the estimate $|\tilde v_1^0(\xi',0)|\le C\||x'|v_1^0\||\xi'|$. Now, for the function $v_1(x,t)$ we estimate the Fourier coefficient $\hat v(x',0,t)$. From the last estimate and (4), we have
$$|\hat v_1(x',0,t)|\le C\left|\int_{R^2}e^{i(x',\xi')}e^{-\nu|\xi'|^2 t}\tilde v_1^0(\xi',0)d\xi'\right|\le$$
$$\le C_1\||x'|v_1^0\|\int_{R^2}e^{-\nu|\xi'|^2 t}|\xi'|d\xi'=\frac{C_2}{(\nu t)^{3/2}}. \qquad (12)$$

Let us estimate the terms of the series $\frac{2}{h}\sum_{n=1}^{\infty}\hat{v}_1(x',n,t)\cos\lambda_n x_3$. Evidently, for $t \ge t_0 > 0$ and $n \ge 1$ the estimate for the derivatives is valid:
$\left|D^{\alpha}\hat{v}_1(x',n,t)\cos\lambda_n x_3\right| \le C_0 n^{\alpha_3}\int_{R^2} e^{-\nu t\left(|\xi'|^2+\lambda_n^2\right)}|\xi'|^{|\alpha'|}\sum_{j=1,2,4}\left|\tilde{v}_j^0\right|d\xi' \le$

$\le C n^{\alpha_3} t_0^{-\left(1+\frac{|\alpha'|}{2}\right)} e^{-\nu\lambda_n^2 t_0}\left\|\vec{U}^0\right\| \le C_1 n^{\alpha_3} e^{-\nu\lambda_n^2 t_0}$.

In this way, for $\hat{v}_1(x',n,t)$ we obtain the inequality: $\left|\hat{v}_1(x',n,t)\right| \le C\left\|\vec{U}^0\right\| t^{-1}e^{-\nu\lambda_n^2 t}$. Now, from the integral criterion of Cauchy-McLaurin ([14]), we have that for $t \ge t_0 > 0$ the estimate holds:

$$\begin{aligned}&\left|\sum_{n=1}^{\infty}\hat{v}_1(x',n,t)\cos\lambda_n x_3\right| \le Ct^{-1}\sum_{n=1}^{\infty}e^{-\nu\lambda_n^2 t} \le \\ &\le Ct^{-1}e^{-\nu\lambda_1^2 t} + Ct^{-1}\int_1^{\infty} e^{-\nu\left(\frac{\pi}{h}\right)\xi^2 t}d\xi \le C_1 t^{-1}e^{-\nu\lambda_1^2 t}\end{aligned} \quad (13)$$

From (12), (13) we obtain the first statement of the Theorem for $v_1(x,t)$. For the rest of the components of the solution, the asymptotic estimates can be obtained similarly.

Now, let us prove the second part of the Theorem. We will use the values of the integrals from [15]:

$$\int_0^{\infty} x^{\nu+1}e^{-\alpha x^2}J_{\nu}(\beta x)dx = \frac{\beta^{\nu}}{(2\alpha)^{1+\nu}}e^{-\frac{\beta^2}{4\alpha}};$$

$$\int_0^{\infty} e^{-px^2}J_1(cx)dx = \frac{1}{c}\left[1-e^{-\frac{c^2}{4p}}\right].$$

We also use (9), (10), and thus obtain the following representation for the considered Fourier coefficients:

$$\hat{v}_i(x',0,t) = \frac{1}{2\pi\nu t}\int_{\Omega} e^{-\frac{|x'-y'|^2}{4\nu t}}v_i^0(y)dy,\ i=1,2,$$

$$\begin{aligned}\hat{p}(x',0,t) &= \frac{\omega}{\pi}\int_{\Omega}\left(1-e^{-\frac{|x'-y'|^2}{4\nu t}}\right)\frac{1}{|x'-y'|^2}\cdot \\ &\cdot\left[(x_1-y_1)v_2^0(y)-(x_2-y_2)v_1^0(y)\right]dy.\end{aligned} \quad (14)$$

Let us study the function $v_1(x,t)$. We decompose the function $\exp\left\{-|x'-y'|^2/(4\nu t)\right\}$ according to Taylor formula with the residual term in integral form and use the estimates (13).

In this way, we have

$$\begin{aligned}v_1(x,t) &= \frac{1}{4\pi\nu t h}\sum_{n=0}^{k}\frac{(-1)^n}{n!(4\nu t)^n}\int_{\Omega}|x'-y'|^{2n}v_1^0(y) + \\ &+\frac{(-1)^{k+1}}{\pi k!(4\nu t)^{k+2}}\int_0^1\int_{\Omega}(1-\xi)^k|x'-y'|^{2k+2}e^{-\frac{|x'-y'|^2}{4\nu t}\xi}v_1^0(y)dyd\xi + \\ &+O\left(t^{-1}e^{-\nu\lambda_1^2 t}\right).\end{aligned} \quad (15)$$

From Lemma 1, it follows that the first term in (15) is zero. The double integral in (15) can be easily estimated by $C(1+|x'|)^{2k+2}(\nu t)^{-k-2}$, and thus we obtain the required asymptotical decomposition for $v_1(x,t)$. The procedure for the components $v_2(x,t)$, $p(x,t)$ is analogous.

Let us obtain the asymptotical decompositions for the components $v_3(x,t)$, $v_4(x,t)$.

Proceeding from (4) and making calculations which are similar to (13), we obtain:

$$\begin{aligned}v_3(x,t) &= \frac{2}{h}\hat{v}_3(x',1,t)\sin\lambda_1 x_3 + O\left(t^{-3/2}e^{-\nu\lambda_2^2 t}\right), \\ v_4(x,t) &= \frac{2}{h}\hat{v}_4(x',1,t)\sin\lambda_1 x_3 + O\left(t^{-1}e^{-\nu\lambda_2^2 t}\right).\end{aligned} \quad (16)$$

We consider the function $v_3(x,t)$. It can be easily seen that it is sufficient to study the Fourier coefficient $\hat{v}_3(x',1,t)$. Similarly to (11), we decompose the functions $\tilde{v}_i^0(\xi',1), i=1,2,4$ in a vicinity of the point $\xi'=0$. For $t\to\infty$ we will have

$$\begin{aligned}v_3(x,t) &= \frac{\sin\lambda_1 x_3}{2\pi^2 h}\int_{R^2} e^{i(x',\xi')}\left\{\lambda_1[\omega\tilde{U}_2^0(0,1)\Psi_1(\xi',1,t) - \right.\\ &-\tilde{U}_1^0(0,1)\Psi_2(\xi',1,t)] + \\ &\left.+|\xi'|^2\tilde{U}_3^0(0,1)\Psi_1(\xi',1,t)\right\}d\xi' + O\left(\frac{(1+|x'|)}{t^2}e^{-\nu\lambda_1^2 t}\right).\end{aligned}$$

From the explicit form of $\tilde{U}_j^0, \Psi_j$ (4), and also from Lemma 3, we obtain the required asymptotical decomposition for the component $v_3(x,t)$. Finally, let us consider the function $v_4(x,t)$. Repeating the previous calculations, we can represent it as

$$\begin{aligned}v_4(x,t) &= \frac{\sin(\lambda_1 x_3)}{2\pi h\nu t}e^{-\nu\lambda_1^2 t}\cdot \\ &\cdot\int_{\Omega} v_4^0(y)e^{-\frac{|x'-y'|^2}{4\nu t}}\sin\lambda_1 y_3 dy + O\left(t^{-3/2}e^{-\nu\lambda_1^2 t}\right).\end{aligned} \quad (17)$$

Once again, we decompose the function $e^{-\frac{|x'-y'|^2}{4\nu t}}$ from (17) according to Taylor formula with the residual term in integral form and thus, we obtain the required asymptotical decomposition for the function $v_4(x,t)$. In this way, the Theorem is proved.

III. The Spectrum of the Inner Vibrations

Now, let us consider the initial system of fluid dynamics for the corresponding inviscid fluid

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} + N\rho = 0 \\ \operatorname{div}\vec{u} = 0 \\ \dfrac{\partial \rho}{\partial t} - Nu_3 = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \tag{18}$$

in the same layer domain Ω with the same boundary conditions (3).

Let us consider the problem of normal vibrations

$$\vec{u}(x,t) = \vec{v}(x)e^{-\lambda t}$$
$$\rho(x,t) = v_4(x)e^{-\lambda t} \tag{19}$$
$$p(x,t) = v_5(x)e^{-\lambda t} \quad , \quad \lambda \in C.$$

We denote $\tilde{v} = (\vec{v}, v_4, v_5)$ and write the system (5) in the matrix form

$$L\tilde{v} = 0\,, \tag{20}$$

where $L = M - \lambda I_4$ and

$$M = \begin{pmatrix} 0 & -\omega & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ \omega & 0 & 0 & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \dfrac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}.$$

Let us define the domain of the differential operator *M* with the boundary condition (3) as follows.

$$D(M) = \left\{ \begin{matrix} \vec{v} \in \left(\overset{0}{W}{}_2^1(\Omega) \right)^3, v_4 \in \overset{0}{W}{}_2^1(\Omega), \\ v_5 \in L_2(\Omega) : M\tilde{v} \in \left(L_2(\Omega) \right)^5 \end{matrix} \right\}.$$

The consideration of the separated variables of the form (19) allows to consider every non-stationary process as a linear superposition of the normal oscillations. The spectrum of normal vibrations may be used for studying the properties of the stability of the flows. As well, the spectral properties of M may be used in the investigation of weakly non-linear flows, since the bifurcation points are exactly the points of the spectrum of the operator M .

Let us denote by $\sigma_{ess}(M)$ the essential spectrum of a closed linear operator *M*. We recall that, according to the definition of the essential spectrum [16],

$$\sigma_{ess}(M) = \left\{ \lambda \in C : (M - \lambda I)^{-1} \text{ is not of Fredholm type} \right\},$$

it consists of the points of the continuous spectrum, eigenvalues of infinite multiplicity, and limit points of the point spectrum.

Therefore, the spectral points outside of the essential spectrum, are eigenvalues of finite multiplicity. For calculating the essential spectrum of *M*, we would like to use the property which is attributed to Weyl [16], [17]: a necessary and sufficient condition for an imaginary finite value λ to be a point of the essential spectrum of a skew-selfadjoint operator M is that there exist a sequence of elements $v_n \in D(M)$ such that $\|v_n\| = 1, v_n \to 0$ weakly, and $\|(M - \lambda I)v_n\| \to 0.$

Evidently, the operator *M* is skew-selfadjoint and its spectrum belongs to the imaginary axis.

Theorem 2 Let $b = \min\{\omega, N\}$, $B = \max\{\omega, N\}$. Then, the essential spectrum of *M* is the symmetrical set of the imaginary axis: $\sigma_{ess}(M) = [-iB, -ib] \cup \{0\} \cup [ib, iB].$

Proof. For the operator *L* we observe that its main symbol $\tilde{L}(\xi)$ is represented by

$$\tilde{L}(\xi) = \begin{pmatrix} -\lambda & -\omega & 0 & 0 & \xi_1 \\ \omega & -\lambda & 0 & 0 & \xi_2 \\ 0 & 0 & -\lambda & N & \xi_3 \\ 0 & 0 & -N & -\lambda & 0 \\ \xi_1 & \xi_2 & \xi_3 & 0 & 0 \end{pmatrix}$$

and thus

$$\det \tilde{L}(\xi) = \lambda\left[(\lambda^2 + N^2)(\xi_1^2 + \xi_2^2) + (\lambda^2 + \omega^2)\xi_3^2 \right].$$

In this way we can see that if the spectral parameter λ does not belong to $[-iB, -ib] \cup \{0\} \cup [ib, iB]$, then the operator *L* is elliptic in sense of Douglis-Nirenberg.

Now, we consider $\lambda_0 \in \pm(ib, iB) \setminus \{0\}$ and choose a vector $\xi \ne 0$ such that $({\lambda_0}^2 + N^2)(\xi_1^2 + \xi_2^2) + ({\lambda_0}^2 + \omega^2)\xi_3^2 = 0$. Therefore, there exists the vector η such that $\tilde{L}(\xi)\eta = 0.$ After solving the obtained system with respect to η, we can represent one of the solutions in the form:

$$\eta_1 = \frac{\lambda_0\xi_1 - \omega\xi_2}{\lambda_0^2 + \omega^2}\ , \ \eta_2 = \frac{\lambda_0\xi_2 + \omega\xi_1}{\lambda_0^2 + \omega^2}\ ,$$

$$\eta_3 = \frac{\lambda_0\xi_3}{\lambda_0^2 + N^2}\ , \ \eta_4 = \frac{-N\xi_3}{\lambda_0^2 + N^2}\ , \ \eta_5 = 1.$$

Let us observe that $\eta_i \ne 0$ for all i . We choose a function

$$\psi_0 \in C_0^\infty(\Omega) \, , \, \int\limits_{\|x\|\le 1} \psi_0^2(x)dx = 1 .$$

Let us fix $x_0 \in \Omega$ and put

$$\psi_k(x) = k^{3/2}\psi_0\left(k(x - x_0)\right) \, , \, k = 1,2,...$$

Now, we can define the Weyl sequence as follows:

$$\begin{cases} \tilde{v}_j^k = \eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k + \dfrac{i}{k^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right), j = 1,2,3 \, , \, \tilde{v}_4^k = \eta_4\psi_k e^{ik^3\langle x,\xi\rangle} , \\ \tilde{v}_5^k = -\dfrac{i}{k}\psi_k e^{ik^3\langle x,\xi\rangle} \, , \, \langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \, , \, k = 1,2,... \end{cases}$$

It is easy to see that the above sequence satisfies the Weyl conditions.

Since the essential spectrum of a linear operator is always a closed set, the points $\{0\}, \pm\{ib\}, \pm\{-iB\}$, also belong to it and thus the Theorem is proved.

IV. Conclusion

The explicit form of the solution of three-dimensional rotating stratified flows in the Ocean, as well as the obtained exact estimates of vanishing of the amplitude for large values of *t* can be used directly for numerical calculations and programming. The constructed Weyl sequence is an explicit example of non-unique solutions for the case when the frequency of internal vibrations belongs to the spectrum. Since the bifurcation points belong to the essential spectrum, it can be used for investigation of the small nonlinear solutions. The practical case of a layer in the Ocean suggests that the obtained results may find their applications in nonlinear dynamic modeling, computational fluid dynamics and weather forecasting, since the obtained results are also valid for the Atmosphere.

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