The Kirchhoff Transformation for convective-radiative thermal problems in fins

[1]Jonatas Motta Quirino, [1]Eduardo Dias Correa and [2]Rodolfo do Lago Sobral

[1]State University of Rio de Janeiro - Rua São Francisco Xavier, 524, Rio de Janeiro, RJ
[2]Federal Center of Technological Education - Est. Adrianopolis, 1317, Nova Iguaçu, RJ
Brazil

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Abstract- The present work describes the thermal profile of a single dissipation fin, where their surfaces reject heat to the environment. The problem happens in steady state, which is, all the analysis occurs after the thermal distribution reach heat balance considering that the fin dissipates heat by conduction, convection and thermal radiation. Neumann and Dirichlet boundary conditions are established, characterizing that heat dissipation occurs only on the fin faces, in addition to predicting that the ambient temperature is homogeneous. Heat transfer analysis is performed by computational simulations using appropriate numerical methods. The most of solutions in the literature consider some simplifications as constant thermal conductivity and linear boundary conditions, this work addresses this subject. The method applied is the Kirchhoff Transformation, that uses the thermal conductivity variation to define the temperatures values, once the thermal conductivity variate as a temperature function. For the real situation approximation, this work appropriated the silicon as the fin material to consider the temperature function at each point, which makes the equation that governs the non-linear problem. Finally, the comparison of the results obtained with typical results proves that the assumptions of variable thermal conductivity and heat dissipation by thermal radiation are crucial to obtain results that are closer to reality.

Keywords- Convection-Radiation, Extended surface, Kirchhoff Transformation, Thermal distribution

I. Introduction

THE use of extended surfaces as a way to optimize or even control heat exchange exists even before human interference with nature. The application of such surfaces, hereinafter also called fins, can be observed in nature, such as the ears of *Vulpes zerda* (or Fennec fox) [1], which work as fins in order to dissipate heat from the running blood on them.

The low or mistaken thermal control is the reason for the failure of a huge part of industrial components. NASA (National Aeronautics and Space Administration) estimates that 100% of the failures of its monolithic microwave integrated circuits (MMIC) could be prevented if there was adequate thermal control [2]. Temperature is the environmental factor that most causes failure in electronic components [3].

In industry there are specific applications in which, the employed materials have properties that are closely dependent on temperature. Some of them at high temperatures, such as blast furnaces and others at low temperatures, as aerospace components. For these materials, it is essential to perform certain thermal analyzes for an adequate dimensioning.

Moreira *et. al* [4] study the heat transfer coefficient for convective problems, without considering radiant effects. Its study aims to establish criteria for the selection of the heat transfer coefficient according to the specific parameters of the problem to which they are applied.

Mazlaghani *et. al* [5] propose an empirical technique for measuring thermal conductivity dependent on the temperature of materials with low thermal conductivity. They analyze unidimensional non-linear inverse problems, and looking for that in their experiment there is no rejection of heat by radiation.

Gama [6] seeks to solve problems of heat transfer by radiation, which generates a non-linear PDE. The solution reached is given by the proposal to impose an upper limit, estimated for the general equation of the problem.

Kim [7] uses the Kirchhoff Transformation in some examples of functions to describe the variation of the thermal conductivity of a surface, considering thermal conductivity as a combination of known functions, which seek to determine its coefficients.

Lesnic *et. al* [8] adopt a one-dimensional model considering the thermal transient, in which the thermal capacity is an established value. For his work, the thermal conductivity and thermal capacity of a material are directly proportional and the proportionality constant is known.

Bonani & Ghione [9] apply the Kirchhoff Transformation method to provide a thermal analysis of a semiconductor material, which presents variation in thermal conductivity, which in turn is described as piecewise heterogeneous.

Zhou *et. al* [10] analyze the thermal behavior of a body of irregular geometry, applying the effects of thermal convection. Mathematical modeling is done using the Weighted Least Squares Method without Mesh, applying the Robin boundary condition.

Aziz & Bouaziz [11] use the Least Squares Method in the treatment of non-linearities associated with the internal generation of heat and thermal conductivity dependent on temperature in a longitudinal fin.

Suk & Park [12] study a porous medium in hydrological application, where the Richards Equation is solved through the Kirchhoff Transform. Using the Finite Volume Method, the Kirchhoff Transformation adapted with expansion of the Taylor series is applied to investigate the thermal behavior at an interface between two different materials.

Bagnall *et. al* [13] present a specific application of the Kirchhoff Transform, considering the effects of thermal convection in electronic components, comparing the Kirchhoff Transformation with the Finite Element Method, being aided by a multiphysical model developed by COMSOL.

Although there are studies on the heat dissipation by fins, the mathematical models used are complex and difficult to manipulate. For this reason, such models tend to neglect some parameters that the present model intends to analyze. Of these parameters, this work prioritizes the variation of the thermal conductivity according to the variation of the temperature values, looking for a method that delivers a simple and effective analysis.

There are difficulties in modeling thermal systems in which variable thermal conductivity is considered. Such an obstacle is faced by some authors by mathematical methods. The present work presents the mathematical modeling using the Kirchhoff Transformation that allows treating the nonlinear equations that govern the thermal distribution, as a linear model. The model is applied to a longitudinal fin, with rectangular geometric profile conected to a thermal source.

The central objective of this analysis is to observe the thermal behavior of fins that dissipate heat from a given primary surface through the heat transfer processes of conduction, convection and thermal radiation. This latter aspect is largely neglected in mathematical modeling studies and will be addressed in this work.

In summary, this work seeks to collaborate with the theme through studies on the influence of the effects of thermal radiation, commonly neglected, in addition to the analysis of the variation in the thermal conductivity of the material.

II. Problem Formulation

As stated earlier, this work is based on the fact that in several applications, thermal conductivity cannot be considered constant, but rather as a temperature dependent function.

Several machines and equipment used both in industry and in everyday life bring with it the need to dissipate the heat produced by combustion, friction or by electric currents, which, when moving, transform part of their kinetic energy into heat.

Therefore, this work proposes a mathematical method for analyzing thermal behavior along a fin that dissipates heat by convection and radiation.

The Heat Equation is given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \quad (1)$$

Initially, it is necessary to define the mathematical modeling of the analyzed problem. For this, some conditions must be established so that the modeling is close to the real situation.

Some considerations must be made:

- The thermal transient is neglected, that is, the entire analysis is performed after the thermal equilibrium is reached. This removes the time term from the general heat equation. Mathematically $\frac{\partial T}{\partial t}=0$;
- The fin is not a source of its own heat, this causes the heat removed by the fin to come from the primary surface to which the fin is attached. Mathematically $\dot{q}=0$;
- There is no heat absorption in the fin, as it is considered that the temperature of the environment will always be lower than the temperature in any point of the fin. Mathematically, the vectorial direction of heat is constant outside the body.

Statements 2 and 3 seek to ensure that the fin is exclusively a heat sink, not generating or absorbing heat.

The PDE for heat conduction problems in isotropic media with temperature-dependent thermal conductivity can be expressed as

$$\nabla\cdot(k(T(x,y,z))\nabla T(x,y,z))=0. \quad (2)$$

By the definitions of fins, the width and height are much larger than the thickness, therafter it is concluded that only in the axis y the temperature differences between their points are considerable. This formulation suggests that the problem is analyzed in a one-dimensional approach.

According to the such definitions, some boundary conditions will be established.

The Fig.1 presents a Single Fin which has one surface extended to the primary surface.

Two mathematical boundary conditions (b.c.) were used. These conditions are very common in mathematical problems involving differential equations.

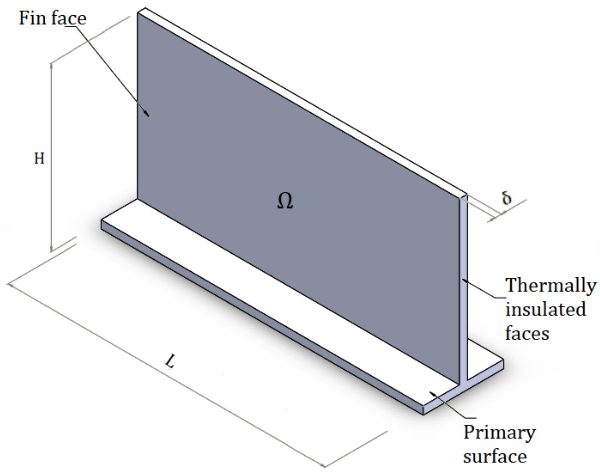

Fig. 1: Boundary and domain of fins.

- The face Γ_1 $(y = 0)$ is conditioned with a certain temperature imposed on it. (Dirichlet b.c.)
- The faces Γ_2 $(x = 0)$, Γ_3 $(x = L)$ and Γ_4 are thermally isolated. (Neumann b.c.)
- In the faces $z = 0$ e $z = \delta$, heat dissipation is considered by thermal convection and radiation [14, 15].

For the characterization of the thermal conductivity profile, it is necessary to trace parameters that relate the thermal conductivity (k) and temperature (T).

Experimental values of the variation of thermal conductivity as a function of different temperature values can be observed in the work of Nayar [16] from where the values for a body of silicon were taken. Such values are shown in the Table 1.

Table 1: Thermal conductivity values (W/m.K)

$T(K)$	$k(W/m \cdot K)$
20	4980
40	3530
60	2110
80	1340
100	884
150	410
200	260
250	190
300	150
400	99
500	76
600	62
800	42
1000	31
1200	26
1400	24
1500	23

This function can be expressed as an equation that relates the temperature values to their respective thermal conductivity values, as shown by (3).

$$k(T) = aT^b \tag{3}$$

Where the constants a and b were defined by the Least Square Method. Bonani & Ghione [9] define these constants as $a = 259000$ and $b = -1.3$ while Bagnall *et. al* [13] conclude that $a = 249085.75$ and $b = -1.3$.

III. Kirchhoff Transformation

With (3) and the values from a and b, the values of k must be defined by the Finite Difference Method discussed above. From the determination of the interesting values to the thermal conductivity variation function, a mathematical method can be applied that performs the inverse description, where the temperature values are defined, once the values of k are known [17].

Kirchhoff Transform is defined by (4) [18].

$$\omega = \hat{f}(T) = \int_0^T \hat{k}(\xi)d\xi \tag{4}$$

This work characterizes that the lower bound of the work domain is limited to the value of $0K$. Such a restriction is not generic, being used for convenience in this application.

It is noteworthy that the value of T_0 is the reference temperature, which in turn represents the primary surface temperature, which is fixed, known and upper limit. This is to say that no fin point can have a higher temperature than T_0. The Equations (5) describes the correct formulation to a thermal heat sink .

$$\omega_1 = \int_0^{T_0} \hat{k}(\xi)d\xi = \int_0^{T_0} aT^b dT \Rightarrow \omega_1 = \frac{aT_0^{b+1}}{b+1} \tag{5a}$$

$$\omega_2 = \int_0^{T} \hat{k}(\xi)d\xi = \int_0^{T} aT^b dT \Rightarrow \omega_2 = \frac{aT^{b+1}}{b+1} \tag{5b}$$

In order to make it possible to limit the working region between T and T_0, it is necessary to algebraically treat Equations (5). It should be noted that $grad\ T_0 = 0$, therefore only $grad\ \omega_2$ is analyzed

$$grad\ \omega_2 = k\ grad\ T \tag{6}$$

The final format of Kirchhoff Equation given by (4) will be given by subtracting the Equations (5a) and (5b), thus reaching the integration interval that matters for the application.

This is why integration limits range from T to T_0, unlike the definitions described by the authors Sobral [19] and Gama [20]. This ensures that the inverse of (4) is within the working domain.

Such a maneuver could be avoided in the application, since, as $b < 0$, the algebraic inconvenience has already been overcome. Therefore, this maneuver was used in order to facilitate generic solutions in the application of the method.

Since the function of k given by (3) is applied to the Kirchhoff Transformation, one has to

$$\omega = \omega_1 - \omega_2 = \frac{aT_0^{b+1}}{b+1} - \frac{aT^{b+1}}{b+1} \tag{7}$$

Whose inverse expression is

$$T = \left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} \tag{8}$$

When plotting the Table 1 points on a T vs k graph, a trend curve depicting a negative exponential function is observed.

Some studies [9, 13] analyze silicon and how its thermal conductivity varies with temperature. These works conclude, just like this present work, that the mathematical modeling that describes such variation presents the format of a power function.

An effective way to treat the non-linear function of dependence between thermal conductivity and temperature is the Least Squares Method [11]. These results will serve to validate the Kirchhoff Method by comparative analysis. In the

LSM the empirical data were stipulated [21] and the approximation is illustrated in Figure 2 for silicon data.

The numerical result applied to the data in the Table 1, conclude that $a = 438900$ and $b = -1,3759$.

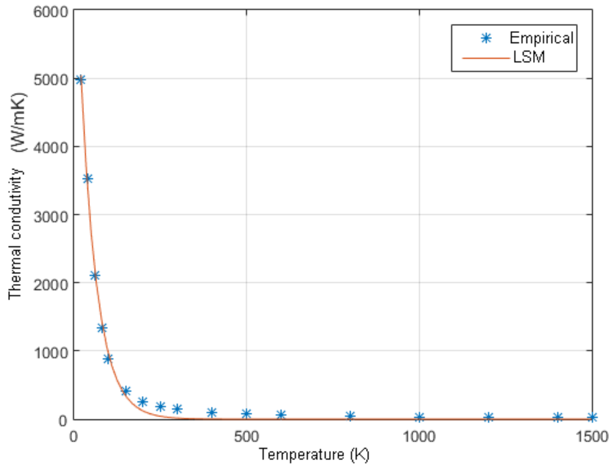

Fig. 2: Curve fitting by LSM in exponential type.

Therefore, the Figure 2 shows that thermal conductivity variation describes a trend curve represented by a negative exponential function.

For the solution of the problem, the considerations presented in the Section II. will be applied: The analysis is stationary after the thermal equilibrium and the fin is not a heat source.

Based on such definitions, the thermal distribution in the directions of each Cartesian axis of the fin, is

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0 \tag{9}$$

As stated earlier, the problem is being formulated in a one-dimensional approach, making the above PDE in (9) become an ODE,as presented in (10). Besides that, the conditions presented in the Section II., when applied to the Kirchhoff Transform mathematical model, take the form

$$k(T)\frac{dT}{dz} = \frac{d\omega}{dz} \tag{10}$$

It is worth mentioning that (10) can be applied at $z = 0$ and $z = \delta$, differentiating only by the vectorial direction.

The first consideration given to heat dissipation by the fin is that both faces of the fin reject heat exclusively by convection, where such an effect occurs when a body at a temperature higher than ambient rejects heat to the environment once it is surrounded by some fluid.

Therefore convection heat dissipation on each side of the fin is given by (11)

$$z = 0 \Rightarrow \frac{d\omega}{dz} = h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) \tag{11a}$$

$$z = \delta \Rightarrow -\frac{d\omega}{dz} = h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) \tag{11b}$$

When working with extended surfaces, one should consider an essential fin feature, which is its geometry as a very thin plate. Thus, by integrating the differential equations above, we conclude through *Average Value Theorem* that

$$\frac{d^2\omega}{dz^2} = -\frac{2}{\delta}\left[h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right)\right]. \tag{12}$$

When considering the effects of thermal radiation, it is assumed that the boundary conditions on the fin faces will be altered, since it is precisely on both faces of the fin that the thermal dissipation occurs. Until here the analytical solution was exposed only with the effects of convection.

This insertion is given by adding the equation of the *Stefan-Boltzmann's Law*, which represents the heat flux emitted by thermal radiation.

For the total analysis of the problem, one has to:

$$\begin{aligned} z = 0 \Rightarrow \frac{d\omega}{dz} =& h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) \\ &+ \varepsilon\sigma\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{4}{b+1}} \end{aligned} \tag{13a}$$

$$\begin{aligned} z = \delta \Rightarrow -\frac{d\omega}{dz} =& h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) \\ &+ \varepsilon\sigma\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{4}{b+1}} \end{aligned} \tag{13b}$$

Equations (13) are similarly symmetrical, since the geometric conditions imposed on the problem produce a symmetrical heat dissipation on both faces of the fin. Thus, the equation governing heat transfer in case of dissipation considering convection and radiation is given by

$$\begin{aligned} \frac{d^2\omega}{dz^2} = -\frac{2}{\delta}\Bigg\{ & h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) \\ & + \varepsilon\sigma\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{4}{b+1}} \Bigg\} \end{aligned} \tag{14}$$

Relating (2) and (14), we have a new formulation for the temperature profile with the equation as a function of T.

$$\begin{aligned} k(T)\frac{\partial^2 T}{\partial x^2} + k(T)\frac{\partial^2 T}{\partial y^2} + \left(\frac{\partial k(T)}{\partial y}\cdot\frac{\partial T}{\partial y}\right) - \\ \frac{2}{\delta}\left[h(T - T_\infty) + \varepsilon\sigma T^4\right] = 0. \end{aligned} \tag{15}$$

In the situation involving thermal radiation dissipation and variable thermal conductivity, the thermal phenomenon presents itself as non-linear, considerably hindering an analytical solution.

This happens for two reasons:

- While the temperature varies depending on the position of the fin, the thermal conductivity varies depending on the temperature $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$.
- The nonlinear term of temperature at the fourth power, derived from the Stefan Boltzmann's law for heat transfer by radiation. T^4.

If it is still considered that the thermal conductivity of the system is constant, then the solution of (15) can be obtained by minimizing the following functional

$$I[v] = \frac{1}{2}\int_0^H \int_0^L \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dxdy + \int_0^H \int_0^L \left[\frac{h}{\delta k}(v - T_\infty)^2 + \frac{2\varepsilon\sigma}{5\delta k}v^5\right] dxdy \quad (16)$$

Where v represents the vector field which, when applying the boundary condition at $y = 0$, says that the temperature at that point is given as a prescribed fixed temperature.

The existence of a solution by minimizing functional I is guaranteed as long as it is proven that I is a convex, coercive and convergent functional [22].

Equation (16) is shown to be equivalent to (15) through its first variation, which means that (16) is evaluated as being the minimization of the functional,considering the constant thermal conductivity.

Equation (15), therefore, it is the equation that models the thermal distribution with radiance in the fin, still in a two-dimensional approach. As previously explained, the studied situation considers that the entire primary surface is at a prescribed homogeneous temperature, ie, along this surface, all points are at the same temperature, which leads to the conclusion that along the axis x of the fin , there is no variation of temperature, so the derivative of temperature in relation to x equals zero.

$$k(T)\frac{\partial^2 T}{\partial y^2} + \left(\frac{\partial k(T)}{\partial y}\cdot\frac{\partial T}{\partial y}\right) - \frac{2}{\delta}[h(T - T_\infty) + \varepsilon\sigma T^4] = 0, \quad (17)$$

where k is a function of T, which in turn is a function of y, so k will be treated as a variable in function of y.

Whose representation in the functional given by (16) can be written as

$$I[v] = \frac{1}{2}\int_0^H \left(\frac{\partial v}{\partial y}\right)^2 dy + \int_0^H \left[\frac{h}{\delta k}(v - T_\infty)^2 + \frac{2\varepsilon\sigma}{5\delta k}v^5\right] dy \quad (18)$$

Resuming the representation of the problem in the Kirchhoff Method, which is the focus of this work and applying the boundary conditions, (2) takes the form

$$\frac{d^2\omega}{dy^2} = \frac{2}{\delta}\left[h\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{1}{b+1}} - T_\infty\right) + \varepsilon\sigma\left(\left[T_0^{b+1} - \frac{(b+1)}{a}\omega\right]^{\frac{4}{b+1}}\right)\right] \text{ for } 0 < y < H \quad (19)$$

In order to solve numerically the partial derivatives in the problem studied, is used the Finite Differences Method (FDM). Given these relations, after algebraic adjustments takes the following form.

$$\omega_j = \frac{2H^2}{\delta}\left[h\left(\frac{(b+1)}{a}\omega_j + T_0^{b+1}\right)^{\frac{1}{b+1}} - T_\infty + \varepsilon\sigma\left(\frac{(b+1)}{a}\omega_j + T_0^{b+1}\right)^{\frac{4}{b+1}}\right] + \frac{(\omega_{j+1} + \omega_{j-1})}{2} \quad (20)$$

IV. RESULTS

The analysis and processing of all the data, methods and processes exposed in this work have resulted in some extremely relevant conclusions.

Since this work argues that certain phenomena can not be neglected, all simulation procedures occur contemplating the most varied situations so that it is possible to make comparisons of the results and determine the relevance of the study.

The simulation environment was maintained in all situations, except for the parameters that characterize the preponderant differences that will be compared.

It was used to carry out this work the commercial software *Matlab*, which allowed the creation of an algorithm that contains all the mathematical data and process the required information and calculations.

Table 2: Applied parameters

Symbol	Concept	Value
T_∞	Environment Temperature	$300K$
T_0	Primary Surface Temperature	$500K$
j_{max}	Number of vertically oriented nodes	100
L_y	Fin height	$10mm$
δ	Fin thickness	$1mm$
tol_T	Specified accuracy	10^{-6}

It is worth mentioning that the parameters stipulated above can be handled in a convenient manner in order to investigate results in other circumstances.

In order to allow better visualization and understanding of the obtained results, some graphs were generated.

To evaluate the behavior of thermal conductivity at different temperature values, the data were processed in four different ways from the combinations that will be exposed in the following subsections.

A. Without radiation

One of the parameters proposed in this work is the insertion of the effects generated by the phenomenon of variation of thermal conductivity, as a function of temperature. Such an analysis has been described in the section II.. Kraus *et. al* [23] have suggested that the Murray-Gardner hypothesis that evaluates the constant thermal conductivity in all directions, when disregarded, proposes results that are closer to a real model.

Most engineering projects still prefer to disregard the effects of thermal radiation, since mathematical modeling is complex. This produces effects in which unknown or neglected parameters are coupled, generating erroneous dimensions.

For purposes of illustration, Gorla & Bakier [24] presents a comparative analysis of how the radiation effects considerably affect the thermal dissipation profile in the fin.

Since the effect of varying thermal conductivity is applied to the thermal distribution of the body, an intense change in temperatures throughout the body can be noted, as seen in the Fig.3.

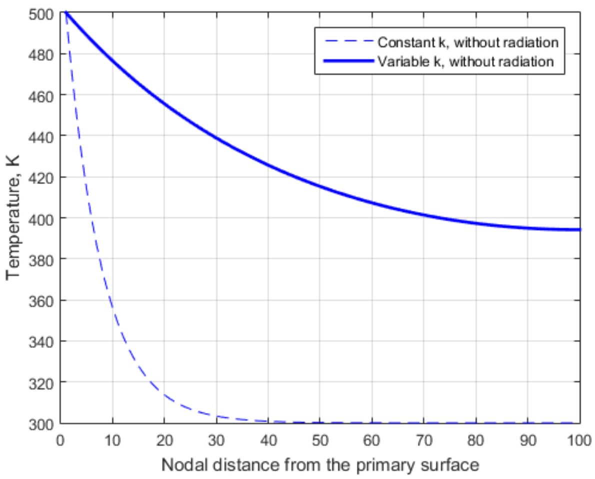

Fig. 3: Thermal profile without radiation and variable *k*.

The importance of this analysis can be seen through the Table 3, where it can be seen that the thermal profiles present values very distant from each other at each point. In the analysis carried out, a maximum percentage difference of 31.3642% is noted in the twenty-third node. On the other hand, as in most applications, the maximum heat dissipation, which is given in the last node, is sought, it can be noted that the percentage variation in this node is 23.8991%.

B. With radiation

The combination of the effects generated by the heat dissipation by thermal radiation, besides the evaluation of the thermal conductivity variation as a function of the temperature at each point was proposed by Cohen [25] and brings a closer approximation to real results. To the non-linear differential equation (as a function of the radiation term) the approximation calculation of thermal conductivity values is added by means of appropriate numerical methods. Furthermore, this work considers that the effects of the variation in thermal conductivity can still be added to thermal radiation, where the body loses even more heat, thus changing the profile generated. These two added phenomena can be analyzed in the Fig.4, where the profiles with and without variation of k are superimposed.

Table 3: Constant and variable k without radiation

Node	Constant k	Variable k	% Error
1	500	500	0,0000%
5	413,6476	488,9148	15,3948%
10	356,069	476,4604	25,2679%
15	327,6621	465,3793	29,5925%
20	313,6473	455,506	31,1431%
30	303,3218	438,8529	30,8830%
40	300,8085	425,6417	29,3282%
50	300,1968	415,2598	27,7087%
60	300,0479	407,2673	26,3265%
70	300,0116	401,3496	25,2493%
80	300,0028	397,2872	24,4872%
90	300,0007	394,9356	24,0381%
100	300,0003	394,2136	23,8991%

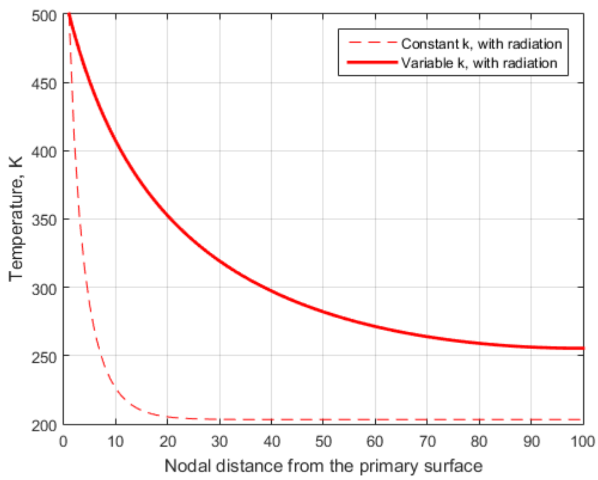

Fig. 4: Thermal profile of single fin, with radiation and variable *k*

As can be seen in Fig.4, the analysis of the phenomenon of variation in thermal conductivity as a function of temperature, when added to the effects of thermal radiation, generate a considerable displacement of the thermal profile of the fin. For further analysis, the Table 4 presents that a maximum percentage difference of 44.8210% is noted in the twelfth node.

As stated earlier, it is very important to analyze the full thermal dissipation, which is given at the last node of the fin. To this end, it is clear that the case of comparison between situations with radiation generates a percentage difference of 20.4363%.

In order to visualize the difference in the thermal profiles of each of the four cases analyzed in this work, Fig. 5 was built in an superimpose way. It is worth mentioning that this work does not seek an arrangement whose heat rejection is maximum, but rather the most realistic case when phenomena intrinsic to the heat equation are considered.

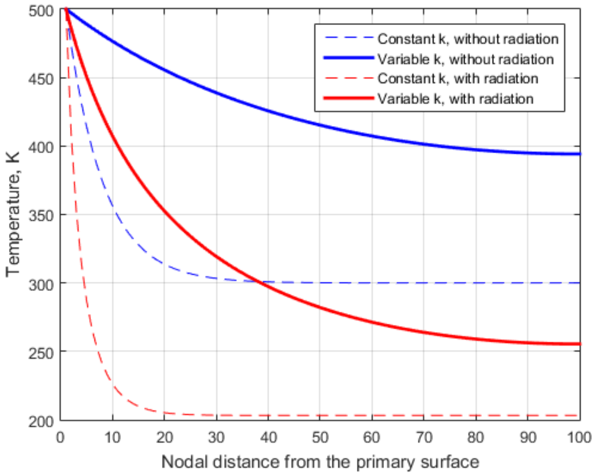

Fig. 5: Comparison between all analyzed cases

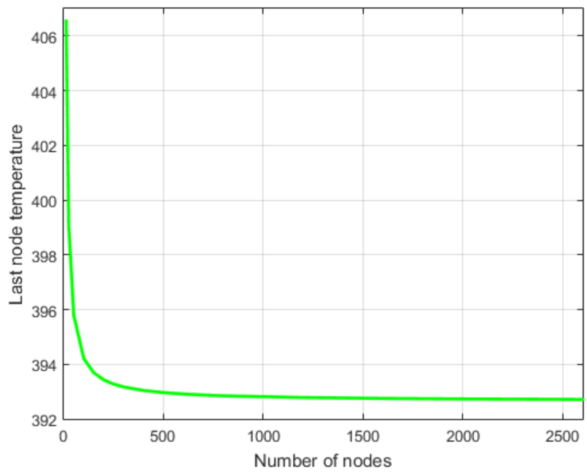

Fig. 6: Convergence of the last node

Table 4: Constant and variable k with radiation

Node	Constant k	Variable k	% Error
1	500	500	0,0000%
5	286,5776	450,3872	36,3708%
10	226,0266	407,0929	44,4779%
15	209,9219	375,9749	44,1660%
20	205,2388	352,4497	41,7679%
30	203,4269	319,2875	36,2872%
40	203,2638	297,3025	31,6306%
50	203,249	282,0702	27,9438%
60	203,2477	271,356	25,0992%
70	203,2476	263,9188	22,9886%
80	203,2476	259,0351	21,5367%
90	203,2476	256,2857	20,6949%
100	203,2476	255,4526	20,4363%

C. *Numerical Convergence*

The heat equation (14) that governs the problem described in this work is presented in a non-linear model, which makes any analytical solution approach very difficult.

For this reason, numerical methods are used to approach the solution. However, this approach brings with it some demands, such as the guarantee of convergence of the solution.

Bearing in mind that along the fin the last node is the one with the highest thermal gradient compared to the primary surface, therefore, this node is considered as a critical point. Therefore, the numerical convergence analysis evaluated the behaviour of the temperature of this node as a function of the mesh refining, which can be seen in Fig.6.

The above convergence analysis shows that the temperature of the last node converges to a value, which can be approximated by the Least Squares Method that generates the (21).

$$f(x) = ax^b + c. \tag{21}$$

Whose solution presents the coefficients $a = 186,2158$, $b = -1,045802$ and $c = 392,6822$.

Since the coefficient c represents the linear displacement of the curve in Fig.6, it can be interpreted that the value of c is the value to which the sequence converges.

D. *Comparison between FDM and Kirchhoff Method*

Having results obtained previously with the numerical processing by the Finite Differences Method, a comparative analysis with the Kirchhoff Method proposed in this work can be performed.

The comparative analysis can be seen in the Tables 5 and6, which shows the values obtained for processing with FDM and Kirchhoff. Since the exposure of the values of all 50 nodes used would be very extensive, it was decided to expose the values of the first five and the last five nodes.

Table 5: FDM vs Kirchhoff without radiation

Without radiation			
Node	Kirchhoff	FDM	Dif. %
1	500,000	500,000	0,000%
2	494,358	494,345	0,003%
3	488,981	488,955	0,005%
4	483,854	483,818	0,008%
5	478,964	478,918	0,010%
.	.	.	.
.	.	.	.
.	.	.	.
46	396,195	396,085	0,028%
47	395,998	395,889	0,028%
48	395,867	395,758	0,028%
49	395,801	395,692	0,028%
50	395,801	395,692	0,028%

In addition, a graph in Fig.7 was generated that overlapped all the results analyzed by this work in order to demonstrate the discrepancy between the results in all situations analyzed in this work.

From the illustration brought by the Fig. 7, a very interesting effect is revealed. It can be noted that, although the ambient temperature is 300K, according to the Table 2, the cases in which the radiation effect is applied allow the fin to reach temperatures below the ambient temperature.

This phenomenon can be explained by the fact that the

Table 6: FDM vs Kirchhoff with radiation

With radiation			
Node	Kirchhoff	FDM	Dif. %
1	500,000	500,000	0,000%
2	473,026	472,880	0,030%
3	450,464	450,224	0,049%
4	431,261	430,961	0,062%
5	414,687	414,348	0,071%
.	.	.	.
.	.	.	.
.	.	.	.
46	257,160	256,982	0,045%
47	256,930	256,753	0,045%
48	256,777	256,601	0,045%
49	256,701	256,525	0,044%
50	256,701	256,525	0,044%

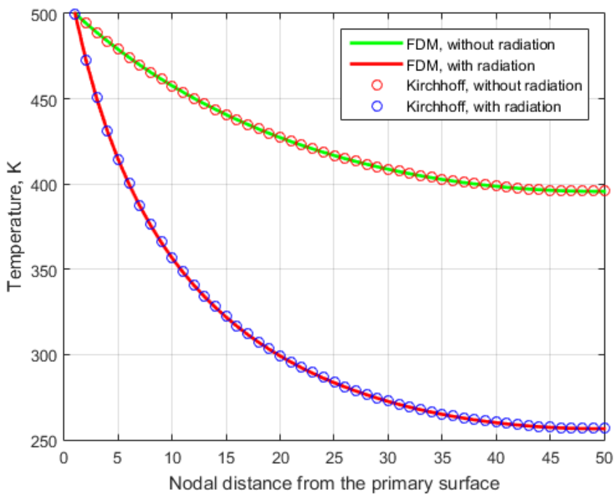

Fig. 7: Comparison of thermal profiles.

problem is radiant being analyzed in an environment without atmosphere, such as a vacuum. Such a situation allows the heat dissipated by the fin to be transferred to infinity, thus not generating the lower limitation in heat dissipation.

However, still in this analysis, another situation can be seen. If the applied ambient temperature is $0K$, T will never be below that environment temperature. That kind of situation should be considered, for example, in aerospace applications, where T_{amb} is just above $0K$.

To ensure the convergence of the numerical analysis used to process the heat equation by the Kirchhoff Transform, a stop criterion was applied [26]. This criterion establishes that there is a Stopping criterion defined by (22)

$$r^{(k)} = \frac{\max |x_i^{(k)} - x_i^{(k-1)}|}{\max\limits_{1\leq i\leq n} |x_i^{(k)}|}. \tag{22}$$

In the iteration in which the criterion is verified, the process stops, otherwise the iterations continue. If it is possible to guarantee that the (22) represents a decreasing series at all points, then the numerical method used converges, where convergence is guaranteed if it is possible to show that r tends to zero.

The Fig. 8 illustrates the stabilization of the iterations, confirming that the method converges.

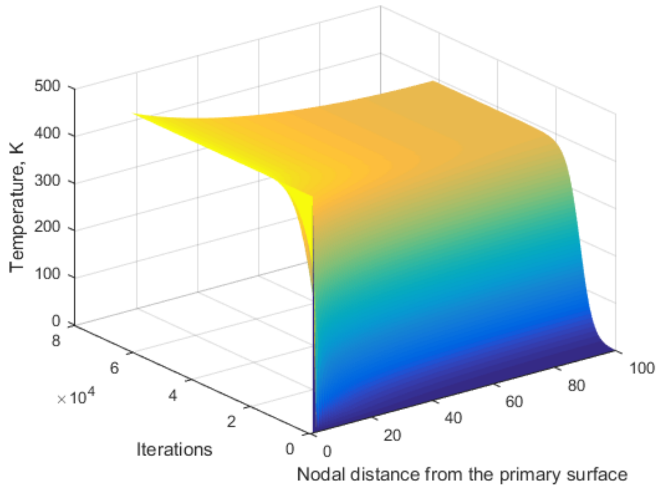

Fig. 8: Final thermal profile.

The convergence study of the method can be done by analyzing the mathematical behavior of each point, verifying whether such points approach defined values or not. However, for the purpose of simplifying the problem, it is noted that the last nodal point is the one that presented the greatest distance in values when compared case by case.

Therefore, the convergence analysis was taken by observing each nodal point in relation to the temperature value of the previous point, thus concluding if the temperature values tend to stabilize. The analysis performed on the behavior of the relative error can also be seen in Table 7.

Table 7: Error analysis

k	r
1	not applicable
2	65535
3	0,9838924
4	0,487936686
5	0,322625087
10	0,1160175
20	0,047182763
30	0,027496305
40	0,018111285
50	0,012548592
60	0,008790923
70	0,005999022
80	0,003752786
90	0,001808115
99	0,000178564

Since the final objective of the analysis is based on the results with thermal radiation dissipation and variable thermal conductivity, the numerical results of the first five nodes and the last five nodes were compared to that, in Tables 5 and 6, where such comparisons are expressed in the percentage difference to the conclusive result.

V. DISCUSSIONS

When looking at Figs.3 and 4, it is clear how the effects of variation in thermal conductivity affect the temperature distribution along the fin.

This shows that this phenomenon should not be neglected in fin heat dissipation projects, because when removing such effects, considering that the thermal conductivity of the material is constant, incorrect dimensioning and insufficient models are caused.

It is worth mentioning that the thermal profiles studied in situations where there is radiation heat dissipation, the minimum temperature values exceed the environment temperature value. This phenomenon is explained by the fact that the mathematical processing of such situations does not take into account the minimum boundary that the environment temperature causes.

Physically speaking, it is considered that the body is capable of rejecting heat to the vacuum and is therefore not limited by the environment from the point of view of radiant dissipation.

It can also be noted that variations in values of k are much more evident in a context in which low and medium temperatures are employed, since the higher the values of T, the more the thermal conductivity profile approaches a straight line, in asymptotic way.

VI. CONCLUSIONS

The analysis of such temperature profile aims to present the importance of considering the effects of thermal radiation on heat dissipation and thermal conductivity variation, so that, compared to the thermal behavior in the absence of such effects, there is a considerable discrepancy in the results.

Since the majority of the works that involve the topic considered consider the parameter of thermal conduction constant, this study proposes to consider the variation of such parameter to bring even greater applicability and approximation of reality. For this purpose will be applied a appropriate mathematical method, that will be approached forward.

This study shows the feasibility of using the Kirchhoff Transformation in problems whose thermal conductivity is a known function of temperature. It was necessary to establish what kind of function this would be and the coefficients related to it, as well as its values. This can be seen in Section II. of this paper, where (3) and Table 1 determine such parameters.

It can be concluded, therefore, that thermal dissipation analysis, in order to approach a real model, should never neglect the variation of thermal conductivity, the effects of thermal radiation.

In the future, this study is still seeking to improve the analysis through comparison with COMSOL's multiphysics model; efficiency and effectiveness analysis; insert internal heat generation; extend the analysis to double fins.

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