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A New Hybrid Method for Solving Inverse Heat Conduction Problems

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Abstract— Solving the inverse problems, especially in the field of heat transfer, is one of the challenges of engineering due to its importance in industrial applications. It is well-known that inverse heat conduction problems (IHCPs) are severely ill-posed, which means that small disturbances in the input may cause extremely large errors in the solution. This paper introduces an accurate method for solving inverse problems by combining Tikhonov's regularization and the genetic algorithm. Finding the regularization parameter as the decisive parameter is modelled by this method, a few sample problems were solved to investigate the efficiency and accuracy of the proposed method. A linear sum of fundamental solutions with unknown constant coefficients assumed as an approximated solution to the sample IHCP problem and collocation method is used to minimize residues in the collocation points. In this contribution, we use Morozov's discrepancy principle and Quasi-Optimality criterion for defining the objective function, which must be minimized to yield the value of the optimum regularization parameter.

Keywords— Inverse Heat Transfer Problems, Tikhonov regularization, Genetic algorithms, III-Posed Problems, Morozov's discrepancy principle and Quasi-Optimality.

I. INTRODUCTION

INVERSE heat conduction problems have broad applications in technological and scientific fields [1]. The primary purpose of solving these types of problems is to obtain solution indirectly. The main reason for the emergence of the inverse heat transfer problems is not knowing boundary conditions or difficulty in accessing boundaries. Therefore, to solve the problem without having boundary conditions, it is necessary to have additional information, which is usually obtained by the sensors installed in an accessible place. Therefore, with empirical data, it is possible to estimate the conditions needed to solve the problem without direct measurements or access to boundary locations. In direct heat transfer problems, geometry, boundary conditions, initial conditions, and thermo-physical properties are known, and the purpose is to calculate the temperature distribution inside the solution domain. In the case of inverse heat transfer, one or some information are unknown, and the objective is to estimate them by using additional information such as the measured temperatures inside the solution domain.

The main difficulty in solving inverse problems is that they are almost always severely ill-posed. According to Hadamard [2], a problem is called well-posed if it has a unique solution which continuously changes with input data. The inverse solution is extremely sensitive to measurement errors, and even the smallest in inputs may cause a significant error in the final approximation of the boundary conditions, therefore, regularization is required for solving inverse problems.

Since IHCPs are incredibly diverse, their solution also requires different strategies. The solution of IHCPs can be classified into three different classes: analytical, numerical, and experimental solutions. Of course, in some cases, a combination of the mentioned methods can be used for solving the problem. Analytical methods are often useful in solving linear problems, but numerical methods such as the finite difference method, finite element method, and boundary elements are applied in solving nonlinear and multidimensional problems.

In 1960, Stols solved the problem of transient heat conduction analytically and showed that frequent use of tiny time steps results in instability in the solution of such problems [3]. It can be seen that using small time steps has the opposite effect on inverse heat conduction problems (IHCPs) compared to numerical solutions of the direct heat conduction equation.

Tikhonov and Arsenin introduced Tikhonov and Iterative Regularization Methods [4]. This method is usually provided as a whole domain solution in which all the parameters of heat flux are estimated for all times and spatial locations simultaneously.

Another method that uses the regularization technique is the conjugate gradient method with an adjoint problem, which is developed and suggested in detail by Alifanov[5], Zisnik and Orlande [6]. The conjugate gradient method minimizes an objective function at each iteration through choosing a new guess by taking the old assumption and tacking on an additional term that pushes the solution closer to the optimal one. This regularization method can also be used to solve linear and nonlinear inverse problems or parametric estimations.

There are various analytical and numerical approaches in the literature for solving IHCPs. Moreover, a specific solution for a particular problem cannot be applied to other problems. The special sequential function developed by Beck [7], is a sequential method stepping forward in time, based on least-squares method and Duhamel theorem. The complicated mathematics in estimating components of the heat flux in different times and spatial locations is a fundamental problem in the mentioned algorithm. Hensel did some research on the analytical transfer function to solve inverse heat conduction problems. He presented an inverse heat method for a one-dimensional case using an adjoint algorithm with a frequency domain [8].

Lesnic et al. proposed another way to solve the IHCPs. In this method, the least-squares regularization and energy method have been introduced into the boundary element method (BEM) formulation. The numerical results obtained using this technique has the advantage of not needing to mesh generation in all domain, unlike the finite element method or finite difference method [9].

Yeun et al. dealt with the smooth fitting problem using the genetic programming algorithm, they presented a novel approach for choosing the regularization parameter and compared the result with general cross-validation (GCV) B-Splines [10].

Several researchers have proposed combinations of the method to minimize the problems involved in measuring errors [11]. Keynini et al. [12] have proposed a modified sequential function for solving the stability of parabolic thermal conduction problem. This method uses computational steps that are larger than the sample intervals, and future time intervals are all set equal to the time interval in the data.

Slota et al. [13] combined the Tikhonov regulation method and the particle swarm optimization algorithm, which is a stochastic optimization method, for approximating the heat source without the prior information of the functional form in temperature-dependent unsteady heat conduction problem and compared the results with the conjugate gradient method.

Ajith et al. [14] attempted to develop a general guideline for designing the high-temperature heaters by investigating modeling tools through different approaches such as lattice Boltzmann method (LMB), finite volume method and genetic algorithm. Stephany et al. [15] formulated the inverse radiative transfer problem and solved it through ant colony optimization (ACO) with the Levenberg–Marquardt (LM) method. Their result shows that this hybridization method results in a better reconstruction at a lower processing time.

Singh and Das analyzed the thermal behavior of the fin when thermophysical parameters are varied. They used approximate analytical Adomian decomposing method to solve the nonlinear problem along with the Newton–Raphson method [16]. Some researchers used the conjugate gradient method to estimate the unknown time-dependent heat flux and time and spatially dependent heat fluxes at the interface of two contacting surfaces and even parallel plate channel [17-20]. Dong et al. applied coupled methods (least square QR decomposition (LSQR)-Genetic Algorithm (GA) and truncated singular value decomposition (TSVD) method- Genetic Algorithm (GA) to investigate the performance of this method on temperature distribution in the participating medium [21]. For some samples, even combinational methods with non-optimal regularization parameters can be more accurately solved than results obtained by LSQR or TSVD [17, 22]

In 2015, Udayraj et al. [23] compared the efficiency and feasibility of three metaheuristic algorithms for a class of heat transfer problems. The result showed that, Ant colony optimization algorithm has the best performance for estimation of transient heat flux boundary condition, It followed by Particle swarm optimization and Cuckoo search algorithms. Sun et al. [24] applied krill herd (KH) algorithms for solving inverse geometry design problems. Based on reported result, the KH algorithm has better performance and efficiency for solving this kind of problem in comparison with micro genetic algorithm and particle swarm optimization algorithms.

In this research, the method of Tikhonov regularization is combined with the genetic algorithm to solve the inverse problem. A genetic algorithm is used to find the regularization parameter, which is the main problem of regularization methods.

II. Theory

As mentioned in the previous section, the definition of a well-posed problem was introduced by Jacques and Hadamard for the first time, in order to understand what kind of boundary conditions should be used for different types of differential equations [25]. Based on this definition, a well-posed mathematical problem has a unique solution which changes continuously with initial conditions. Therefore, if one of these conditions is not satisfied, the problem will be ill-posed. Stable numerical differentiation of noisy data, stable inversing of ill-posed matrices, parameter determination in a partial differential equation, first order homogenous differential equations are examples of ill-posed problems.

Consider the following ill-posed problem in which K is a linear bounded operator from X into Y

$$K\theta = W, K: X \to Y \quad (1)$$

Suppose that the right side is given with its approximation W_δ in such a way that $\|W - W_\delta\| \leq \delta$. Naturally, we need to find the approximate answer in the set $Q_\delta: \{\theta \in X: \|K\theta - W_\delta\| \leq \delta\}$. In any case, in an ill-posed problem, we cannot take an arbitrary element$x_\delta \in Q_\delta$ as an approximate solution for problem (1), because θ_δ does not change continuously as W_δ changes. Satisfying equation $\|K\theta - W_\delta\| \leq \delta$ does not guarantee that θ_δ is close to the desired response

Theorem 1: Suppose that, K is a bounded linear operator between Hilbert spaces X to Y [26] and then J_α has a unique minimum in $\theta_\delta \in X$, the minimum is the unique solution of the normal equation $\lambda\theta_\lambda + K^*K\theta_\lambda = K^*W$ In which, for all $x \epsilon X$, $J_\lambda(\theta) = \|K\theta_\lambda - W\|2_2 + \lambda\|\theta_\lambda\|_2^2$ is defined as Tikhonov's function. $R_\lambda: Y \to X, R_\lambda = (\lambda I + K^*K)^{-1}K^*$. It can be proved that the operators form a regularization strategy with $\lim_{\lambda\to 0}\|R_\lambda K\theta_e - \theta_e\| \le \lim_{\lambda\to 0}\frac{\|z\|\sqrt{\lambda}}{2}, \theta_e = K^*z \in K*(Y), z \in Y$. This method is called Tikhonov's regularization. After approximation, the result will be:

$$\|\theta_{\lambda,\delta} - \theta_e\| \le \frac{\delta}{2\sqrt{\lambda}} + \frac{\|z\|\sqrt{\lambda}}{2} := E(\lambda) \tag{2}$$

Theoretically, although $\|z\|$ is not known, we can minimize the $E(\lambda)$ function to find the optimal value for the $\|z\|$ regularization parameter, *e.g.,* in the posteriori method for choosing parameter λ which is called Morozov's discrepancy principle the value of is not required.

Choosing an appropriate regularization parameter is a critical part of achieving an optimal response. The most commonly used methods for selecting the regularization parameter are as follows.

A. Morozov's discrepancy principle

In this method, it is proposed that $\lambda(\delta) > 0$ be calculated in such a way that the Tikhonv solution which corresponds to the following equation

$$\lambda\theta_{\lambda,\delta} + K^*K\theta_{\lambda,\delta} = K^*W_\delta \tag{3}$$

which is the minimizer of the following functional

$$J_{\lambda,\delta}(\theta_{\lambda,\delta}) := \|K\theta_{\lambda,\delta} - W_\delta\|^2 + \lambda\|\theta_{\lambda,\delta}\|^2 \tag{4}$$

Satisfies

$$\|K\theta_{\lambda,\delta} - W_\delta\| = \delta \tag{5}$$

Therefore, λ choosing in this condition is sufficient to ensure that, on othe ne hand, the difference is equal to δ and, on the other hand, λ is not too small [23].

B. Quasi-Optimality criterion

The Quasi-Optimality criterion [27] determines the value $\lambda > 0$ in such a way that

$$\|K\theta_{\lambda,\delta} - W_\delta\| = \delta \tag{6}$$

To obtain the regularization parameter, Morozov's discrepancy principle and Quasi-Optimality criterion are used which the former requires the disturbance amplitude that the second does not require. We can use derivatives, or different numerical root finding can be used to optimize the objective functions of these two criterions, but doing this for any criterion requires separate calculations and derivation and root finding that complicates the work and raises computational costs, at the same time, there is no guarantee that the algorithm implemented converges. This paper presents a new high-precision meta-heuristic algorithm, which is easy to modify and doesn't produce complications when the objective function changes, which then is applied to a sample problem. Clearly, the solution in Tikhonov's regularization depends on the regularization parameter, which directly affects the degree of approximation and the stability of the solution. In terms of approximation, the smaller the λ is the better and $\|\theta_\lambda - \theta_e\|$ will have a smaller value in a stable solution; but from the stability point of view, the bigger the λ is the better. The key in solving Tikhonov's regularization method is to achieve optimal value for the regularization parameter.

The regularization parameter in Morozov's discrepancy principle is chosen in a way that:

$$\|K\theta_\lambda - W_\delta\|_2^2 = \delta^2 \tag{7}$$

On the other hand, in the Quasi-Optimality criterion, the optimal regularization parameter is the minimizer of the following objective function

$$\Lambda(\lambda) = \frac{1}{\lambda^2 {W_\delta}^T K(K^TK + \lambda I)^{-4}K^TW_\delta} \tag{8}$$

Unlike Morozov's discrepancy principle, in the Quasi-Optimality criterion, the magnitude of perturbations is not required. In this paper, a genetic algorithm is used to find the optimal regularization parameter. The aim is to optimize one of the two following objective functions for obtaining the optimal value of the regularization parameter.

$$\Gamma(\lambda) = |\|K\theta_\lambda - W_\delta\|_2^2 - \delta^2| \tag{9}$$

$$\Lambda(\lambda) = \frac{1}{\lambda^2 {W_\delta}^T K(K^TK + \lambda L)^{-4}K^TW_\delta} \tag{10}$$

Initially, a population of monogenic chromosomes which their gene value is λ is created, then by having the value of λ for each chromosome, $\alpha_\lambda$is calculated for each chromosome. Using θ_λ, the value of the objective functions $\Gamma(\lambda)$ and $\Lambda(\lambda)$ are calculated for each chromosome. Crossover and mutation operations are performed to create offspring and mutated chromosomes and the value of the objective functions are calculated for them. The chromosomes are ranked according to the value of their objective function then, the best chromosomes make up the second generation according to their rank, all of these operations are carried out again for the second generation. The genetic algorithm continues till the termination criterion is satisfied. After stopping the algorithm, the chromosome which has the lowest value of the cost function in the last generation is the solution and its gene is the optimal regularization parameter λ_{opt}. Figure.1 depicts the schematic of solving an inverse heat conduction problem using the proposed algorithm in this paper.

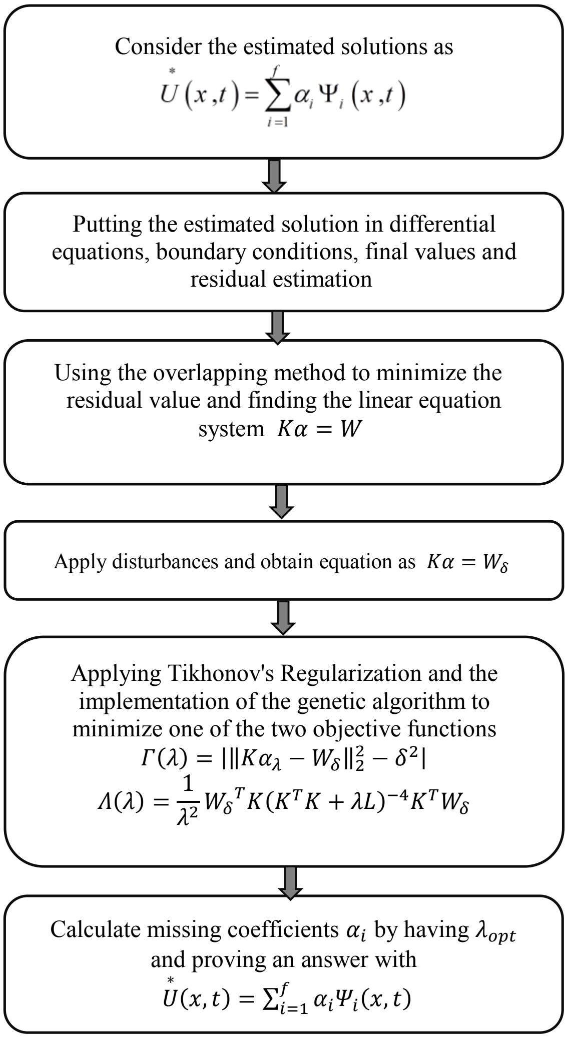

Fig. 1 Schematic of solving an inverse heat conduction problem using the algorithm provided in this paper

C. *Solving the sample heat conduction problem using the proposed algorithm*

Figure 2 illustrates an inverse heat conduction problem investigated in this paper. The energy equation, boundary and initial conditions for this problem is written as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \le x \le 1, t \ge 0 \\ u(0,t) = e^{-\pi^2 t}, t \ge 0 \\ u(1,t) = -e^{-\pi^2 t}, t \ge 0 \\ u(x,0) = cos(\pi x), 0 \le x \le L = 1 \end{cases} \qquad (11)$$

Analytical solution of this problem is $u(x,t) = cos(\pi x)e^{-\pi^2 t}, 0 \le x \le 1, t \ge 0$. For two reasons, this problem has been used to define the inverse problem; First, it has a relatively large coverage factor $e^{-\pi^2 t}$ which, by increasing the value of final time, *i.e.*, increasing the value of τ, makes solving the problem more difficult and the matrix of coefficients much more ill-conditioned and as time increases, and the response approaches to its steady state quickly. Second, this problem has been used as a standard example in research papers to investigate the accuracy and stability of the regularization algorithms. Using the exact solution of problem (11) in order to obtain the additional conditions, the inverse problem is defined as:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \le x \le 1, 0 \le t \le \tau \\ u(0,t) = f(t) = e^{-\pi^2 t}, 0 \le t \le \tau \\ u(1,t) = g(t) = -e^{-\pi^2 t}, 0 \le t \le \tau \\ u(x,\tau) = \Omega(t) = cos(\pi x)\, e^{-\pi^2 \tau}, 0 \le x \le 1 \end{cases} \qquad (12)$$

The approximate answer is formed using fundamental solutions as follows [28]

$$\overset{*}{U}(x,t) = \sum_{l=1}^{f} \alpha_l \Psi_l(x,t) = \sum_{l=1}^{f} \alpha_l \frac{H(t+t_0)}{2\sqrt{\pi(t+t_0)}} e^{\frac{(x-x_l)^2}{4(t+t_0)}} \qquad (13)$$
$$x_l = \left(\frac{l-1}{f-1}\right), l = 1, \ldots, f$$

t_0 is a parameter that is equal to the value of final time τ in our calculations. x_l will be uniformly distributed in [0 1]. To investigate the stability of the problem, the noise level entered into the additional condition is considered to be 0, 1,3 and 10 percent.

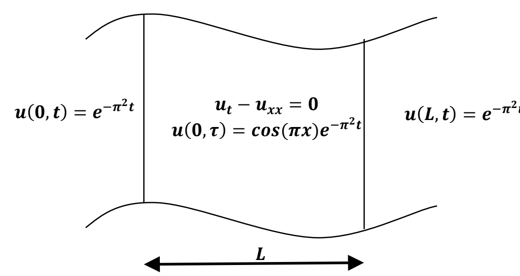

Fig. 2 Schematic of the IHCP examined in this paper

The collocation points are defined as

$$(x_i, t_i) = \begin{cases} \left(0, \left(\dfrac{e^{-s(i)} - 1}{e^{-1} - 1}\tau\right)\right), s(i) = \dfrac{i-1}{m-1}, i = 1, .., m \\ \left(1, \left(\dfrac{e^{-s(i)} - 1}{e^{-1} - 1}\tau\right)\right), s(i) = \dfrac{i-m-1}{m-1}, i = m+1, \ldots, 2m \\ \left(\dfrac{i-2m-1}{r-1}, \tau\right), i = 2m+1, \ldots, 2m+r \end{cases} \qquad (14)$$

The genetic algorithm is used to minimize objective functions Γ, Λ. The initial population consists of 25 chromosomes, each chromosome has only one gene, which is the value of regularization parameter λ. For the selection process, we use the roulette wheel, the maximum number of generations is 500, which is considered as the stopping algorithm criteria.

In order to investigate the accuracy of the approximate solution, the error is defined on the collocated points on the boundary at $t = 0$ as:

$$error(x_i) = \overset{*}{U}(x_i, 0) - u_{exc}(x_i, 0) \tag{15}$$

In which $u_{exc}(X_i, 0) = \cos(\pi X_i)$ the above equation shows the error distribution as a function of X. The average error in the entire domain is calculated as:

$$error(x_i) = \overset{*}{U}(x_i, 0) - u_{exc}(x_i, 0) \tag{16}$$

Where r is the number of collocation points on the boundary $t = 0$.

III. Result

Regularization is done for $m = 18$, $r = 18$ ($n = 54$ collocation points), 54 no. of trial functions ($f = 54$) and final time $\tau = 0.1$. The result of genetic calculations and convergence of objective functions are discussed in this paper. The performance of the Quasi-Optimality criterion and the Morozov's discrepancy principle are also compared at the desired time. The approximated solutions will be compared with each other and with the exact solution.

A. Results for different values of noise levels

The value of converged Morozov and Quasi- Optimality objective functions using a genetic algorithm for noise = 1% are respectively equal to $\Gamma(\lambda_{opt}) \approx O\ (10^{-16})$ and $\Lambda(\lambda_{opt}) \approx O\ (10^{-10})$, But this does not mean that the optimal regularization parameter of the Morozov's discrepancy principle is better, But in general, the objective function of the Quasi-Optimality criterion converges to larger amounts in respect to the Morozov's objective function. Despite better convergence patterns for Morozov's objective function in comparison with Quasi-Optimality criteria, it is clearly evident in Figure 3 that using regularization parameter λ_{opt} obtained from the quasi-optimal criterion, the approximate solution is generally closer to the exact one.

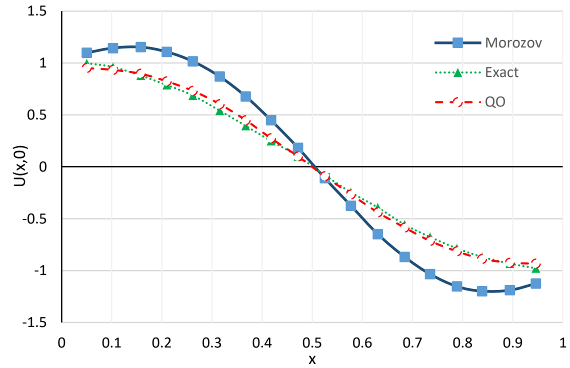

Fig. 3 Comparison of the approximate solution obtained using Quasi-Optimality and Morozov's regularization parameter with the exact solution for noise = 1% and $\tau = 0.1$ and $n = 54$ and $f = 54$

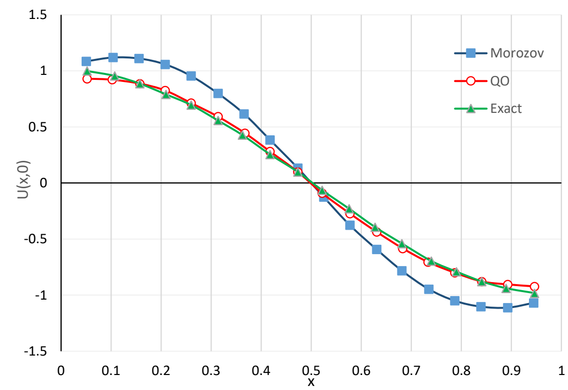

Fig. 4 Comparison of the approximate solution obtained using Quasi-Optimality and Morozov's regularization parameter with the exact solution for noise = 3% and $\tau = 0.1$ and $n = 54$ and $f = 54$

Same as disturbance level at noise = 1%; with an increase to 3%, the accuracy of the Quasi-Optimality criterion is much higher than the Morozov's discrepancy principle. In calculating Tikhonov's regularization coefficient using a genetic algorithm, increasing the population size will increase the required time it takes to calculate each generation, at the same time it reduces the number of generations needed to achieve optimal achievable value for regularization parameter. In various runs of the code, it can be seen that initial population growth did not affect improving the final value of the objective function. As a result, there was no need to increase the population size and examine its impact on noise levels.

The Quasi-optimality criterion results in less error and is a more precise method. The optimal parameters of the two methods differ significantly, as shown in the Table. 1. By increasing the noise level to 3%, the accuracy of the Quasi - optimality criterion is still much better than Morozov's discrepancy principle as shown in Table. 2.

The value of average error increases as the norm of noise increases. Interestingly, by the rise in disturbance norms, the average error of the Morozov method has decreased, although the Quasi -optimality criterion is more precise. Various methods for choosing the regularization parameter in different problems have different accuracies.

Table1. Comparison of the quasi-optimality criterion and the Morozov's discrepancy principle for $m = r = 18, f = 54$, noise $= 1\%$, and $\tau = 0.1$

Criterion	λ_{opt} in genetic code	Objective function in genetic code	ME, $\|\|Error\|\|$	δ
Morozov	9.32×10^{-9}	5.20×10^{-17}	0.2397, 1.1025	0.0645
quasi-optimality	0.0419	2.79×10^{-9}	0.0334, 0.1570	0.0645

Table2. Comparison of the quasi-optimality criterion and the Morozov's discrepancy principle for $m = r = 18, f = 54$, noise $= 3\%$, and $\tau = 0.1$

Criterion	λ_{opt} in genetic code	Objective function in genetic code	ME, $\|\|Error\|\|$	δ
Morozov	0.0021	0	0.18, 0.8276	0.1872
quasi-optimality	0.0301	3.06×10^{-9}	0.0271, 0.1346	0.1872

Naturally, increasing the level of disturbance in a continuous solution will increase the error in the output. But using Morozov's discrepancy principle by increasing the level of disturbances, the error value decreases in the most collocation points. In Quasi -optimality criterion, by increasing levels of disturbances, the error value increases subsequently and at the same time, remains at an acceptable level. Although this criterion does not require the extent of disturbance range or $\|W - W_\delta\|_2$, it has a great accuracy.

As shown in Figure. 5a, using Morozov's discrepancy principle to find the optimal regularization parameter, in the case of a disturbance of 10% the error value in most collocation points will be less than 1 or 3 percent cases. Interestingly, the error value is the highest in the case where the disturbance level is 1%. Additionally, the error value is not acceptable at any level of disturbance. In Figure. 5b it can be seen by increasing levels of disturbances, the error value increases subsequently and at the same time remains at an acceptable level. The use of Quasi -optimality criterion can also be more practical since it might not be possible to find the norm of error in the measured data used as an additional condition.

It can be seen in Figure. 6 that, even in the presence of 10% noise level, which is very high and in practice in inverse engineering, the measurement errors are much lower than this, the approximate solution follows exact solution accurately which demonstrates the successful implementation of our algorithm.

Figure 7 shows the approximate solution error on all collocation points. In general, the error in the internal points is less than the boundary collocation points.

B. Validation

The exact IHCP solved in this paper has not been solved in any paper. Therefore, to verify the method, the problem studied by Lesnic et al.[28] , which they used fundamental functions method using 60 collocation points and 20 guessed functions, has been solved using out algorithm. Lesnic et al. [28] assumed 5% disturbance level and the final time $\tau = 0.25$ and applied Tikhonov regularization technique and the L-curve method to their problem. This problem has been analysed again based on method presented here, the problem is defined as

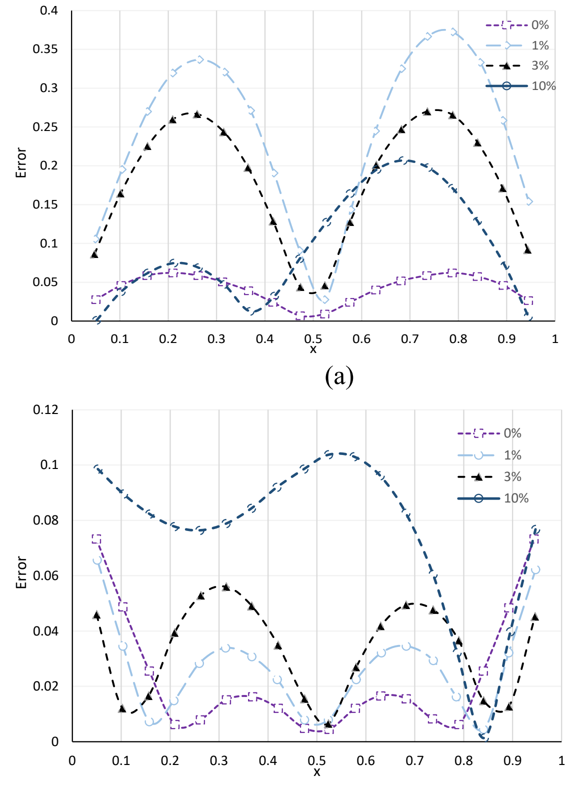

Fig. 5 The error in the collocation points on the boundary $t = 0$ for $n = 54, f = 54$, and at different noise levels, a) using the Quasi - optimality criterion b) using the Morozov's discrepancy principle.

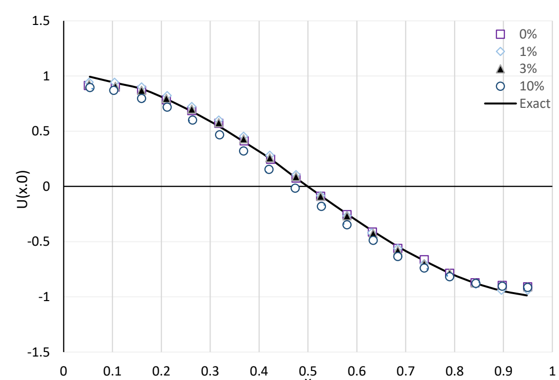

Fig. 6 Accurate and approximate solution on the collocation points on the boundary $t = 0$ for $n = 54, f = 54, \tau = 0.1$ at different noise levels using Quasi-optimality criterion

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \le x \le 1, 0 \le t \le \tau \\ u(0,t) = f(t) = 0, 0 \le t \le \tau \\ u(1,t) = g(t) = 0, 0 \le t \le \tau \\ u(x,\tau) = \Omega(t) = sin(\pi x)\, e^{-\pi^2 \tau}, 0 \le x \le 1 \end{cases} \tag{17}$$

The above problem was solved using the same parameters which were used by Lesnic. Figure 8 illustrates that the method used in this study has an accurate solution and is in proper compliance with the results of Lesnic et al. [28] and has a precise solution.

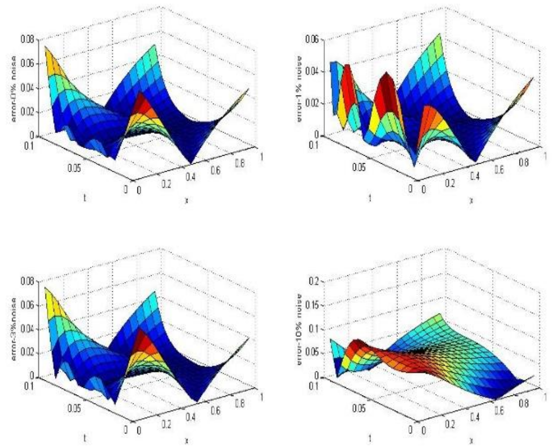

Fig. 7 Comparison between approximate solution error on all collocation points for $n = 54, f = 54, \tau = 0.1$ in different noise levels using quasi-optimally criterion

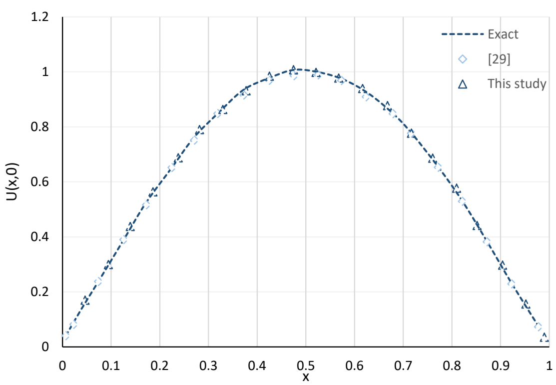

Fig. 8 Validation of the problem using quasi-optimally criterion assuming $\tau = 0.25, n = f = 60$ and noise = 5%.

IV. Conclusion

The primary purpose of this paper is to introduce an effective method for solving inverse problems in combination with Tikhonov's Regularization and genetic algorithms. Finding the optimal regularization parameter in Tikhonov regularization has been modeled to investigate the efficiency and accuracy of its application in solving sample IHCPs. Fundamental solutions have been used to guess estimate solution with constant unknowns' coefficients, and the collocation method is applied to minimize the residue on the collocation points.

The Morozov's discrepancy Principle and the Quasi-Optimality criterion are used to define the objective functions which minimizing them gives the optimal parameter. Results show that the parameters of the Genetic Algorithm (like mutation rate, crossover, operator, …) should be chosen appropriately according to the dynamic of the problem. Otherwise, the results will not be sufficiently precise. Crossover and mutation operators play the main role in minimizing and changing the selection operator did not have any practical effect on minimizing the objective function. By increasing the number of collocation or nodal points, the condition number of the matrix of coefficients increased, and it became severely ill-conditioned, however, if regularization applied successfully, the increase of nodal or collocated points results in less error in the estimated solution. The quasi-optimality criterion was more effective at smaller final times while Morozov's discrepancy principle was better at larger final times. The objective function of the Quasi-Optimality criterion minimized to lower values with respect to Morozov's objective function. Comparing the results of the proposed hybrid method presented in this paper with the analytical solution and the results of other researchers indicates the efficiency and accuracy of this method in solving inverse problems.

Nomenclature

$Error$	Absolute error
F	Number of fundamental functions (nodal points)
f(t)	Boundary condition at x=0
g(t)	Boundary condition at x=L
H(t)	Heaviside step function
I	Identity matrix
$J_\lambda(\theta)$	Tikhonov's functional
$J_{\lambda,\delta}(\theta)$	Tikhonov's functional using noisy data
K	Matrix of coefficients
K^*	Adjoint of K
L	Length of the space domain
M	Number of collocation points at x=0 and x=L
ME	Mean error
Q_δ	Solution set
R	Number of collocation points at $t = \tau$
T	Time variable
$u(x,t)$	Analytical solution of temperature distribution
$u_{exc}(x,t)$	Exact Temperature distribution
$U^*(x,t)$	Approximate temperature distribution
W	Right-hand side matrix (data)
W_δ	Right-hand side matrix (noisy data)
X	Spatial variable
X, Y	Hilbert spaces
X_δ	Arbitrary element in Q_δ
δ	Noise norm
θ	Solution matrix
θ_δ	Regularized solution
λ	Regularization parameter
λ_{opt}	Optimum regularization parameter
$\theta_{\delta,\lambda}$	Regularized solution using noisy data
τ	Final time
$\Lambda(\lambda)$	Quasi-optimality objective function
$\Gamma(\lambda)$	Morozov objective function
$\Psi_i(x,t)$	Fundamental functions
$\Omega(x)$	Additional condition at $t = \tau$

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