Nonlinear dynamic response of a fractionally damped suspension bridge subjected to small external force*

Yury A. Rossikhin and Marina V. Shitikova

Abstract – Nonlinear force driven coupled vertical and torsional vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance, are investigated. The method of multiple time scales is used as the method of solution. The damping features are described by the fractional derivative, which is interpreted as the fractional power of the differentiation operator. The influence of the fractional parameters (orders of fractional derivatives) on the motion of the suspension bridge model is investigated.

Keywords – Suspension bridge, nonlinear force driven vibrations, fractional damping, method of multiple time scales.

I. Introduction

The experimental data obtained in [1] during ambient vibration studies of the Golden Gate Bridge show that different vibrational modes feature different amplitude damping factors, and the order of smallness of these coefficients tells about low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another. Besides, as natural frequencies of vibrations increase, the corresponding damping ratios decrease.

Nonlinear free damped vibrations of suspension bridges in the cases of the one-to-one internal resonance (when the natural frequency of a certain mode of vertical vibrations is close to the natural frequency of a certain mode of torsional vibrations) and the two-to-one internal resonance (when one natural frequency is nearly twice as large as another natural frequency) have been examined in [2] when damping features of the system are prescribed by the first derivative of the displacement with respect to time. In this study, the equations of motion due to Abdel-Ghaffar and Rubin [3] were modified by adding damping terms, and then solved by the method of multiple scales. It has been shown that for the both types of the internal resonance the damping coefficient does not depend on the natural frequency of vibrations, but this result is in conflict with the experimental data presented in [1].

To lead the theoretical investigations in line with the experiment, fractional derivatives were introduced in [4] for describing the processes of internal friction proceeding in suspension combined systems at nonlinear free vibrations. The nonlinear suspension bridge model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations. The model suggested in [4] has been generalized in [5] by using two different fractional parameters for analyzing vertical and torsional modes of nonlinear damped vibrations of suspension bridges.

In the present paper, the model described in [5] is used for investigating nonlinear forced vibrations of suspension bridges for the case when only two interacting modes predominate in the vibrational process, i.e., when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance. The influence of the fractional parameters (orders of the fractional derivatives) on the motion of the suspension bridge model is investigated.

Before we proceed to the detail analysis of the given problem, it should be emphasized following Lacarbonara [6] that "within the framework of analytical techniques, non-linear vibrations of continuous (distributed-parameter) systems can be studied either by attacking directly the original partial-differential equations and boundary conditions with a reduction method (e.g., the method of multiple scales) or by discretizing the system, first, and, then, by constructing, via a reduction method, approximations of the obtained reduced-order systems. With the first approach (direct treatment), the reduction procedure acts on the temporal dependence of the system without any a priori assumption of the form of the solution. With space discretization, the spatial condensation, also referred to as system order reduction, achieved by means of one of the many versions of the method of weighted residuals, is a crucial step. It is a common practice to project via the standard or "flat" Galerkin procedure the original infinite-dimensional non- linear system on to a basis forming a complete set

*Some of the results had been presented at the 7th WSEAS International Conference on Continuum Mechanics (CM'12), Kos Island, Greece, July 14–17, 2012. This research was supported by the joint Grant from the Russian Foundation for Basic Research No. 10-01-92004-HHC-a and the National Science Council of Taiwan No. RP10E02.

of functions usually consisting of the eigenfunctions of the associated linearized system when the boundary conditions are homogeneous. Then, truncation to a finite number of basis functions generates classical low-order models."

After the publication of this excellent work [6], wherein it has been shown "that the approximate solutions obtained with the direct treatment, the full-basis Galerkin discretization, and the rectified Galerkin procedure are equivalent either in the case of no internal resonances or in the case of a sub-harmonically excited two-to-one internal resonance", there was a discussion of these approaches between the author of [6] and [7] and the supporter of the direct perturbation technique [8], who wrote previously that "discretizing the nonlinear partial differential equations of vibration problems and then solving the resulting ordinary differential equations by perturbation techniques is a quite common method, the discretization process simplifies the equations much and the ordinary differential equations are easier to handle. However *the process may lead to inaccurate results in the case of quadratic and cubic nonlinearities*" [9].

In order to show that the approximate displacement fields obtained with the three methods are equivalent, Lacarbonara [6] has shown that "the complex-valued amplitude of the only excited mode or the amplitudes of the interacting modes are the same regardless of the method employed and the higher-order spatial corrections due to the non-linearities are the same". The authors of this paper agree completely with the opinion and the results of Lacarbonara [6], [7].

In the given paper it will be shown that the application of the method of multiple time scales directly to the nonlinear partial integro-differential equations of suspension bridge involving conventional damping and vertical force term, as it was carried out in [10], has not clarify so many advantages as it was claimed by the authors of [10]. The main drawback of the model used in [10] is the employment of the ordinary first-order time-derivatives of displacements for evaluating the damping features of the suspension bridges. The authors have not even mentioned that this approach for damping inclusion had been previously implemented in [2]. Moreover, application of the method of multiple time scales directly to the partial integro-differential equations of suspension bridges results in solving separately differential equations for defining the symmetric and asymmetric mode shapes of vertical and torsional vibrations on each level of solution approximation, what is inconvenient in engineering applications. The solution for eigenfunctions has been carried out only for the first order of approximation, while for the next level of approximation the authors of [10] wrote the following: "However, the transcendental equations yielding the solution functions for the symmetric modes at this order are very complicated. Therefore we solved the functions at this order only for the asymmetric modes." It has been also claimed in this paper that the coefficients arised in the solvability conditions depend "on the converged mode functions and hence more accurate numerical values can be obtained compared to the discretization perturbation technique" [10].

Commenting this, we could follow Lacarbonara [6] referring to the paper by Pakdemirli and Boyaci [11]: "However, in their analysis, one of the fundamental results was postulated instead of proved", since this could be concerned with the paper [10] as well.

II. Problem Formulation

To analyze the forced damped vibrations of suspension bridges we will use its classical scheme involving a bisymmetrical thin-walled stiffening girder connected with two suspended cables by virtue of vertical suspensions [5]. The cables are thrown over the pilons and are tensioned by anchor mechanisms. The suspensions are considered as inextensible and uniformly distributed along the stiffening girder. The cables are parabolic, and the contour of the girder's cross-section is underformable. It is assumed that the girder's contour translates as a rigid body vertically (in the y-axis direction) on the value of $\eta(z,t)$ and rotates with respect to the girder's axis (the z-axis) through the angle of $\varphi(z,t)$ (Fig. 1). The origin of the frame of references is in the center of gravity of the cross section.

It is known for suspension bridges [3], [12] that some natural modes belonging to different types of vibrations could be coupled with each other, i.e., the excitation of one natural mode gives rise to another one. Two modes interact more often that not, although the possibility for interaction of a greater number of modes is not ruled out.

Below we consider the case when only two modes predominate in the vibrational process, namely: the vertical n-th mode with linear natural frequency ω_{0n}, and the torsional m-th mode with the natural frequency Ω_{0m}. Under such an assumption the functions $\eta(z,t)$ and $\varphi(z,t)$ could be approximately defined as (using the eigenbase of the associated linear undamped unforced problem)

$$\eta(z,t) \sim v_n(z)x_{1n}(t), \quad \varphi(z,t) \sim \Theta_m(z)x_{2m}(t), \tag{1}$$

where $x_{1n}(t)$ and $x_{2m}(t)$ are the generalized displacements, and $v_n(z)$ and $\Theta_m(z)$ are natural shapes of the two interacting modes of vibrations.

When the harmonic force $F = \hat{F}\cos(\omega_F t)$ is applied at the center of the suspension bridge, then the equations of its forced vibrations are written in the dimensionless form as (what is the immediate generalization of the approach proposed in [5] by adding the external vertical excitation with amplitude $\hat{F} = \mathrm{const}$ and frequency ω_F)

$$\begin{aligned}\ddot{x}_{1n} + \omega_{0n}^2 x_{1n} + \beta D_{0+}^{\gamma_1} x_{1n} + a_{11}^n x_{1n}^2 + a_{22}^{nm} x_{2m}^2 \\ +(b_{11}^n x_{1n}^2 + b_{22}^{nm} x_{2m}^2)x_{1n} = \hat{F}\cos(\omega_F t),\end{aligned} \tag{2}$$

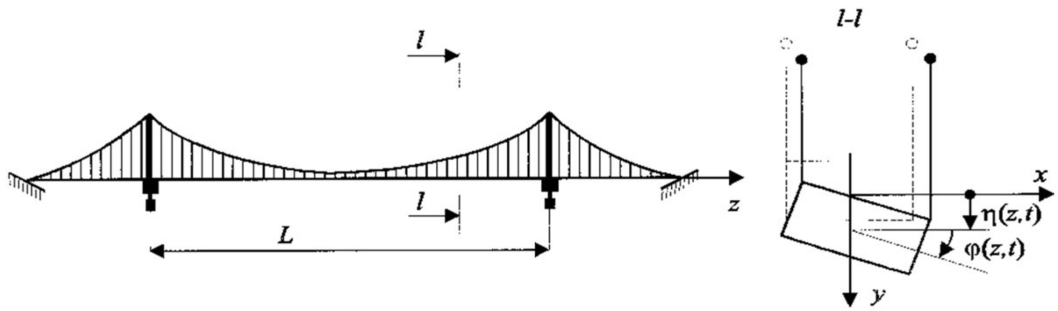

Fig.1 Scheme of a suspension bridge

$$\ddot{x}_{2m} + \Omega_{0m}^2 x_{2m} + \beta D_{0+}^{\gamma_2} x_{2m} + a_{12}^{nm} x_{1n} x_{2m} + (c_{11}^{nm} x_{1n}^2 + c_{22}^{m} x_{2m}^2) x_{2m} = 0,$$

where a_{ij}, b_{ii}, and c_{ii} ($i = 1, 2$, $j = 2$) are certain dimensionless coefficients which are defined in [3] (subsequently the indices n and m are omitted for ease of presentation), dots denote differentiation with respect to time, the terms $\beta D_{0+}^{\gamma_1} x_1$ and $\beta D_{0+}^{\gamma_2} x_2$ characterize inelastic reaction of the system, β is the viscosity coefficient, the fractional derivative $D_{0+}^{\gamma} x$ ($\gamma = \gamma_1$ or γ_2) is defined as follows [13]

$$D_{0+}^{\gamma} x = \frac{d}{dt} \int_0^t \frac{x(t - t')dt'}{\Gamma(1-\gamma) t'^{\gamma}} \quad (0 < \gamma \le 1), \tag{3}$$

γ is the order of the fractional derivative (fractional parameter), and $\Gamma(1 - \gamma)$ is the Gamma-function.

Let us consider the case of the one-to-one internal resonance, as well as suppose that the frequency of the external force is close to the natural frequency of the interacting modes, i.e.,

$$\omega_0 \approx \Omega_0 \approx \omega_F. \tag{4}$$

Note that the influence of the detuning parameter characterizing the small difference in magnitudes of the natural frequencies ω_0 and Ω_0 has been investigated in [4] and [12].

Since for finding the solution of Eqs. (2) we will use the method of multiple time scales, where the functions $e^{\pm i\omega t}$ are utilized as the main harmonic functions, then in order to carry out the calculations the following formulas will be in demand [14]

$$D_{0+}^{\gamma} e^{\pm i\omega t} = D_{+}^{\gamma} e^{\pm i\omega t} + \frac{\sin \pi\gamma}{\pi} \int_0^{\infty} \frac{u^{\gamma} e^{-ut} du}{u \pm i\omega}, \tag{5}$$

$$D_{+}^{\gamma} e^{\pm i\omega t} = (\pm i\omega)^{\gamma} e^{\pm i\omega t}, \tag{6}$$

where D_{+}^{γ} is obtained from (3) changing the low limit to $-\infty$.

It has been shown in [15] and [16] that the second term in formula (5) does not produce secular terms in the method of multiple time scales under the limitation of the zero- and first-order approximations. In other words, this term could be neglected in further consideration, and it is possible to use the approximate formula

$$D_{0+}^{\gamma} e^{\pm i\omega t} \approx D_{+}^{\gamma} e^{\pm i\omega t}. \tag{7}$$

If we take into account formula (5.82) from [13]

$$D_{+}^{\gamma} e^{\pm i\omega t} = \left(\frac{d}{dt}\right)^{\gamma} e^{\pm i\omega t}, \tag{8}$$

then from the combination of (7) and (8) it follows the relationship

$$D_{0+}^{\gamma} e^{\pm i\omega t} \approx \left(\frac{d}{dt}\right)^{\gamma} e^{\pm i\omega t}, \tag{9}$$

which will be used in further calculations.

III. Method of Solution

We will seek the solution for two cases: (1) $\beta = \varepsilon\mu$ and that $\hat{F} = \varepsilon^2 f$, and (2) $\beta = \varepsilon^2\mu$ and that $\hat{F} = \varepsilon^3 f$. In these cases, an approximate solution of equations (2) for small amplitudes weakly varying with time can be represented by an expansion in terms of different time scales

$$\begin{aligned} x_1(t) &= \varepsilon x_{11}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{12}(T_0, T_1, T_2, \ldots) \\ &\quad + \varepsilon^3 x_{13}(T_0, T_1, T_2, \ldots) + \ldots \\ x_2(t) &= \varepsilon x_{21}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{22}(T_0, T_1, T_2, \ldots) \\ &\quad + \varepsilon^3 x_{23}(T_0, T_1, T_2, \ldots) + \ldots \end{aligned} \tag{10}$$

where $T_n = \varepsilon^n t$ ($n = 0, 1, 2, \ldots$) are new independent variables, ε is a small parameter which is of the same order of magnitude as the amplitudes, and μ and f are finite values. Here, $T_0 = t$ is a fast scale, characterizing motions with the natural frequencies ω_0 and Ω_0, while $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow scales, characterizing the modulations of the amplitudes and phases.

Considering that

$$d/dt = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \tag{11a}$$

$$d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \tag{11b}$$

as well as following [4] that

$$(d/dt)^\gamma = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots)^\gamma \tag{12}$$

$$= D_+^\gamma + \varepsilon\gamma D_+^{\gamma-1} D_1 + \frac{1}{2}\,\varepsilon^2\gamma(\gamma-1) D_+^{\gamma-2} D_1^2 + \ldots$$

where $D_n = \partial/\partial T_n$,

$$D_+^{\gamma-n} x = \frac{d}{dt}\int_{-\infty}^{t} \frac{x(t-t')dt'}{\Gamma(1-\gamma+n)t'^{\gamma-n}} \quad (n = 0, 1, 2, ...)$$

substituting (10) into (2), and equating the coefficients at like powers of ε to zero, we obtain to order ε:

$$D_0^2 x_{11} + \omega_0^2 x_{11} = 0, \tag{13a}$$

$$D_0^2 x_{21} + \Omega_0^2 x_{21} = 0; \tag{13b}$$

to order ε^2:

$$D_0^2 x_{12} + \omega_0^2 x_{12} = -2D_0 D_1 x_{11} - \mu(2-k) D_+^{\gamma_1} x_{11}$$
$$-a_{11} x_{11}^2 - a_{22} x_{21}^2 + (2-k) f \cos(\omega_0 T_0), \tag{14a}$$

$$D_0^2 x_{22} + \Omega_0^2 x_{22} = -2D_0 D_1 x_{21} - \mu(2-k) D_+^{\gamma_2} x_{21}$$
$$-a_{12} x_{11} x_{21}; \tag{14b}$$

to order ε^3:

$$D_0^2 x_{13} + \omega_0^2 x_{13} = -2D_0 D_1 x_{12} - (D_1^2 + 2D_0 D_2) x_{11}$$
$$-\mu(2-k) D_+^{\gamma_1} x_{12} - \mu(2-k)\gamma_1 D_+^{\gamma_1 - 1} D_1 x_{11}$$
$$-\mu(k-1) D_+^{\gamma_1} x_{11} - 2a_{11} x_{11} x_{12} - 2a_{22} x_{21} x_{22}$$
$$-b_{11} x_{11}^3 - b_{22} x_{21}^2 x_{11} + (k-1) f \cos(\omega_0 T_0), \tag{15a}$$

$$D_0^2 x_{23} + \Omega_0^2 x_{23} = -2D_0 D_1 x_{22} - (D_1^2 + 2D_0 D_2) x_{21}$$
$$-\mu(2-k) D_+^{\gamma_2} x_{22} - \mu(2-k)\gamma_2 D_+^{\gamma_2 - 1} D_1 x_{21}$$
$$-\mu(k-1) D_+^{\gamma_2} x_{21} - a_{12}(x_{11} x_{22} + x_{12} x_{21})$$
$$-c_{11} x_{11}^2 x_{21} - c_{22} x_{21}^3. \tag{15b}$$

At $k = 1$ and $k = 2$, we obtain governing equations for the first and second cases, respectively.

Integrating equations (13a) and (13b) yields

$$x_{11} = A_1(T_1, T_2) e^{i\omega_0 T_0} + \bar{A}_1(T_1, T_2) e^{-i\omega_0 T_0}, \tag{16a}$$

$$x_{21} = A_2(T_1, T_2) e^{i\Omega_0 T_0} + \bar{A}_2(T_1, T_2) e^{-i\Omega_0 T_0}, \tag{16b}$$

where A_1 and A_2 are unknown complex functions, and $\bar{A}_1$ and $\bar{A}_2$ are the complex conjugates of A_1 and A_2, respectively.

In order to integrate the sets of equations (14) and (15), it is necessary to consider each case separately.

A. The case $k = 1$

Now let us substitute (16) into the right-hand sides of equations (14) putting there $k = 1$, then gather all terms standing at $e^{i\omega_0 T_0}$ and $e^{-i\omega_0 T_0}$ with due account for (4) and vanish them in order to exclude secular terms. As a result we obtain [17]

$$D_1 A_1 + \frac{1}{2}\,\mu(i\omega_0)^{\gamma_1 - 1} A_1 - \frac{f}{4i\omega_0} = 0, \tag{17a}$$

$$D_1 A_2 + \frac{1}{2}\,\mu(i\omega_0)^{\gamma_2 - 1} A_2 = 0, \tag{17b}$$

$$D_0^2 x_{12} + \omega_0^2 x_{12} = -\left(a_{11} A_1^2 + a_{22} A_2^2\right) e^{2i\omega_0 T_0}$$
$$-a_{11} A_1 \bar{A}_1 - a_{22} A_2 \bar{A}_2 + cc, \tag{18a}$$

$$D_0^2 x_{22} + \omega_0^2 x_{22} = -a_{12} A_1 A_2 e^{2i\omega_0 T_0}$$
$$-a_{12} A_1 \bar{A}_2 + cc, \tag{18b}$$

where cc is the complex conjugate part to the preceding terms.

Integrating Eqs. (17), we find

$$A_1(T_1, T_2) = a_1(T_2) \exp\left[-\frac{1}{2}\,\mu(i\omega_0)^{\gamma_1 - 1} T_1\right]$$
$$+\frac{f}{2\mu(i\omega_0)^{\gamma_1}}, \tag{19a}$$

$$A_2(T_1, T_2) = a_2(T_2) \exp\left[-\frac{1}{2}\,\mu(i\omega_0)^{\gamma_2 - 1} T_1\right]. \tag{19b}$$

Substituting (19) in Eqs. (18) and integrating, we obtain the expressions for x_{12} and x_{22}. Then substituting found x_{12} and x_{22} in Eqs. (15a) and (15b) and using the standard procedure for eliminating the secular terms, we have

$$D_2 a_1 + \left[\frac{1}{8}\,\mu^2(i\omega_0)^{2\gamma_1 - 3}(1 - 2\gamma_1)\right.$$
$$\left. + \frac{1}{4}\,i\,\frac{f^2(a_{11}^2 - 3b_{11})}{\mu^2\omega_0^{2\gamma_1 + 1}}\,e^{-2\pi i\gamma_1}\right] a_1 = 0, \tag{20a}$$

$$D_2 a_2 + \left[\frac{1}{8}\,\mu^2(i\omega_0)^{2\gamma_2 - 3}(1 - 2\gamma_2)\right.$$
$$\left. + \frac{1}{4}\,i\,\frac{f^2(a_{11}a_{12} - 2c_{11} - \frac{1}{3}\,a_{12}^2\omega_0^{-2})}{\mu^2\omega_0^{2\gamma_1 + 1}}\,e^{-2\pi i\gamma_1}\right] a_2 = 0. \tag{20b}$$

Integrating Eqs. (20) yields

$$a_1 = a_1^0 \exp\left\{T_2\left[-\frac{1}{8}\,\mu^2(1 - 2\gamma_1)(i\omega_0)^{2\gamma_1 - 3}\right.\right.$$
$$\left.\left. -\frac{1}{4}\,\frac{f^2(a_{11}^2 - 3b_{11})}{\mu^2\omega_0^{2\gamma_1 + 1}}\,(i\cos 2\pi\gamma_1 + \sin 2\pi\gamma_1)\right]\right\}, \tag{21a}$$

$$a_2 = a_2^0 \exp\left\{T_2\left[-\frac{1}{8}\,\mu^2(1-2\gamma_2)(i\omega_0)^{2\gamma_2-3} -\frac{1}{4}\,\frac{f^2(a_{11}a_{12}-2c_{11}-\frac{1}{3}\,a_{12}^2\omega_0^{-2})}{\mu^2\omega_0^{2\gamma_1+1}} \times\left(i\cos 2\pi\gamma_1+\sin 2\pi\gamma_1\right)\right]\right\}, \qquad (21b)$$

where a_1^0 and a_2^0 are arbitrary constants.

Considering formulas (10), (16), (19), and (21), we finally obtain

$$x_1 = \varepsilon\left[2a_1^0 e^{-\alpha_1 t}\cos\Omega_1 t +\frac{f}{\mu\omega_0^{\gamma_1}}\,\cos\left(\omega_0 t-\frac{\pi}{2}\,\gamma_1\right)\right]+O(\varepsilon^2), \qquad (22a)$$

$$x_2 = \varepsilon 2a_2^0 e^{-\alpha_2 t}\cos\Omega_2 t+O(\varepsilon^2), \qquad (22b)$$

where

$$\alpha_1 = \frac{1}{2}\,\varepsilon\mu\omega_0^{\gamma_1-1}\sin\left(\frac{\pi\gamma_1}{2}\right) \times\left[1+\frac{1}{2}\,\varepsilon\mu(2\gamma_1-1)\omega_0^{\gamma_1-2}\cos\left(\frac{\pi\gamma_1}{2}\right)\right] -\frac{1}{4}\,\varepsilon^2\,\frac{f^2(a_{11}^2-3b_{11})}{\mu^2\omega_0^{2\gamma_1+1}}\,\sin(2\pi\gamma_1),$$

$$\Omega_1 = \omega_0\left[1+\frac{1}{2}\,\varepsilon\mu\omega_0^{\gamma_1-2}\cos\left(\frac{\pi\gamma_1}{2}\right) +\frac{1}{8}\,\varepsilon^2\mu^2(2\gamma_1-1)\omega_0^{2(\gamma_1-2)}\cos\left(\pi\gamma_1\right) -\frac{1}{4}\,\varepsilon^2\,\frac{f^2(a_{11}^2-3b_{11})}{\mu^2\omega_0^{2(\gamma_1+1)}}\,\cos(2\pi\gamma_1)\right],$$

$$\alpha_2 = \frac{1}{2}\,\varepsilon\mu\omega_0^{\gamma_2-1}\sin\left(\frac{\pi\gamma_2}{2}\right) \times\left[1+\frac{1}{2}\,\varepsilon\mu(2\gamma_2-1)\omega_0^{\gamma_2-2}\cos\left(\frac{\pi\gamma_2}{2}\right)\right] -\frac{1}{4}\,\varepsilon^2\,\frac{f^2(a_{11}a_{12}-2c_{11}-\frac{1}{3}\,a_{12}^2\omega_0^{-2})}{\mu^2\omega_0^{2\gamma_1+1}}\,\sin(2\pi\gamma_1),$$

$$\Omega_2 = \omega_0\left[1+\frac{1}{2}\,\varepsilon\mu\omega_0^{\gamma_2-2}\cos\left(\frac{\pi\gamma_2}{2}\right) +\frac{1}{8}\,\varepsilon^2\mu^2(2\gamma_2-1)\omega_0^{2(\gamma_2-2)}\cos\left(\pi\gamma_2\right) -\frac{1}{4}\,\varepsilon^2\,\frac{f^2(a_{11}a_{12}-2c_{11}-\frac{1}{3}\,a_{12}^2\omega_0^{-2})}{\mu^2\omega_0^{2(\gamma_1+1)}}\,\cos(2\pi\gamma_1)\right].$$

Reference to the found analytical solution (22) shows that it involves two parts: the first corresponds to the damping vibrations with damping coefficients and nonlinear frequencies dependent on the fractional parameters and describes the transient process, while the second one is nondamping in character and describes forced vibrations with the frequency of the exciting force and with the phase difference depending on the fractional parameter.

B. The case $k = 2$

Let us substitute relations (16) in the right-hand parts of equations (14) at $k = 2$. Eliminating secular terms and integrating the equations obtained, we have

$$D_1A_1 = D_1A_2 = 0, \qquad (23)$$

$$x_{12} = \frac{1}{3\omega_0^2}\,A_1^2e^{2i\omega_0T_0}+\frac{a_{22}}{3\omega_0^2}\,A_2^2e^{2i\omega_0T_0} -\left(a_{11}A_1\bar{A}_1+a_{22}A_2\bar{A}_2\right)\omega_0^2+cc, \qquad (24a)$$

$$x_{22} = \frac{a_{12}}{3\omega_0^2}\,A_1A_2e^{2i\omega_0T_0}-\frac{a_{12}}{\omega_0^2}\,A_1\bar{A}_2+cc. \qquad (24b)$$

From (23) it follows that the functions A_1 and A_2 are T_1-independent.

Substituting then (16) and (24) in equations (15) and utilizing the standard procedure for eliminating secular terms, we obtain

$$-iD_2A_1-\frac{1}{2}\,\mu\omega_0^{-1}(i\omega_0)^{\gamma_1}A_1-\lambda_1A_1^2\bar{A}_1-\lambda_2A_1A_2\bar{A}_2 +\frac{1}{4}\,\Gamma_1\bar{A}_1A_2^2+\frac{1}{4}\,\frac{f}{\omega_0}=0, \qquad (25a)$$

$$-iD_2A_2-\frac{1}{2}\,\mu\omega_0^{-1}(i\omega_0)^{\gamma_2}A_2-\lambda_3A_1\bar{A}_1A_2-\lambda_4A_2^2\bar{A}_2 +\frac{1}{4}\,\Gamma_2A_1^2\bar{A}_2=0, \qquad (25b)$$

where the coefficients λ_i and Γ_j ($i = 1,2,3,4$ and $j = 1,2$) are presented in [3], [12].

Now we multiply (25a) and (25b) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. Adding every pair of the mutually adjoint equations and subtracting one from another, and after all manipulations representing the functions A_1 and A_2 in their polar form, i.e.,

$$A_1(T_2) = a_1(T_2)\exp\left[i\varphi_1(T_2)\right],$$

$$A_2(T_2) = a_2(T_2)\exp\left[i\varphi_2(T_2)\right],$$

as a result we obtain the modulation equations

$$\dot{a}_1+\frac{1}{2}\,s_1a_1-\frac{1}{4}\,\Gamma_1a_1a_2^2\sin\delta+\frac{1}{4}\,f\omega_0^{-1}\sin\varphi_1=0, \qquad (26a)$$

$$\dot{a}_2+\frac{1}{2}\,s_2a_2+\frac{1}{4}\,\Gamma_2a_1^2a_2\sin\delta=0, \qquad (26b)$$

$$\dot{\varphi}_1-\frac{1}{2}\,\psi_1-\lambda_1a_1^2-\lambda_2a_2^2+\frac{1}{4}\,\Gamma_1a_2^2\cos\delta +\frac{1}{4}\,f\omega_0^{-1}a_1^{-1}\cos\varphi_1=0, \qquad (26c)$$

$$\dot{\varphi}_2-\frac{1}{2}\,\psi_2-\lambda_3a_1^2-\lambda_4a_2^2+\frac{1}{4}\,\Gamma_2a_1^2\cos\delta=0, \qquad (26d)$$

where $\delta = 2(\varphi_2-\varphi_1)$, a dot denotes differentiation with respect to T_2, and

$$\psi_1 = \mu\omega_0^{\gamma_1-1}\cos\left(\frac{1}{2}\,\pi\gamma_1\right),\ \psi_2 = \mu\omega_0^{\gamma_2-1}\cos\left(\frac{1}{2}\,\pi\gamma_2\right),$$

$$s_1 = \mu\omega_0^{\gamma_1 - 1} \sin\left(\frac{1}{2}\,\pi\gamma_1\right), \; s_2 = \mu\omega_0^{\gamma_2 - 1} \sin\left(\frac{1}{2}\,\pi\gamma_2\right).$$

Note that when $\gamma_1 = \gamma_2 = 1$, the solvability conditions (25) and modulation equations (26) coincide with those presented in [10] within an accuracy of notations and coefficients, if one puts detuning parameters equal to zero in solvability conditions (28) and modulation equations (31)-(33) of [10], which were solved numerically.

The system of equations (26) could be rewritten in another form if we suppose that the values a_1 and a_2 are the functions in φ_1 and φ_2, i.e.,

$$a_1 = a_1\left[\varphi_1(T_2), \varphi_2(T_2)\right], \quad a_2 = a_2\left[\varphi_1(T_2), \varphi_2(T_2)\right]. \tag{27}$$

Differentiating (27) with respect to T_2 yields

$$\dot{a}_1 = \frac{\partial a_1}{\partial \varphi_1}\,\dot{\varphi}_1 + \frac{\partial a_1}{\partial \varphi_2}\,\dot{\varphi}_2, \tag{28a}$$

$$\dot{a}_2 = \frac{\partial a_2}{\partial \varphi_1}\,\dot{\varphi}_1 + \frac{\partial a_2}{\partial \varphi_2}\,\dot{\varphi}_2. \tag{28b}$$

Using relationships (26a)-(26d) for $\dot{a}_1$, $\dot{a}_2$, $\dot{\varphi}_1$, and $\dot{\varphi}_2$, let us rewrite equations (28) as

$$\frac{\partial a_1}{\partial \varphi_1}\,B_1(a_1, a_2, \varphi_1, \varphi_2) + \frac{\partial a_1}{\partial \varphi_2}\,B_2(a_1, a_2, \varphi_1, \varphi_2) = f_1(a_1, a_2, \varphi_1, \varphi_2), \tag{29a}$$

$$\frac{\partial a_2}{\partial \varphi_1}\,B_1(a_1, a_2, \varphi_1, \varphi_2) + \frac{\partial a_2}{\partial \varphi_2}\,B_2(a_1, a_2, \varphi_1, \varphi_2) = f_2(a_1, a_2, \varphi_1, \varphi_2), \tag{29b}$$

where

$$B_1(a_1, a_2, \varphi_1, \varphi_2) = \frac{1}{2}\,\psi_1 + \lambda_1 a_1^2 + \lambda_2 a_2^2 - \frac{1}{4}\,\Gamma_1 a_2^2 \cos\delta - \frac{1}{4}\,f\omega_0^{-1} a_1^{-1} \cos\varphi_1, \tag{29c}$$

$$B_2(a_1, a_2, \varphi_1, \varphi_2) = \frac{1}{2}\,\psi_2 + \lambda_3 a_1^2 + \lambda_4 a_2^2 - \frac{1}{4}\,\Gamma_2 a_1^2 \cos\delta, \tag{29d}$$

$$f_1(a_1, a_2, \varphi_1, \varphi_2) = -\frac{1}{2}\,s_1 a_1 + \frac{1}{4}\,\Gamma_1 a_1 a_2^2 \sin\delta - \frac{1}{4}\,f\omega_0^{-1} \sin\varphi_1, \tag{29e}$$

$$f_2(a_1, a_2, \varphi_1, \varphi_2) = -\frac{1}{2}\,s_2 a_2 - \frac{1}{4}\,\Gamma_2 a_1^2 a_2 \sin\delta. \tag{29f}$$

The characteristics of equations (29a) and (29b) have the form

$$B_1 d\varphi_2 - B_2 d\varphi_1 = 0, \tag{30}$$

while equations along the characteristics are written as

$$f_1 d\varphi_2 - B_2 da_1 = 0, \tag{31a}$$

or, what is the same thing,

$$B_1 da_1 - f_1 d\varphi_1 = 0, \tag{31b}$$

and

$$f_2 d\varphi_2 - B_2 da_2 = 0, \tag{32a}$$

or, what is the same thing,

$$B_1 da_2 - f_2 d\varphi_1 = 0. \tag{32b}$$

From relationships (30), (31b), and (32b) we find

$$\frac{d\varphi_2}{d\varphi_1} = \frac{B_2}{B_1}, \tag{33a}$$

$$\frac{da_1}{d\varphi_1} = \frac{f_1}{B_1}, \tag{33b}$$

$$\frac{da_1}{d\varphi_2} = \frac{f_1}{B_2}, \tag{33c}$$

$$\frac{da_2}{d\varphi_1} = \frac{f_2}{B_1}, \tag{33d}$$

$$\frac{da_2}{d\varphi_2} = \frac{f_2}{B_2}. \tag{33e}$$

From equations (33b), (33d), and (33c), (33e) it follows that

$$\frac{da_1^2}{d\varphi_1} = \frac{2}{B_1}\,f_1 a_1, \tag{34a}$$

$$\frac{da_2^2}{d\varphi_1} = \frac{2}{B_1}\,f_2 a_2, \tag{34b}$$

$$\frac{da_1^2}{d\varphi_2} = \frac{2}{B_2}\,f_1 a_1, \tag{34c}$$

$$\frac{da_2^2}{d\varphi_2} = \frac{2}{B_2}\,f_2 a_2. \tag{34d}$$

It is convenient to rewrite equations (33a) and (33c), as well as (33b) and (33d) in the following form:

$$\frac{da_1^2}{d\delta} = \frac{f_1 a_1}{B_2 - B_1}, \tag{34e}$$

$$\frac{da_2^2}{d\delta} = \frac{f_2 a_2}{B_2 - B_1}. \tag{34f}$$

Multiplying (34a) by Γ_2 and (34b) by Γ_1 and adding the obtained relationships, we find

$$\frac{d}{d\varphi_1}\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right) = \frac{2}{B_1}\left(\Gamma_2 f_1 a_1 + \Gamma_1 f_2 a_2\right). \tag{35a}$$

Carrying out the same procedure for equations (34c) and (34d) yields

$$\frac{d}{d\varphi_2}\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right) = \frac{2}{B_2}\left(\Gamma_2 f_1 a_1 + \Gamma_1 f_2 a_2\right). \tag{35b}$$

Rewriting equations (35a) and (35b) in the form

$$B_2 d\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right) = 2\left(\Gamma_2 f_1 a_1 + \Gamma_1 f_2 a_2\right) d\varphi_2, \tag{36a}$$

$$B_1 d\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right) = 2\left(\Gamma_2 f_1 a_1 + \Gamma_1 f_2 a_2\right) d\varphi_1, \quad (36b)$$

and subtracting (36b) from (36a), we obtain

$$\frac{d}{d\delta}\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right) = \frac{\Gamma_2 f_1 a_1 + \Gamma_1 f_2 a_2}{B_2 - B_1}. \quad (37)$$

Introducing the value ξ according to the formula

$$\xi = \frac{\Gamma_2 a_1^2}{\Gamma_2 a_1^2 + \Gamma_1 a_2^2} \quad (38)$$

and considering (34e), let us rewrite equation (37) in the form

$$\frac{d\xi}{d\delta} = \frac{\Gamma_1 \Gamma_2 a_1 a_2 (a_2 f_1 - a_1 f_2)}{(B_2 - B_1)\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right)^2}. \quad (39)$$

C. The case of free damped vibrations at $\gamma_1 = \gamma_2 = \gamma$

In order to consider the case of free damped vibrations, we put $f = 0$ and suppose that $\gamma_1 = \gamma_2 = \gamma$. Then $s_1 = s_2 = s$ and $\psi_1 = \psi_2 = \psi$, and it is possible to find the first integral of the set of equations (29). To show this, we introduce the energy of the mechanical system under consideration

$$E = \Gamma_2 a_1^2 + \Gamma_1 a_2^2. \quad (40)$$

Considering (29e) and (29f), we rewrite (35a) in the form

$$\frac{dE}{d\varphi_1} = -\frac{s}{B_1} E. \quad (41)$$

Integrating (41) and considering that $B_1 = d\varphi_1/dT_2$ yields

$$E = E_0 e^{-sT_2}, \quad (42)$$

where E_0 is the initial energy of the system.

From (40) and (42) it follows that

$$a_1^2 = \Gamma_2^{-1} E_0 \xi e^{-sT_2}, \quad (43a)$$

$$a_2^2 = \Gamma_1^{-1} E_0 (1-\xi) e^{-sT_2}, \quad (43b)$$

where $\xi = \xi(T_2)$ is a new function.

Note that the substitution of formulas (43) in (38) results in the identity. Thus, considering (29c)-(29f) and (40)-(43), we have

$$B_2 - B_1 = \Big[(\lambda_3 - \lambda_1)\Gamma_2^{-1}\xi + (\lambda_4 - \lambda_2)\Gamma_1^{-1}(1-\xi) + \frac{1}{4}(1-2\xi)\cos\delta\Big] E_0 e^{-sT_2}, \quad (43c)$$

$$a_1 a_2^2 f_1 - a_2 a_1^1 f_2 = \frac{1}{4}\Gamma_1^{-1}\Gamma_2^{-1}\xi(1-\xi) E_0^3 e^{-3sT_2}\sin\delta, \quad (43d)$$

$$\left(\Gamma_2 a_1^2 + \Gamma_1 a_2^2\right)^2 = E_0^2 e^{-2sT_2}. \quad (43e)$$

Substituting (43c)-(43e) in (39), we are led to the differential equation

$$\frac{d\cos\delta}{d\xi} + \frac{1-2\xi}{\xi(1-\xi)}\cos\delta - \frac{4(\lambda_1-\lambda_3)}{\Gamma_2(1-\xi)} - \frac{4(\lambda_2-\lambda_4)}{\Gamma_1 \xi} = 0. \quad (44)$$

Integrating equation (44) yields

$$G(\delta,\xi) = \xi(1-\xi)\cos\delta - \frac{2(\lambda_1-\lambda_3)}{\Gamma_2}\xi^2 + \frac{2(\lambda_2-\lambda_4)}{\Gamma_1}(1-\xi)^2 = G_0, \quad (45)$$

where $G_0 = G(\delta_0, \xi_0)$ is an arbitrary constant depending on the initial magnitudes of $\delta_0 = \delta|_{T_2=0}$ and $\xi_0 = \xi|_{T_2=0}$.

Relationship (45) is also the first integral of the set of equations (29), in so doing by its physical meaning the function $G(\delta,\xi)$ is the stream-function of the phase fluid on the plane $\delta - \xi$, since the velocities of displacement of this fluid are determined as

$$v_\xi = \dot{\xi} = -\frac{1}{2}\Gamma_2 E_0 \frac{\partial G}{\partial \delta}\exp(-sT_2), \quad (46a)$$

$$v_\delta = \dot{\delta} = \frac{1}{2}\Gamma_2 E_0 \frac{\partial G}{\partial \xi}\exp(-sT_2). \quad (46b)$$

Thus, in this case, we have a steady-state motion of the system under consideration.

The detailed analysis of this case with numerical examples could be found in [4]. Based on the experimental data for the Golden Gate suspension bridge [1], it has been proved that “the nonlinear models with fractional derivative damping are more preferred over the models with integer derivatives for describing damping features of suspension bridges. The vibrating regimes investigated are combined from two interacting processes: the process of energy transfer and the process of damping” [4]. To illustrate the interaction of these two processes, the method of vector diagrams has been suggested, which allows one to trace not only the change in the energy of the whole system, but also to follow the redistribution of the partial energy of the two vibrating subsystems during this process.

One of the examples considered with due account for damping in [2] and [4], and without damping in [12] has been studied also in [10] without any references to [2], [4] and [12], although the authors of [10] considered the excitation of only one mode or the interaction of two modes as it was carried out in [2], [4], [12], aa well as in the present study.

D. The case of force driven vibrations at $\gamma_1 = \gamma_2 = \gamma$

Putting $\gamma_1 = \gamma_2 = \gamma$ in (29c)-(29f) and substituting these relations in (39), we have

$$\frac{d\xi}{d\delta} = -E^{-2}\left[\Gamma_1\Gamma_2 a_1^2 a_2^2 E\sin\delta - f\omega_0^{-1} a_1 a_2^2 \Gamma_1\Gamma_2 \sin\varphi_1\right] \times \left[\left(\Gamma_2 a_1^2 - \Gamma_1 a_2^2\right)\cos\delta + 4(\lambda_1-\lambda_3)a_1^2 + 4(\lambda_2-\lambda_4)a_2^2 - f\omega_0^{-1}a_1^{-1}\cos\varphi_1\right]^{-1}. \quad (47)$$

When $f=0$, we could obtain from (47) the known equation [4]

$$\frac{d\xi}{d\delta} = -\xi(1-\xi)\sin\delta\left[(1-2\xi)\cos\delta - 4(\lambda_1-\lambda_3)\Gamma_2^{-1}\xi \right.$$
$$\left. -4(\lambda_2-\lambda_4)\Gamma_1^{-1}(1-\xi)\right]^{-1}. \qquad (48)$$

Now imagine that at the initial instant of time a phase fluid point locates on a certain stream-line and has the coordinates δ_0, ξ_0. At the time $T_2=0$, the force f begins to act on the given mechanical system, and the phase point leaves the stream-line and describes its own trajectory. To determine the angle, under which the phase fluid point leaves the stream-line at the initial instant of time, we put $T_2=0$ in (47) and consider formulas (43). As a result we obtain

$$\left.\frac{d\xi}{d\delta}\right|_{T_2=0} = \left[E_0^{3/2}\xi_0(1-\xi_0)\sin\delta_0\right.$$
$$\left. -f\omega_0^{-1}(1-\xi_0)\xi_0^{1/2}\Gamma_2^{1/2}\sin\varphi_{10}\right]\left\{E_0^{3/2}\left[(1-2\xi_0)\cos\delta_0\right.\right.$$
$$\left. -4(\lambda_1-\lambda_3)\Gamma_2^{-1}\xi_0 - 4(\lambda_2-\lambda_4)\Gamma_1^{-1}(1-\xi_0)\right]$$
$$\left. +f\omega_0^{-1}\xi_0^{-1/2}\Gamma_2^{1/2}\cos\varphi_{10}\right\}^{-1}, \qquad (49)$$

where $\varphi_{10} = \varphi_1\big|_{T_2=0}$.

Using the angle of inclination of the tangent to the trajectory of the phase fluid point motion to the δ-axis, it is possible to determine the character of vibrational process of the given system being under the conditions of the internal and external resonances (4) at a time. This character is influenced essentially by the magnitude of the external vertical force amplitude f.

E. The case of free damped vibrations at $\gamma_1 \neq \gamma_2$

Substituting (29c)-(29f) at $f=0$ in (39) yields

$$\frac{d\xi}{d\delta} = -E^{-2}\left[\Gamma_1\Gamma_2 a_1^2 a_2^2 E\sin\delta - 2\Gamma_1\Gamma_2(s_1-s_2)a_1^2a_2^2\right]$$
$$\times\left[\left(\Gamma_2 a_1^2 - \Gamma_1 a_2^2\right)\cos\delta + 4(\lambda_1-\lambda_3)a_1^2\right.$$
$$\left. +4(\lambda_2-\lambda_4)a_2^2 + 2(\psi_1-\psi_2)\right]^{-1}. \qquad (50)$$

At $\gamma_1=\gamma_2=\gamma$, formula (50) goes over into (48).

Imagine that at the initial instant of time a phase fluid point locates on a certain stream-line and has the coordinates δ_0, ξ_0. At the time $T_2=0$, the increment $\Delta\gamma = \gamma_1-\gamma_2 \neq 0$ is imparted to the given mechanical system, and the phase point leaves the stream-line and describes its own trajectory. To determine the angle, under which the phase fluid point leaves the stream-line at the initial instant of time, we put $T_2=0$ in (50) and consider formulas (43). As a result we obtain

$$\left.\frac{d\xi}{d\delta}\right|_{T_2=0} = \xi_0(1-\xi_0)\left[E_0\sin\delta_0 - 2(s_1-s_2)\right]$$
$$\times\left\{E_0\left[(1-2\xi_0)\cos\delta_0 - 4(\lambda_1-\lambda_3)\Gamma_2^{-1}\xi_0\right.\right.$$
$$\left.\left. -4(\lambda_2-\lambda_4)\Gamma_1^{-1}(1-\xi_0)\right] + 2(\psi_2-\psi_1)\right\}^{-1}. \qquad (51)$$

Thus, it has been found that as distinct to the case $\gamma_1=\gamma_2=\gamma$ when the fractional derivative favors damped steady-state motions, the presence of fractional derivatives of two different orders in equations (26) results, contrary to the expectations, in the destabilization of all vibratory motions of the suspension combined system, i.e., quasistable motion of the suspended combined system goes over into the transient one.

This case was also discussed in [5] using the experimental data for the Golden Gate suspension bridge [1].

IV. Conclusion

Nonlinear force driven coupled vertical and torsional vibrations of a suspension bridge subject to the combination of external and internal resonances have been investigated for the case when its damping features are described by the fractional derivatives. From the above discussion the following conclusions could be reached.

If the external force is of order of ε^2 and the viscosity coefficients are of order of ε, then it is possible to obtain the approximate analytical solutions for the generalized displacements. As this takes place, the solution for the vertical displacement x_1 involves two parts: the first corresponds to the damping vibrations with damping coefficients and nonlinear frequencies dependent on the fractional parameters and describes the transient process, while the second one is nondamping in character and describes the steady-state regime, i.e., forced vibrations with the frequency of the exciting force and with the phase difference depending on the fractional parameter. The solution for the torsional displacement x_2 consists only from one term describing the transient process. Moreover, in the transient processes, the damping coefficients and the frequencies of nonlinear vibrations depend on the square of the exciting force amplitude.

If the external force is of order of ε^3 and the viscosity coefficients are of order of ε^2, then it is impossible to obtain the analytical expressions for the generalized displacements x_1 and x_2, since the differential modulation equations could be solved only numerically. However, in the case of absence of the external force and when the orders of fractional derivatives are equal to each other $\gamma_1=\gamma_2=\gamma$, the nonlinear set of equations describing the vibratory motion possesses two first integrals, namely: the integral of energy and the integral representing itself the stream-function, along which phase fluid interpreting the vibratory process moves on the phase plane $\delta-\xi$. If at some point in time an external force affects on the mechanical system or the orders of the fractional derivatives begin to differ little in magnitude, then at this moment the stream-lines of the phase fluid become unstable and disappear at further instants of time according to formulas (49) and (51).

REFERENCES

[1] A. M. Abdel–Ghaffar and R. H. Scanlan, "Ambient vibration studies of Golden Gate Bridge. I: Suspended structure," *ASCE Journal of Engineering Mechanics*, vol. 111 (4), pp. 463–482, 1985.

[2] Yu. A. Rossikhin and M. V. Shitikova, "Effect of viscosity on the vibrational processes in a combined suspension system," *Mechanics of Solids*, vol. 30 (1), pp. 157–166, 1995.

[3] A. M. Abdel-Ghaffar and L. I. Rubin, "Nonlinear free vibration of suspension bridge: Theory and application," *ASCE Journal of Engineering Mechanics*, vol. 109 (1), pp. 313–345, 1983.

[4] Yu. A. Rossikhin and M. V. Shitikova, "Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges," *ASCE Journal of Engineering Mechanics*, vol. 124 (9), pp. 1029–1036, 1998.

[5] Yu. A. Rossikhin and M. V. Shitikova, "Nonlinear free damped vibrations of suspension bridges with uncertain fractional damping," *RS-Journal Europeen des Systemes Automatises*, vol. 42 (6-8), pp. 879–894, 2008.

[6] W. Lacarbonara, "Direct treatment and discretizations of non-linear spatially continuous systems," *Journal of Sound and Vibration*, vol. 221 (5), pp. 849–866, 1999.

[7] W. Lacarbonara, "Author's reply," *Journal of Sound and Vibration*, vol. 235 (5), pp. 875–876, 2000.

[8] M. Pakdemirli, "Comments on "Direct treatment and discretizations of non-linear spatially continuous systems"," *Journal of Sound and Vibration*, vol. 235 (5), pp. 874-875, 2000.

[9] M. Pakdemirli, "A comparison of two perturbation methods for vibrations of systems with quadratic and cubic nonlinearities," *Mechanics Research Communications*, vol. 21 (2), pp. 203–208, 1994.

[10] M. Cevik and M. Pakdemirli, "Non-linear vibrations of suspension bridges with external excitation," *International Journal of Non-Linear Mechanics*, vol. 40, pp. 901–923, 2005.

[11] M. Pakdemirli and H. Boyaci, "Comparison of direct-perturbation methods with discretization-perturbation methods for non-linear vibrations," *Journal of Sound and Vibration*, vol. 186, p. 837–845, 1995.

[12] Yu. A. Rossikhin and M. V. Shitikova, "Analysis of nonlinear free vibrations of suspension bridges," *Journal of Sound and Vibration*, vol. 186 (3), pp. 369–393, 1995.

[13] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications* (in Russian), Nauka i Tekhnika, Minsk 1988 (Engl. transl. by Gordon and Breach Science Publ.)

[14] Yu. A. Rossikhin and M. V. Shitikova, "New approach for the analysis of damped vibrations of fractional oscillators," *Shock and Vibration*, vol. 16, pp. 365–387, 2009.

[15] Yu. A. Rossikhin and M. V. Shitikova, "Forced vibrations of a nonlinear oscillator with weak fractional damping," *Mechanics of Materials and Structures*, vol. 4 (9), pp. 1619–1636, 2009.

[16] Yu. A. Rossikhin and M. V. Shitikova, "On fallacies in the decision between the Caputo and Riemann-Liouville fractional derivatives for the analysis of the dynamic response of a nonlinear viscoelastic oscillator," *Mechanics Research Communications*, vol. 45, pp. 22–27, 2012.

[17] Yu. A. Rossikhin and M. V. Shitikova, "Analysis of nonlinear forced vibrations of fractionally damped suspension bridges," in *Recent Researchers in Environmental & Geological Sciences* (N. Altawell et al., Eds), Energy, Environmental and Structural Engineering Series, vol. 4, pp. 404–407, 2012, WSEAS Publisher.