Track Defects and the Dynamic Loads due to Non-Suspended Masses of Railway Vehicles

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Abstract— The motion of a railway vehicle on the rail running table is a forced oscillation with a forcing excitation (force), and damping expressed by a random, non-periodic function. The motion is described by formulas and it is illustrated through diagrams which have the form of a "signal". The railway vehicle has the Suspended (Sprung) and the Non Suspended (Unsprung) Masses. The track defects/ faults are the random excitation for the rolling wheels of the vehicle. In the case of the Sprung Masses of the vehicles, the forces resulting from the excitation (track defects) are not large and have small effect on the rolling of the wheel. In the case of the Unsprung Masses the forces resulting from the excitation (track defects) are large and have grate effect on the rolling of the wheel. The track, is simulated (with the observer situated on the wheel) as an elastic means with damping. The general equation that describes the problem is the second order differential equation of motion. In this paper the aforementioned equation is presented for the case of a railway vehicle rolling on a railway track and its solution is presented for the Unsprung (Non-Suspended) Masses of the vehicle that act directly on the track without the presence of any spring or damper.

Keywords: *Static Stiffness Coefficient, Sprung Masses, Unsprung Masses, Fourier Transform, Spectral Density Introduction, Variance, Standard Deviation, Dynamic Component of Actions*

I. The Non-Suspended Masses and the Suspended Masses of a Railway Vehicle

The railway vehicles consist of (a) the car-body, (b) the primary and the secondary suspension with the bogie in between the axles, and (c) the wheels. The heaviest vehicles are the locomotives which are "motive units" and have electric motors on the axles and/or the frame of the bogie. In Fig. 1 a locomotive with three-axle bogie is depicted while the three axle bogie with the springs of the primary and secondary suspensions is depicted in Fig. 2.

Electric motors are either suspended totally from the frame of the bogie or they are suspended on the frame of the bogie at one end and supported on the axle at the other end. In the second case the electric motor is semi-suspended (Figure 3) and a part of it is considered as Non Suspended Mass as it will be clarified below.

If we try to approach mathematically the motion of a vehicle on a railway track, we will end up with the model shown in Fig. 4, where both the vehicle and the railway track are composed of an ensemble of masses, springs and dashpots.

Figure 1 Diesel Locomotive of the Greek railways with three-axle bogies, on a railway track. The three-axle bogie is marked with the black elipse.

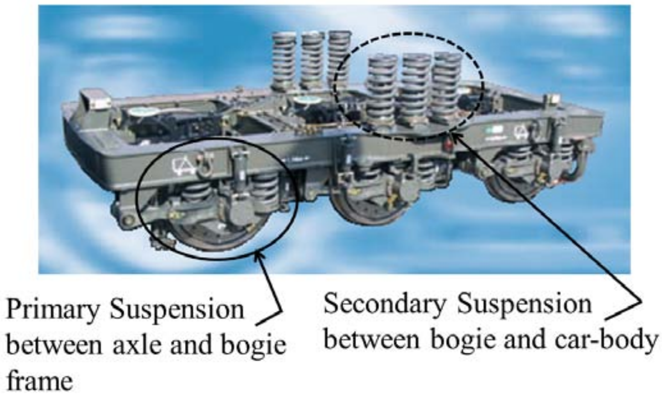

Figure 2 A three-axle bogie with the springs of the primary suspension inside the black elipse with continuous line and the springs of the secondary suspension in the black elipse with the dashed line.

As we can observe, the car body is supported by the secondary suspension that includes two sets of "springs-dashpots", seated on the frame of the bogie. The loads are transfered to the truss and the side frames of the bogie. Underneath the bogie there is the primary suspension, through which the bogie is seated onto the carrying axles and the wheels. Below the contact surface, between the wheel and the rail, the railway track also consists of a combination of masses-springs-dampers that simulates the rail, the sleepers, the elastic pad, the rail fastenings, the ballast and the ground.

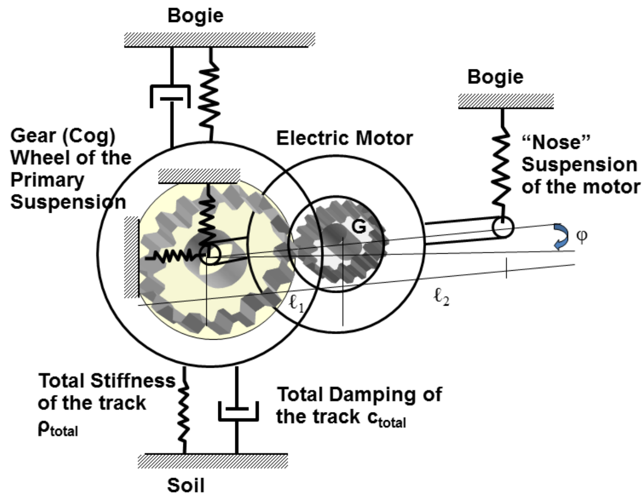

Figure 3 Schematic depiction of an Electric Motor "semi-suspended" from the bogie's frame at one end and supported on the axle at the other end.

The masses of the railway vehicle located under the primary suspension (axles, wheels and a percentage of the electric motor weight in the case of locomotives) are the Non Suspended Masses (N.S.M.) of the Vehicle, that act directly on the railway track without any damping at all. Furthermore a section of the track mass (m_{TRACK}) also participates in the motion of the vehicle's Non Suspended Masses, which also highly aggravates the stressing on the railway track (and on the vehicle too).

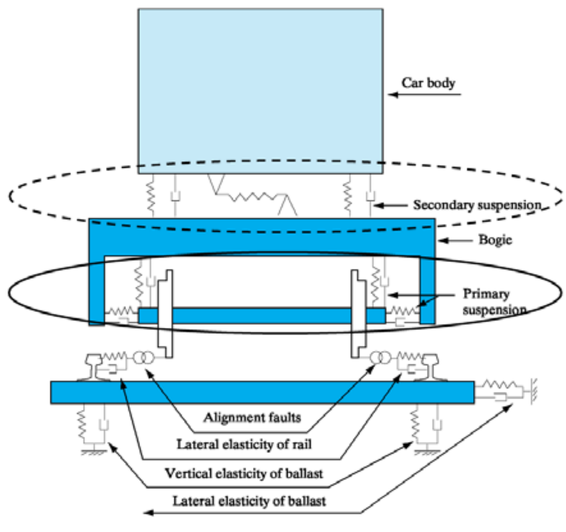

Figure 4 Model of the Railway system "Vehicle –Track", as an ensemble of springs and dashpots.

The remaining vehicle masses are called Suspended Masses (S.M.) or Sprung Masses: the car-body, the secondary suspension, the frame of the bogie, a part of the electric motor's weight and the primary suspension.

In the present paper, we analyze the effect of the Non-Suspended Masses, whose contribution to the stressing of the track is of great importance.

II. The Loads on Track

The system operates based on the classical principles of physics: Action-Reaction between the vehicle and the track. It is a dynamic stressing of random, vertical form.

The loading of the railway track from a moving vehicle consists of:

(a) the static load (static load of vehicle axle), as given by the rolling stock's producer.

(b) the semi-static load (cant/ superelevation deficiency at curves, which results in non-compensated lateral acceleration)

(c) the load from the Non-Suspended Masses of the vehicle (the masses that are not damped by any suspension, because they are under the primary suspension of the vehicle) and

(d) the load from the Suspended Masses of the vehicle, that is a damped force component of the total action on the railway track.

For High Speed Lines (V_{max} > 200 km/h), the component of the Load, due to the Non Suspended Masses, is of decisive importance for the Dynamic Load. In order to calculate the total Action/ Reaction on each support point of the rail (pair of fastenings on a sleeper) the static, the semistatic and the dynamic components should be added. The total dynamic component [(c)+(d)] is the square root of the second powers of the (c) and (d).

III. The Second Order Differential Equation of Motion for the Non Suspended Masses

A. *General Form of the Second Order Differential Equation*

The railway track is an infinite beam on elastic foundation, and the elastic foundation can be simulated by a large number of closely spaced translational springs [1]:

$$EI\frac{\partial^4 z}{\partial x^4}+\rho_1\cdot\frac{\partial^2 z}{\partial x^2}+k_1\cdot z=0 \qquad (1)$$

when there is no external force, or [2]:

$$EI\frac{\partial^4 z}{\partial x^4}+\rho_1\cdot\frac{\partial^2 z}{\partial x^2}+k_1\cdot z=-Q\cdot\delta(x) \qquad (2)$$

In these equations z is the deflection of the beam, ρ_1 is the mass of the track participating in the motion, k_1 the viscous damping of the track, E, I, the elasticity modulus and the moment of inertia of the rail and Q the force/ load from the wheel (when the force is present).

The solution of equations (1) and (2) becomes challenging if we want to take into account all the parameters [2]. However, if we make some simplifying hypotheses we will be able to approximate the influence of certain parameters provided that we will verify the theoretical results with experimental measurements.

B. The System Vehicle – Track

The track defects/ faults are the random excitation for the rolling wheels of the vehicle. In the case of the Sprung Masses of the vehicles, the forces resulting from the excitation (track defects) are not large and have small effect on the rolling of the wheel. In the case of the Unsprung Masses the forces resulting from the excitation (track defects) are large and have grate effect on the rolling of the wheel. The track, is simulated (with the observer situated on the wheel) as an elastic means with damping (see Fig. 4). According to the French Railways, the spring constant and the damping coefficient can be measured through various tests: (1) by the direct sinusoidal excitation of the track or a track panel, (2) by the impact caused from the induced fall of an axle upon the track.

The very wide dispersion of the results allows the determination of the stiffness and the total damping of the track. The track exhibits significant nonlinear behavior and the measured values are neither independent of the excitation frequency, nor of the type of the excitation.

In Fig. 4 the system “Railway Track – Railway Vehicle” is depicted as an ensemble of springs and dashpots. The rail running table imposes to the vehicle a forced vibration. It is not smooth but instead it comprises a lot of faults that give to the rail running table a random surface. Furthermore, under the primary suspension there are the Unsprung (Non-Suspended) Masses (axle, wheels and a fraction of the semi-suspended motive electromotor in the locomotives) which act without any dumping directly on the track panel.

On the contrary the Sprung (Suspended) Masses that are cited above the primary suspension of the vehicle, act through a combination of springs and dumpers on the track.

A part of the track mass is also added to the unsprung masses, which participates in their motion [3]. The rail running table has the shape of a wave that is not completely “rectilinear”, that is, it does not form a perfectly straight line but contains faults, varying from a few fractions of a millimeter to a few millimeters, and imposes forced oscillation on the railway vehicles that move on it [4]. The general form of the equation of a vehicle moving on an infinitely long beam, on elastic ground without damping, is described in the paragraph below.

C. Second Order Differential Equation of Motion of the Unsprung Masses on a Railway Track

The Suspended (sprung) Masses of the vehicle –masses situated above the primary suspension– create forces with very small influence on the wheel’s trajectory and on the system’s excitation. This enables the simulation of the track as an elastic media with damping, as shown in Fig. 5[5]. Forced oscillation is caused by the irregularities of the rail running table (like an input random signal) –which are represented by n–, in a gravitational field with acceleration g. As already described, there are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

If the random excitation (track irregularities) is given, it is difficult to derive the response, unless the system is linear and invariable. In this case the input signal can be defined by its spectral density and from this we can calculate the spectral density of the response. The theoretical results confirm and explain the experimental verifications performed in the former British railway network (relevant results in [2], p.39, 71).

The equation for the interaction between the vehicle’s axle and the track-panel becomes [6], [5]:

$$\begin{aligned}&\left(m_{NSM}+m_{TRACK}\right)\cdot\frac{d^2y}{dt^2}+\Gamma\cdot\frac{dy}{dt}+h_{TRACK}\cdot y=\\&=-m_{NSM}\cdot\frac{d^2n}{dt^2}+\left(m_{NSM}+m_{SM}\right)\cdot g\end{aligned}\tag{3}$$

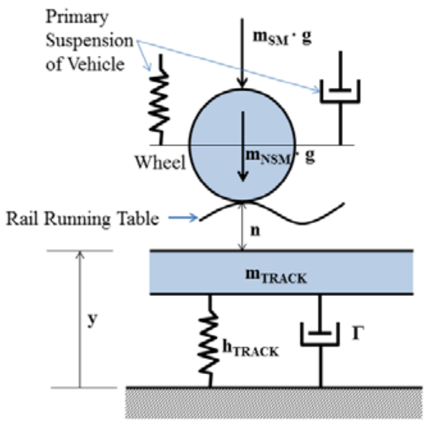

Figure 5 Model of a rolling wheel on the rail running table

where: m_{NSM} the Non-Suspended (Unsprung) Masses of the vehicle, m_{TRACK} the mass of the track that participates in the motion, m_{SM} the Suspended (Sprung) Masses of the vehicle that are cited above the primary suspension of the vehicle, Γ damping constant of the track (for its calculation Ref. [7], [8]), h_{TRACK} the total dynamic stiffness coefficient of the track, n the fault ordinate of the rail running table and y the total deflection of the track.

The phenomena of the wheel-rail contact and of the wheel hunting, particularly the equivalent conicity of the wheel and the forces of pseudo-glide, are non-linear.

In any case the use of the linear system’s approach is valid for speeds lower than the $V_{critical}$≈500 km/h. The integration for the non-linear model (wheel-rail contact, wheel-hunting and pseudoglide forces) is performed through the Runge Kutta method ([2], p.94-95, 80, [9], p.98, see also [10], p.171, 351).

IV. Applying the Fourier Transform to the Second Order Differential Equation

In Fig. 5 the rail running table depicts a longitudinal fault/ defect of the rail surface. In the above equation, the oscillation of the axle is damped after its passage over the defect. Viscous damping, due to the ballast, enters the above equation under the condition that it is proportional to the variation of the deflection dy/dt. To simplify the investigation, if we ignore the track mass (for its calculation Ref. [7], [8]) in relation to the much larger Vehicle's Non Suspended Mass and bearing in mind that y+n is the total subsidence of the wheel during its motion (since the y and n are added algebraically), we can approach the problem of the random excitation, from cosine defect ($V \ll V_{critical}$=500 km/h):

$$\eta = a \cdot \cos \omega t = a \cdot \cos\left(2\pi \cdot \frac{V \cdot t}{\lambda}\right) \tag{4}$$

Where V the speed of the vehicle, $T=2\pi/\omega \rightarrow \rightarrow \omega t=2\pi/(Tt)=2\pi Vt/\lambda$ where λ the length of the defect, run by the wheel in:

$$T = \frac{\lambda}{V} \Rightarrow \lambda = T \cdot V \tag{5}$$

If we set:

$$y = z + \frac{m_{SM} + m_{NSM}}{h_{TRACK}} \cdot g \Rightarrow \frac{dy}{dt} = \frac{dz}{dt} \quad \textit{and} \quad \frac{d^2 y}{dt^2} = \frac{d^2 z}{dt^2}$$

where the quantity $\frac{m_{SM} + m_{NSM}}{h_{TRACK}} \cdot g$ represents the subsidence due to the static loads only, and z random (see [11]) due to the dynamic loads, equation (3) becomes:

$$m_{NSM} \frac{d^2 z}{dt^2} + \Gamma \cdot \frac{dz}{dt} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} \Rightarrow \tag{6a}$$

$$\Rightarrow m_{NSM}\left(\frac{d^2 z}{dt^2} + \frac{d^2 n}{dt^2}\right) + \Gamma \cdot \frac{dz}{dt} + h_{TRACK} \cdot z = 0 \tag{6b}$$

Since, in this case, we are examining only the dynamic loads, in order to approach their effect, we could narrow the study of equation (6b), by changing the variable:

$$u = n + z \Rightarrow \frac{d^2 u}{dt^2} = \frac{d^2 n}{dt^2} + \frac{d^2 z}{dt^2}$$

Equation (6) becomes:

$$m_{NSM} \frac{d^2 u}{dt^2} + \Gamma \cdot \frac{dz}{dt} + h_{TRACK} \cdot z = 0 \Rightarrow \tag{7a}$$

$$\Rightarrow m_{NSM} \frac{d^2 u}{dt^2} + \Gamma \cdot \frac{d(u-n)}{dt} + h_{TRACK} \cdot (u-n) = 0 \tag{7b}$$

Where, u is the trajectory of the wheel over the vertical fault in the longitudinal profile of the rail.

If we apply the Fourier transform to equation (6a) (see relevantly Ref. [12] for solving second order differential equations with the Fourier transform):

$$(i\omega)^2 \cdot Z(\omega) + \frac{\Gamma \cdot (i\omega)}{m_{NSM}} \cdot Z(\omega) + \frac{h_{TRACK}}{m_{NSM}} \cdot Z(\omega) = -(i\omega)^2 \cdot N(\omega) \Rightarrow$$

$$H(\omega) = \frac{Z(\omega)}{N(\omega)}, \quad |H(\omega)|^2 = \frac{m_{NSM}^2 \cdot \omega^4}{\left(m_{NSM} \cdot \omega^2 - h_{TRACK}\right)^2 + \Gamma^2 \cdot \omega^2} \tag{8}$$

H(ω) is a complex transfer function, *called frequency response function* [13], that makes it possible to pass from the fault n to the subsidence Z. If we apply the Fourier transform to equation (7a):

$$(i\omega)^2 \cdot U(\omega) + \Gamma \cdot (i\omega) \cdot Z(\omega) + h_{TRACK} \cdot (i\omega)^0 \cdot Z(\omega) = 0 \Rightarrow$$

$$G(\omega) = \frac{U(\omega)}{Z(\omega)}, |G(\omega)|^2 = \frac{h_{TRACK}^2 + \Gamma^2 \cdot \omega^2}{m_{NSM}^2 \cdot \omega^4} \tag{9}$$

G(ω) is a complex transfer function, *the frequency response function*, that makes it possible to pass from Z to Z+n.

If we name U the Fourier transform of u, N the Fourier transform of n, p=2πiν=iω the variable of frequency and $\hat{\Delta}Q$ the Fourier transform of ΔQ and apply the Fourier transform at equation (7b):

$$Eq.(7) \Rightarrow m_{NSM} \frac{d^2 u}{dt^2} + \Gamma \cdot \frac{du}{dt} + h_{TRACK} \cdot u = \Gamma \cdot \frac{dn}{dt} + h_{TRACK} \cdot n \Rightarrow$$

$$\left(m_{NSM} \cdot p^2 + \Gamma \cdot p + h_{TRACK}\right) \cdot U = \left(\Gamma \cdot p + h_{TRACK}\right) \cdot N \Rightarrow$$

$$U(\omega) = \underbrace{\frac{\Gamma \cdot p + h_{TRACK}}{m_{NSM} \cdot p^2 + \Gamma \cdot p + h_{TRACK}}}_{B(\omega)} \cdot N(\omega) \tag{10a}$$

$$|B(\omega)|^2 = \frac{\Gamma^2 \cdot \omega^2 + h_{TRACK}^2}{\left(m_{NSM} \cdot \omega^2 - h\right)^2 + \Gamma^2 \cdot \omega^2} \tag{10b}$$

B(ω) is a complex transfer function, *the frequency response function*, that makes it possible to pass from the fault n to the u=n+z. Practically it is verified also by the equation:

$$|B(\omega)|^2 = |H(\omega)|^2 \cdot |G(\omega)|^2 = \frac{h_{TRACK}^2 + \Gamma^2 \cdot \omega^2}{\left(m_{NSM} \cdot \omega^2 - h_{TRACK}\right)^2 + \Gamma^2 \cdot \omega^2} \tag{10c}$$

passing from n to Z through H(ω) and afterwards from Z to n+Z through G(ω). This is a formula that characterizes the transfer function between the wheel trajectory and the fault in the longitudinal level and enables, thereafter, the calculation of the transfer function between the dynamic load and the track defect (fault).

The transfer function of the second derivative of (Z+n) in relation to time: $\frac{d^2(Z+n)}{dt^2}$, that is the acceleration γ, will be calculated below (and is equal to *(iω)*^{2}*·B(ω), as second derivative of a Fourier Transform function*).

The increase of the vertical load on the track due to the Non Suspended Masses, according to the principle *force = mass x acceleration*, is given by:

$$\Delta Q = m_{NSM} \cdot \frac{d^2 u}{dt^2} = m_{NSM} \cdot \frac{d^2(n+Z)}{dt^2} \tag{11}$$

If we apply the Fourier transform to equation (11):

$$\hat{\Delta} Q = m_{NSM} \cdot p^2 \cdot U(\omega) = m_{NSM} \cdot p^2 \cdot \hat{f}_{Z+n}(\omega) \Rightarrow \tag{12a}$$

$$\left|\hat{\Delta} Q\right| = m_{NSM} \cdot |p|^2 \cdot |B(\omega)| = m_{NSM} \cdot \beta^2 \cdot \omega_n^2 \cdot |B(\omega)| \cdot |N(\omega)| \tag{12b}$$

The transfer function B(ω) allows us to calculate the effect of a spectrum of sinusoidal faults, like the undulatory wear. If we replace $\omega/\omega_n=\rho$, where $p=i\omega$, ω_n= the circular eigenfrequency (or natural cyclic frequency) of the oscillation, and:

$$\omega_n^2 = \frac{h_{TRACK}}{m_{NSM}}, \quad \omega = \frac{2\pi V}{\lambda}, \quad 2\zeta\omega_n = \frac{\Gamma}{m_{NSM}}, \quad \beta = \frac{\omega}{\omega_n}$$

where ζ is the damping coefficient. Equation (10b) is transformed:

$$|B(\omega)|^2 = |B_n(\beta)|^2 = \frac{1+4\zeta^2 \cdot \beta^2}{\left(1-\beta^2\right)^2 + 4\zeta^2 \cdot \beta^2} \tag{13}$$

V. Power Spectral Density and Variance

The excitation (rail irregularities/ track defects) in reality is random and neither periodic nor analytically defined, like eq. (4).

It can be defined by its autocorrelation function in space and its spectral density ([2], p.58, [14], p.700, [15]). If f(x) is a signal with determined total energy and F(v) its Fourier transform, from Parseval’s modulus theorem [13], the total energy is ([16], [17]):

$$\int_{-\infty}^{+\infty} |f(x)|^2 \cdot dx = \int_{-\infty}^{+\infty} |F(\nu)|^2 \cdot d\nu \tag{14a}$$

where, $F(v)=A(v)\cdot e^{i\varphi(v)}$ and the power spectral density:

$$S(\omega) = |F(\nu)|^2 = A^2(\nu) \tag{15}$$

Reference [13] solves equation (14a) as:

$$\int_{-\infty}^{+\infty} f(t)^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)^2 \cdot d\omega \tag{14b}$$

The square of the modulus F(ω) is called the energy spectrum of the signal because $F^2(\omega)\cdot\Delta(\omega)$ represents the amount of energy in any ΔΩ segment of the frequency spectrum, and the integral of $F^2(\omega)$ over (-∞, +∞) gives the total energy of the signal.

An input signal -like the running rail table- creates through the vehicle an output signal: the wheel trajectory. The output spectral density and the input spectral density of the excitation are related through equation [18], [6]:

$$S_{out}(\bar{\omega}) = |H(i\bar{\omega})|^2 \cdot S_{in}(\bar{\omega}) \tag{15a}$$

In order to relate the temporal spectrum with the spectrum in space we use the following equation:

$$\omega \cdot t = \frac{2\pi V t}{\lambda} \Rightarrow \omega = \frac{2\pi}{\lambda} \cdot V \Rightarrow \omega = \Omega \cdot V \tag{15b}$$

where λ is the wavelength of the defect. This means that circular frequency in space Ω is the wave number k of the equation of oscillation, and ([16], [17]):

$$\begin{gathered} \int_0^{\infty} S(\Omega) \cdot d\Omega = \int_0^{\infty} s(\omega) \cdot d\omega, \\ \mathbb{F}\left[f(ax)\right] = \frac{1}{|a|} \cdot \hat{f}\left(\frac{\nu}{a}\right) \Rightarrow S(\omega) = S\left(\frac{\Omega}{V}\right) = \frac{1}{V} \cdot S(\Omega) \end{gathered} \tag{16}$$

where $\mathbb{F}$ is the symbol for the application of the Fourier transform of f and f□ the function after the transform. This is a property of the Fourier transform.

VI. Variance and Spectrum of Track Defects

The Variance or mean square value $\sigma^2(x)$ of the function is given by [19], [20]:

$$\sigma^2(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S(\omega) \cdot d\omega = \bar{x}^2 \tag{17}$$

where σ(x) is the standard deviation of the function.

The Power Spectral density and the variance of a function are depicted in Fig. 6.

From equation (17) we derive:

$$\begin{gathered} \sigma^2(n) = \frac{1}{\pi} \int_0^{+\infty} S_n(\omega) \cdot d\omega, \quad \sigma^2(z) = \frac{1}{\pi} \int_0^{+\infty} S_z(\omega) \cdot d\omega \\ \sigma^2(\Delta Q) = \frac{1}{\pi} \int_0^{+\infty} S_{\Delta Q}(\omega) \cdot d\omega \end{gathered} \tag{18}$$

where n is the random variable of the defect (input), z the subsidence of the wheel (output) and ΔQ the dynamic component of the Load that is added to the Static Load of the wheel due to the Non Suspended Masses (output also).

From these equations and the analytic form of the spectrum of the defects/ faults, we can calculate the mean square value of the dynamic component of the Load due to the Non Suspended Masses that is added to the Static and Semi-static Components of the Load of the wheel

From the power spectral density and the variance functions and their definitions [6]:

$$S_{\Delta Q}(\omega) = S_n(\omega)\cdot\left|B(\omega)\right|^2 \tag{19}$$

$$\Delta Q = m_{NSM}\cdot\Delta\gamma \Rightarrow \sigma^2(\Delta Q) = m_{NSM}\cdot\sigma^2(\gamma) \Rightarrow \sigma^2(\gamma) = \frac{\sigma^2(\Delta Q)}{m_{NSM}}$$

and using the eqs. (19) and (11-12b):

$$\sigma^2(\gamma) = \frac{1}{m_{NSM}}\cdot\sigma^2(\Delta Q) = \frac{1}{m_{NSM}^2\cdot\pi}\cdot\int_0^{+\infty}\left|B(\omega)\right|^2\cdot S_n(\omega)\cdot d\omega \tag{20}$$

$$\sigma^2(\gamma) = \frac{m_{NSM}^2}{m_{NSM}^2\cdot\pi}\cdot\int_0^{+\infty}\beta^4\cdot\omega_n^4\cdot\left|B(\omega)\right|^2\cdot S_n(\omega)\cdot d\omega =$$

$$\sigma^2(\gamma) = \frac{1}{\pi}\cdot\int_0^{+\infty}\beta^4\cdot\omega_n^4\cdot\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot S_n(\omega)\cdot d\omega \tag{21}$$

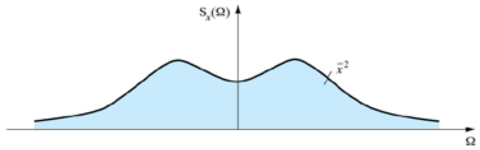

Figure 6 Power Spectral density S(Ω) –the black curve– Variance (mean square value $\overline{x}^2$) –the shaded area– of a function [6].

From the above equations and the analytical form of the spectrum of the longitudinal defects/ faults of the track we could effectively calculate the variance (mean square value) of the dynamic component of the Loads on the track panel due to the Non Suspended Masses. All the results of measurements on track in the French railways network show that the spectrum of defects in the longitudinal level has the form of [5], [21]:

$$S_n(\Omega) = \frac{A}{(B+\Omega)^3} \tag{22}$$

This implies that the mean square value or variance of the defects is given by:

$$\sigma^2(z) = \frac{1}{\pi}\cdot\int_0^{+\infty}\frac{A}{(B+\Omega)^3}\cdot d\Omega = \frac{A}{\pi}\cdot\int_0^{+\infty}\frac{1}{(x)^3}\cdot dx = -\frac{A}{2\pi}\left[\frac{1}{(B+\Omega)^2}\right]_0^{+\infty} \Rightarrow$$

$$\sigma^2(z) = -\frac{A}{2\pi}\cdot\left[\frac{1}{B^2+2B\Omega+\Omega^2}\right]_0^{+\infty} = -\frac{A}{2\pi}\left[0-\frac{1}{B^2}\right] \Rightarrow$$

$$\sigma^2(z) = \frac{1}{2\pi}\cdot\frac{A}{B^2} \tag{23}$$

The most severe case for the magnitude of the acting Load on the railway track, due to the Non Suspended Masses of the railway vehicle, is the case of the defects of short wavelength, consequently large Ω, like the undulatory wear of the rail running table. If we examine this case for the Non Suspended Masses, then we can omit the term B, and using eq. (15b):

$$S_n(\Omega) = \frac{A}{\Omega^3} = \frac{A}{\frac{1}{V^3}\cdot\omega^3} = \frac{A\cdot V^3}{\omega^3} \tag{24}$$

The term B characterizes the defects with large wavelengths, for which the maintenance of track is effective, and when we examine this kind of defects term B should be taken into account.

For the line "*Les Aubrais – Vierzon*", the parameters values are: B=0,36, $A=2{,}1\cdot 10^{-6}$ and S(Ω) is calculated in m^3 and σ(z)=1,57 mm. The eigenfrequency of the Non Suspended Masses of the vehicles is approximately 30 – 40 Hz and even for speeds of 300 km/h there are wavelengths less than 3 m [21].

From equations (16) and (24):

$$S_n(\omega) = \frac{1}{V}\cdot S(\Omega) = \frac{1}{V}\cdot\frac{A\cdot V^3}{\omega^3} = \frac{A\cdot V^2}{\omega_n^3\cdot\beta^3} \tag{25}$$

VII. The Variance of the Vertical Acceleration due to the Non Suspended masses

Using eq. (25), eq. (21) is transformed:

$$\sigma^2(\gamma) = \frac{1}{\pi}\cdot\int_0^{+\infty}\beta^4\cdot\omega_n^4\cdot\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\frac{A\cdot V^2}{\omega_n^3\cdot\beta^3}\cdot d\omega =$$

$$= \frac{A\cdot V^2\cdot\omega_n^2}{\pi}\cdot\underbrace{\int_0^{+\infty}\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\beta\cdot d\beta}_{Int} \tag{26}$$

We will calculate the integral (*Int*) of eq. (26).

$$Int = \underbrace{\int_0^{+\infty}\frac{1}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\beta\cdot d\beta}_{A_1} + \underbrace{\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\beta\cdot d\beta}_{A_2}$$

In order to calculate these two integrals A1 and A2 we put only for this calculation, independently from the previous paragraphs:

$$f = \beta^2 = \left(\frac{\omega}{\omega_n}\right)^2,\quad p = 1-2\zeta^2, q^2 = 4\zeta^2\left(1-\zeta^2\right) \tag{27}$$

consequently:

$$df = 2\beta\cdot d\beta = \frac{1}{\omega_n^2}\cdot 2\cdot\omega\cdot d\omega \Rightarrow \omega\cdot d\omega = \frac{\omega_n^2}{2}\cdot df$$
$$2\beta\cdot d\beta = \frac{1}{\omega_n^2}\cdot 2\cdot\omega\cdot d\omega \Rightarrow \beta\cdot d\beta = \frac{\omega\cdot d\omega}{\omega_n^2} \tag{28}$$

and:

$$A_1=\int_0^{+\infty}\frac{1}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\beta\cdot d\beta=\int_0^{+\infty}\frac{\frac{1}{\omega_n^2}\cdot\omega\cdot d\omega}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\cdot\left(\frac{\omega}{\omega_n}\right)^2}=$$

$$=\int_0^{+\infty}\frac{\frac{1}{\omega_n^2}\cdot\frac{\omega_n^2}{2}\cdot df}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\cdot\left(\frac{\omega}{\omega_n}\right)^2}=\frac{1}{2}\cdot\int_0^{+\infty}\frac{df}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\cdot\left(\frac{\omega}{\omega_n}\right)^2}=$$

$$=\frac{1}{2}\cdot\int_0^{+\infty}\frac{df}{\underbrace{1}_{3}+\underbrace{\left(\frac{\omega}{\omega_n}\right)^4}_{2}\underbrace{-2\left(\frac{\omega}{\omega_n}\right)^2}_{4}+\underbrace{4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}_{4}+\underbrace{4\zeta^2}_{1}\underbrace{-4\zeta^2}_{3}\underbrace{+4\zeta^4}_{3}\underbrace{-4\zeta^4}_{1}}=$$

$$=\frac{1}{2}\cdot\int_0^{+\infty}\frac{df}{\underbrace{4\zeta^2\left(1-\zeta^2\right)}_{q^2}+\underbrace{\left(\frac{\omega}{\omega_n}\right)^4}_{f^2}+\underbrace{\left(1-2\zeta^2\right)^2}_{p^2}-2\underbrace{\left(\frac{\omega}{\omega_n}\right)^2}_{f}\cdot\underbrace{\left(1-2\zeta^2\right)}_{p}}=$$

$$=\frac{1}{2}\cdot\int_0^{+\infty}\frac{\frac{1}{q^2}\cdot df}{\frac{1}{q^2}\cdot\left[q^2+f^2+p^2-2\cdot f\cdot p\right]}=\frac{1}{2\cdot q}\cdot\int_0^{+\infty}\frac{\frac{1}{q}\cdot df}{\left[1+\left[\frac{f-p}{q}\right]^2\right]}=A_1$$

If we put $x=(f-p)/q$ then $dx=df/q$, and

$$A_1=\frac{1}{2\cdot q}\cdot\int_0^{+\infty}\frac{dx}{1+x^2}\Rightarrow\boxed{A_1=\frac{1}{2\cdot q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_2}{\omega_n}}}\qquad(29)$$

For the second term:

$$A_2=\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot\beta\cdot d\beta\Rightarrow$$

$$A_2=\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2\cdot\beta\cdot d\beta}{\underbrace{1}_{3}+\underbrace{\left(\frac{\omega}{\omega_n}\right)^4}_{2}\underbrace{-2\left(\frac{\omega}{\omega_n}\right)^2}_{4}+\underbrace{4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}_{4}+\underbrace{4\zeta^2}_{1}\underbrace{-4\zeta^2}_{3}\underbrace{+4\zeta^4}_{3}\underbrace{-4\zeta^4}_{1}}=$$

$$=\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2\cdot\beta\cdot d\beta}{\left[q^2+f^2+p^2-2\cdot f\cdot p\right]}=\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2\cdot\beta\cdot d\beta}{\left[q^2+\left(f-p\right)^2\right]}=$$

$$=\underbrace{\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2\cdot\beta\cdot d\beta}{\left[q^2+\left(f-p\right)^2\right]}-\int_0^{+\infty}\frac{4\zeta^2\cdot p\cdot\beta\cdot d\beta}{\left[q^2+\left(f-p\right)^2\right]}}_{A_3}+\underbrace{\int_0^{+\infty}\frac{4\zeta^2\cdot p\cdot\beta\cdot d\beta}{\left[q^2+\left(f-p\right)^2\right]}}_{A_4}=A_2$$

$$\boxed{A_4=4\zeta^2\cdot p\cdot A_1=\frac{4\zeta^2 p}{2\cdot q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_2}{\omega_n}}}\qquad(30)$$

we use the new variable: $x=(f-p)^2+q^2=f^2-2fp+p^2+q^2\rightarrow$ $\rightarrow dx=2fdf-2pdf\rightarrow$

$$A_3=\int_0^{+\infty}\frac{4\zeta^2\cdot\beta^2\cdot\overbrace{\beta\cdot d\beta}^{df/2}}{\left[q^2+\left(f-p\right)^2\right]}-\int_0^{+\infty}\frac{4\zeta^2\cdot p\cdot\overbrace{\beta\cdot d\beta}^{df/2}}{\left[q^2+\left(f-p\right)^2\right]}=\zeta^2\cdot\int_0^{+\infty}\frac{2f\cdot df}{\left[q^2+\left(f-p\right)^2\right]}-$$

$$-\zeta^2\cdot\int_0^{+\infty}\frac{2p\cdot df}{\left[q^2+\left(f-p\right)^2\right]}=\zeta^2\cdot\int_0^{+\infty}\frac{2f\cdot df-2p\cdot df}{\left[q^2+\left(f-p\right)^2\right]}=\zeta^2\cdot\int_0^{+\infty}\frac{dx}{x}\Rightarrow$$

$$\boxed{\Rightarrow A_3=\frac{1}{2}\cdot2\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]_0^{\frac{\omega_2}{\omega_n}}}\qquad(31)$$

The Integral (*Int.)* of eq. (26) is equal to the sum of A_1, A_3 and A_4:

$$Int=\frac{1}{2\cdot q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_2}{\omega_n}}+\frac{4\zeta^2 p}{2\cdot q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_2}{\omega_n}}+\frac{1}{2}\cdot2\cdot\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]_0^{\frac{\omega_2}{\omega_n}}=$$

$$=\frac{1+4\zeta^2 p}{2\cdot q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_2}{\omega_n}}+\frac{1}{2}\cdot2\cdot\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]_0^{\frac{\omega_2}{\omega_n}}$$

And the eq. (26) is transformed:

$$\sigma^2\left(\gamma\right)=\frac{AV^2\omega_n^2}{2\pi}\left[\frac{1+4\zeta^2 p}{q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_1}{\omega_n}}+2\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]_0^{\frac{\omega_1}{\omega_n}}\right]\qquad(32)$$

VIII. THE VARIANCE OF THE DYNAMIC COMPONENT OF THE LOAD DUE TO THE NON SUSPENDED MASSES

In reality, it is not necessary to integrate up to infinity because the real condition of the contact area between the wheel and the rail, which resembles an elliptic intersection area (Fig. 7), does not affect defects of very short wavelength.

If the length of the elliptical "*intersection area*" formed by the contact area is $2\lambda_1$, the integration has to stop at the wavelength of the defect, $\lambda = 2\lambda_1$, or for the excitation in the time domain, at frequency $\nu=(V/2\lambda_1)$ or cyclic frequency $\omega=(\pi V/\lambda_1)$.

So $\sigma(\gamma)$ is practically bounded, despite the presence of a logarithmic term.

We have to note that this term increases with speed, even when the contact area decreases, that is when the wheel diameter and the load decrease. These variations are negligible for $V \geq 140$ km/h, speeds for which we are interested in even more since this term is affected by $2\zeta^2$.

The same happens for the term *arctan*, which always tends to $(\pi/2)$ for $\omega=(\pi V/\lambda_1)$. This implies that the function $\sigma(\gamma)$ could be considered approximately as a function of the damping coefficient ζ increasing proportionally to it.

Eq. (32) can be calculated using example values of: (a) damping coefficient ζ of the track that varies between 0,2 – 0,7 ([22], [5]), here *we choose two values* $\zeta=0{,}30$ *and* $\zeta=0.35$, (b) ω_n/ω is finite and with a predominant value of 2/3 ([22], p. 36-37 and [5], p.30) as we calculate from equations: $\omega_n= =(h_{TRACK}/m_{NSM})^{(1/2)}, v=(V/2\lambda_1)\rightarrow\omega=2\pi v=(\pi V/\lambda_1)$, (c) m_{NSM}= 1,5 t for axle-load 20 t, (d) for a superstructure consisting of rail UIC60, concrete sleepers and fastenings W14 with pad Zw700 Saargummi the h_{TRACK} fluctuates from 60 to 90 kN/mm, here we choose 70 kN/mm, (e) λ_1=6 mm ([21] and (f) V=200 km/h:

$$\zeta=0{,}3 \quad f=13.465{,}5, \quad p=1-2\zeta^2=0{,}82, \quad q=\sqrt{4\zeta^2\cdot\left(1-\zeta^2\right)}=0{,}5724$$

$$\left[\left[\frac{1+4\zeta^2 p}{q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_1}{\omega_n}}\right]+2\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]\right]=3{,}5545+4{,}2518=7{,}81$$

And for ζ=0,35:

$$\left[\left[\frac{1+4\zeta^2 p}{q}\cdot\left[\arctan\frac{f-p}{q}\right]_0^{\frac{\omega_1}{\omega_n}}\right]+2\zeta^2\cdot\ln\left[\left(f-p\right)^2+q^2\right]\right]=3{,}2817+5{,}7871=9{,}07$$

The value of the term inside the brackets is dependent on the parameters selected above. For the Greek railway network this term becomes 8,44, the average of the two values.

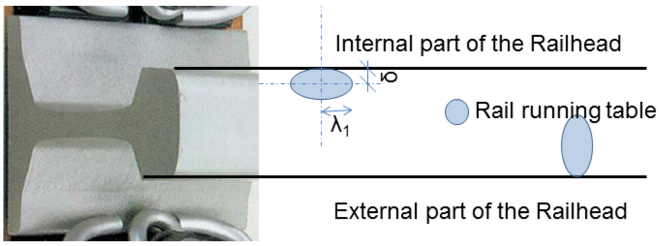

Figure 7 The rail running table bears the elliptical contact area in different positions. The ellipse has semi-axles λ_1, b and the total wavelength of the defect is $2\lambda_1$.

$$\sigma^2\left(\gamma\right)=8{,}44\cdot\frac{AV^2\omega_n^2}{2\pi}=1{,}3433\cdot AV^2\cdot\frac{h_{TRACK}}{m_{NSM}} \tag{33}$$

In order to calculate the variance of the dynamic component of the Load on the track panel due to the Non Suspended Masses the following equation is valid:

$$\sigma^2\left(\Delta Q_{NSM}\right)=m_{NSM}^2\cdot\sigma^2\left(\gamma\right)=\left(\underbrace{1{,}3433\cdot A}_{k_a^2}\right)\cdot V^2\cdot m_{NSM}\cdot h_{TRACK}\Rightarrow \tag{34a}$$

$$\Rightarrow\sigma\left(\Delta Q_{NSM}\right)=\underbrace{\sqrt{1{,}3433\cdot A}}_{k_a}\cdot V\cdot\sqrt{m_{NSM}\cdot h_{TRACK}}\Rightarrow$$
$$\Rightarrow\sigma\left(\Delta Q_{NSM}\right)=k_a\cdot V\cdot\sqrt{m_{NSM}\cdot h_{TRACK}} \tag{34b}$$

The standard deviation of the dynamic component of the Load due to Non Suspended (Unsprung) Masses and the Suspended (Sprung) Masses is given by the following equation [23]:

$$\sigma^2\left(\Delta Q_{dynamic}\right)=\sigma^2\left(\Delta Q_{NSM}\right)+\sigma^2\left(\Delta Q_{SM}\right)\Rightarrow$$
$$\Rightarrow\sigma\left(\Delta Q_{dynamic}\right)=\sqrt{\sigma^2\left(\Delta Q_{NSM}\right)+\sigma^2\left(\Delta Q_{SM}\right)} \tag{35}$$

The above equations give results in [N] Newtons, when V is in [m/sec], m_{NSM} in [kg] and h_{TRACK} in [N/m].

The standard deviation of the Suspended Masses is not calculated here but it is given by equation [6], [23], [24]:

$$\sigma\left(\Delta Q_{SM}\right)\approx\frac{V-40}{1000}\cdot NL\cdot Q_{wheel} \tag{36}$$

Where NL is the mean standard deviation of the longitudinal level condition of the track, on a 300 m length approximately, for both rails and the gauge ≈0,7 -1,5 mm [see [5], [11], p.335-336].

IX. The Variance Of The Dynamic Component And The Rail Defects

Equation (34b) can be transformed in order to give the values in [km/h], [t] of mass and [t/mm] for the dynamic stiffness of the track as in the following, since: *1 [N] = 1 [kg]·[m·sec*$^{-2}$*]*:

$$\sigma\left(\Delta Q_{NSM}\right)\left[N\right]=k_a\cdot V\left[\frac{m}{\sec}\right]\cdot\sqrt{m_{NSM}\left[kg\right]\cdot h_{TRACK}\left[\frac{N}{m}\right]}\Rightarrow$$
$$\Rightarrow\sigma\left(\Delta Q_{NSM}\right)\left[t\right]=\frac{k_a}{10000}\cdot V\left[\frac{km}{h}\right]\cdot\frac{1000}{3600}\cdot\sqrt{m_{NSM}\cdot1000\left[t\right]\cdot h_{TRACK}\cdot\frac{10000}{\frac{1}{1000}}\left[\frac{t}{mm}\right]}=$$
$$=\frac{k_a\cdot10^5}{10000\cdot3{,}6}\cdot V\sqrt{m_{NSM}\cdot h_{TRACK}}\Rightarrow$$

$$\sigma\left(\Delta Q_{NSM}\right)\left[t\right]=\frac{10}{3{,}6}\cdot k_a\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}\left[t\right]\cdot h_{TRACK}\left[\frac{t}{mm}\right]} \tag{37a}$$

Normally the units of the total dynamic stiffness of the track h_{TRACK}, are kN/mm. For this case we derive the following equation:

$$\sigma\left(\Delta Q_{NSM}\right)[kN]=\frac{k_a}{1000}\cdot V\left[\frac{km}{h}\right]\cdot\frac{1000}{3600}\cdot\sqrt{m_{NSM}\cdot 1000[t]\cdot h_{TRACK}\cdot\frac{1000}{\frac{1}{1000}}\left[\frac{kN}{mm}\right]}= \quad (37b)$$

$$=\frac{k_a\cdot\sqrt{10^9}}{1000\cdot 3,6}\cdot V\sqrt{m_{NSM}\cdot h_{TRACK}}=8,7841\cdot k_a\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}[t]\cdot h_{TRACK}\left[\frac{kN}{mm}\right]}$$

According to References [2] and [21], from measurements in the French railway network, $A=2\cdot 10^{-6}$. For this value the coefficient $k_a=\sqrt{(1,3433\cdot A)}=\sqrt{2,6866\cdot 10^{-6}}=1,639\cdot 10^{-3}$.

Equations (37) are transformed to:

$$\sigma\left(\Delta Q_{NSM}\right)[t]=\frac{0,4553}{100}\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}[t]\cdot h_{TRACK}\left[\frac{t}{mm}\right]} \quad (38a)$$

$$\sigma\left(\Delta Q_{NSM}\right)[kN]=\frac{1,440}{100}\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}[t]\cdot h_{TRACK}\left[\frac{kN}{mm}\right]} \quad (38b)$$

During the investigation performed by the author as head of the team of the Greek railways in collaboration with the French Railways' research department "Track", for the case of the cracks in the concrete sleepers, the measurements on the tracks of the French network were used in order to estimate the k_a coefficient ([21], [25], [26]). These tracks were used as basis with V_{max}=200 km/h, m_{NSM}=1,7804 t (including the track mass participating to the motion of the Non Suspended Masses –NSM– of the vehicle (see also Ref. [7], [8], [12])) and total dynamic stiffness coefficient of track 75 kN/mm.

$$\sigma\left(\Delta Q_{NSM}\right)[kN]=\underbrace{\frac{10\cdot 0,3\cdot k_1}{200\cdot\sqrt{1,7804\cdot 75}}}_{k_a}\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}[t]\cdot h_{TRACK}\left[\frac{kN}{mm}\right]}\Rightarrow$$

$$\sigma\left(\Delta Q_{NSM}\right)[kN]=\underbrace{k_1\cdot\sqrt{A_1}}_{k_a}\cdot V\left[\frac{km}{h}\right]\cdot\sqrt{m_{NSM}[t]\cdot h_{TRACK}\left[\frac{kN}{mm}\right]} \quad (39)$$

h_{TRACK} should also include the track mass participating in the motion of NSM, as described in the next paragraph X. The values of k_1 are depicted in Table 1 and $\sqrt{A_1}=1298,081124\cdot 10^{-6}$, for $\sigma(\Delta Q)$ in [kN]. The values of k_1 are dependent on the condition of the rail running table (ground or non-ground), that is the rail defects. In order to calculate coefficient k_a of eq. (34b) we should equate the two formulas, using the same units, i.e. [t] mass, [km/h] speed, and [kN/mm] stiffness:

$$k_a=\frac{10\cdot 0,3\cdot k_1}{200\cdot\sqrt{1,7804\cdot 75}}=\left(\sqrt{A_1}\right)\cdot k_1=1298,081124\cdot 10^{-6}\cdot k_1 \quad (40a)$$

X. TRACK MASS PARTICIPATING IN THE MOTION OF THE NON SUSPENDED MASSES OF THE VEHICLES

Besides the parts of the vehicles under the primary suspension, a part of the track-panel can also be considered as belonging to the Non Suspended Masses, since it is participating to the movement of the Non Suspended Masses of the vehicles ([7], [8]). At this point of the present paper, we cite the results of a theoretical calculation of the mass of the track, which participates in the movement of the Unsprung Masses of the railway vehicles.

The results are depicted in the Figs 8 and 9 below.

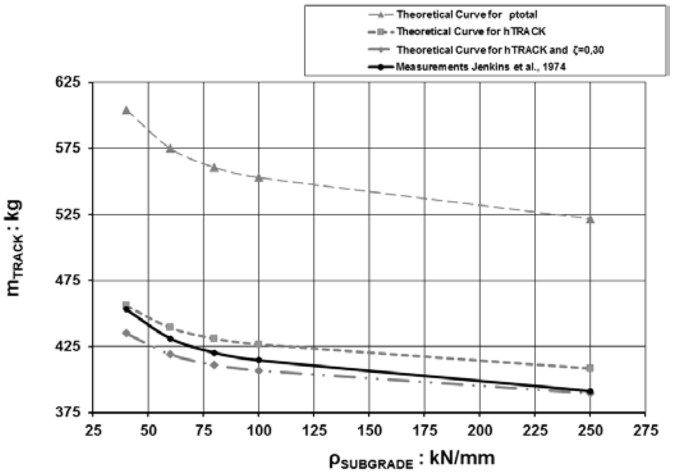

Figure 8 (see [7]) Comparison of theoretical results and measurements for the calculation of the track mass that participates to the motion of the Non Suspended Masses of the railway vehicles: (1) using the ρ_{total} of the track, (2) using the h_{TRACK} and (3) using the h_{TRACK} and coefficient of damping for the track ζ=0,30 in relation to the measurements in the British rail network (see [12]).

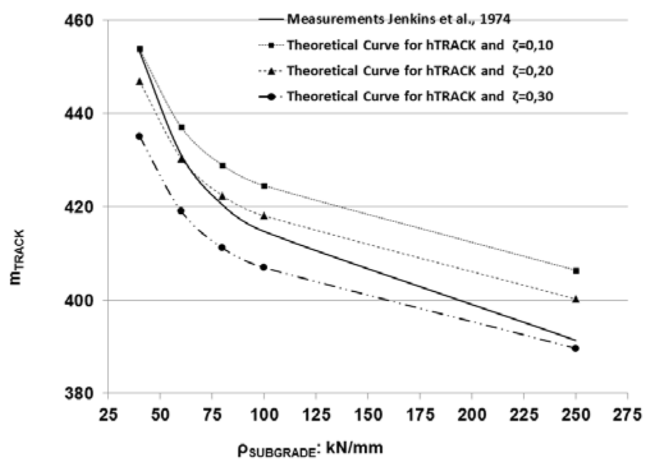

Figure 9 (see [8]) Comparison of theoretical results and measurements for the calculation of the track mass that participates in the motion of the Non Suspended Masses of the railway vehicles: (1) using the h_{TRACK} and damping coefficient of track ζ=0,10, (2) using the h_{TRACK} and damping coefficient of track ζ=0,2, (3) using the h_{TRACK} and damping coefficient of track ζ=0,30, in relation to the measurements (4) (see [12]) in the British Rail network.

These results of the theoretical approach are compared to measurements' results on track. The influence of the track damping coefficient -that appears in real conditions- is also taken into account. The track was modeled as a vibrating string and a theoretical solution was derived. The result of the theoretical calculation was compared to the measurements that were carried out in railway networks (see [12]). Three theoretical calculations are presented: (a) using the ρ_{total}, (b) using the h_{TRACK} and (c) using h_{TRACK} and the influence of damping coefficient (ζ). The influence of the stiffness of the fastenings was also investigated (see [7], [8]).

The Track Mass that participates in the motion of the Non Suspended Masses of the Railway Vehicle fluctuates from 390 to 455 kg approximately, dependent on the static stiffness coefficient of the subgrade.

The comparison of the theoretical prediction and the measured response is less than 4%, hence the proposed approach is very satisfactory.

XI. Coefficient A And Fluctuation Of k_a

Eq. (39) gives σ(ΔQ) in [t] for stiffness h_{TRACK} in [t/mm]:

Table 1: Values of k_1

	Ground rail running table			Rough rail running table (Non-ground)		
	min	medium	max	min	medium	max
k_1=	3	4,5	6	6	9	12

$$k_a = \frac{0,3 \cdot k_1}{200 \cdot \sqrt{1,7804 \cdot 7,5}} = \left(\sqrt{A_1}\right) \cdot k_1 = \underbrace{0,4104893 \cdot 10^{-3}}_{\sqrt{A_1}} \cdot k_1 \quad (40b)$$

The values of coefficient k_a are depicted in Table 2a, in order to calculate $\sigma(\Delta Q_{NSM})$ in [kN] with h_{TRACK} in [kN/mm] and in Table 2b for $\sigma(\Delta Q_{NSM})$ in [t] and h_{TRACK} in [t/mm]. m_{NSM} is in [t] and V in [km/h]. Coefficient k_a is dimensionless. For railway lines in very bad condition and, consequently, low operational speeds, the coefficient k_1 has been measured equal to 25, and k_a= $32,45 \cdot 10^{-3}$. The coefficient 0,01440 in eq. (38b) and 0,004553 in eq. (38b) give k_1= 11,09, divided by the relevant $\sqrt{A_1}$. For SI system [eq.(34b)]: $\sqrt{A_1}=0,147776 \cdot 10^{-3}$.

Table 2a: Values of k_a, for σ(ΔQ) in [kN]in eq. (34b)

	Ground rail running table			Rough rail running table (Non-ground)		
	min	medium	max	min	medium	max
k_a=	$3,89 \cdot 10^{-3}$	$5,84 \cdot 10^{-3}$	$7,79 \cdot 10^{-3}$	$7,79 \cdot 10^{-3}$	$11,68 \cdot 10^{-3}$	$15,58 \cdot 10^{-3}$

XII. Influence of the Defect's Ordinate on the Standard Deviation of the Dynamic Component

The mean square value or variance of the defects is given by eq. (23):

$$\sigma^2(z) = \frac{1}{\pi} \cdot \int_0^{+\infty} \frac{A}{(B+\Omega)^3} \cdot d\Omega = \frac{A}{\pi} \cdot \int_0^{+\infty} \frac{1}{(x)^3} \cdot dx = -\frac{A}{2\pi} \left[\frac{1}{(B+\Omega)^2} \right]_0^{+\infty}$$

and since we are dealing with the defects of short wavelength (undulatory wear) $\lambda_{max} \approx 3\text{-}4$ m, term B can be omitted, hence: $\Omega = \frac{2\pi}{\lambda}, \quad \lambda = 3m$. The calculation of the ordinate n_1 of the vertical defect depends on the probability of occurrence. The defect is on the rail running table with ordinate 2σ(z) and we should cover a 99,7% probability, consequently, n_1 is equal to three times the (two times) standard deviation (see Fig. 10):

$$n = 3 \cdot \left[2 \cdot \sigma(z) \right]$$

$$\sigma^2(z) = \left| -\frac{A}{2\pi} \left[\frac{1}{(B+\Omega)^2} \right]_0^{+\infty} \right| = \frac{A}{2\pi} \cdot \frac{1}{\Omega^2} = \frac{A}{2\pi} \cdot \frac{\lambda^2}{(2\pi)^2} \Rightarrow \sigma(z) = \frac{\lambda}{2\pi} \cdot \sqrt{\frac{A}{2\pi}}$$

Where from $(eq.(34b)) \Rightarrow A = \frac{k_a^2}{1,3433} = \frac{k_1^2 \cdot \left(\sqrt{A_1}\right)^2}{1,3433} \Rightarrow$

$$n_1 = 3 \cdot 2 \cdot \left(\frac{\lambda}{2\pi} \cdot \sqrt{\frac{A}{2\pi}} \right) = 3 \cdot \frac{2 \cdot 3}{2\pi} \cdot \left(\sqrt{\frac{k_1^2 \cdot \left(\sqrt{A_1}\right)^2}{1,3433 \cdot 2\pi}} \right) = k_1 \cdot \sqrt{A_1} \cdot 0,986[m] \quad (41)$$

For k_1=1 (SI system), n_1 corresponds to an ordinate of defect on the rail running table of 0,15 mm measured from a 3 m cord ($\lambda=2\lambda_1$=3m). For 2 times the standard deviation n_1=0,10 mm.

We conclude that the ordinate of the defect fluctuates from 0,44 mm till 1,75 mm for a 3 m cord with a probability of occurrence 99,7% and from 0,29 till 1,17 mm for a probability 95,5%. Eq. (39) can be related with the ordinate of the defect n_1 from eq. (41), in units of SI system:

$$\sigma(\Delta Q_{NSM}) = \underbrace{k_1 \cdot \sqrt{A_1}}_{k_a} \cdot \frac{\sqrt{A_1} \cdot 0,986}{\underbrace{0,15}_{\sqrt{A_1} \cdot 0,986}} \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} = \sqrt{A_1} \cdot \frac{n_1}{0,15} \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \quad (42)$$

where n_1 in [mm] the ordinate of the defect measured from a 3 m cord on two consecutive peaks of the defect, V the speed in [km/h], m_{NSM} the Non Suspended Mass of the vehicle and h_{TRACK} the total dynamic stiffness coefficient of track and $\sqrt{A_1}=0,147776 \cdot 10^{-3}$.

Table 2b: Values of k_a, for σ(ΔQ) in [t] in eq. (34b)

	Ground rail running table			Rough rail running table (Non-ground)		
	min	medium	max	min	medium	max
k_a=	$1,23 \cdot 10^{-3}$	$1,85 \cdot 10^{-3}$	$2,46 \cdot 10^{-3}$	$2,46 \cdot 10^{-3}$	$3,69 \cdot 10^{-3}$	$4,93 \cdot 10^{-3}$

We calculate the case of 1 mm ordinate of defect for 99,7% probability of occurrence (eq. 39, 40a). In SI system:

$$k_a = 1000 \cdot \frac{10 \cdot 0,3 \cdot k_1}{\frac{200}{3,6} \cdot \sqrt{1780,4 \cdot 75000000}} = 0,147776 \cdot 10^{-3} \cdot k_1$$

and from eq. (41): k_1=6,86. For V=200 km/h=55,56 m/sec, $h_{TRACK}=80 \cdot 10^6$ N/m and m_{NSM}=1500 kg, then $\sigma(\Delta Q)$ = 32.96 kN or 3,296 t for 99,7% probability of occurrence. From eq. (42) we can calculate the standard deviation of the dynamic component of the Load for each mm of ordinate of the defect depending on the condition of the rail running table (k_1), V, m_{NSM} and h_{TRACK}. In the case of undulatory wear and of speed approaching the resonance limit, the dynamic response factor for the respective resonance frequencies of the acceleration is $1/(2\zeta\sqrt{(1-\zeta^2)})$ (see Ref. [27], p.83). If we elaborate eq. (38a), equating the ω to ω_n and considering three standard deviations $3\sigma(z)$ that is a probability of occurrence 95,5%:

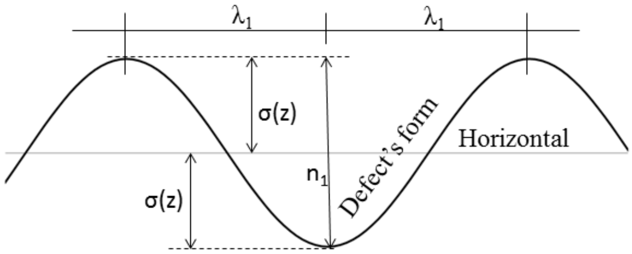

Figure 10 A defect (fault) on the rail running table, wavelength $\lambda=2\lambda_1$ and standard deviation $\sigma(z)$ of the defect. The total ordinate of the fault $n_1=2\cdot\sigma(z)$.

$$T = 2\pi \cdot \sqrt{\frac{m_{NSM}}{h_{TRACK}}} = 2\pi \cdot \sqrt{\frac{1,5[t]}{8000\left[\frac{t}{m}\right] \cdot 9,81}} = 0,00437186 \Rightarrow \nu = 228,7357 Hz$$

The critical speed for resonance is measured for ω_1/ω_n=2/3 (see Ref. [22], p.36, 37):

$$V_{critical} = \frac{3}{2} \cdot \frac{\lambda}{T} = \frac{3}{2} \cdot \frac{\lambda}{0,00437186} = 343,10348 \cdot \underbrace{\lambda}_{\lambda=0,162m} \Rightarrow V_{critical} = 55,56 m/\sec$$

$$\sigma(\Delta Q_{NSM})[N] = \sqrt{A_1} \cdot \frac{n_1}{0,15} \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \cdot \frac{1}{2\zeta \cdot \sqrt{1-\zeta^2}} =$$

$$= 0,147776 \cdot 10^{-3} \cdot \underbrace{343,10348 \cdot 0,162}_{V} \cdot \frac{n_1}{4} \cdot \frac{h_{TRACK}}{\zeta} \cdot \underbrace{\sqrt{\frac{m_{NSM}}{h_{TRACK}}}}_{0,137} \cdot \frac{2}{0,15 \cdot \underbrace{\sqrt{1-\zeta^2}}_{\sim 0,95}} =$$

$\approx 0,85 \cdot \frac{n_1}{4} \cdot \frac{h_{TRACK}}{\zeta}$, for h_{TRACK}=8 t/mm and ζ=0,3 then ΔQ=5,66 t per mm of defect ordinate for 99,7% probability of occurrence, in the resonance area of wavelengths.

XIII. CONCLUSIONS

The calculation of the Loads exerted on track by the railway vehicles, is of decisive importance, for the maintenance and the Life Cycle of the Track itself.

The total acting Load is consistent of (a) the static load (static load of vehicle axle), as given by the rolling stock's producer, (b) the semi-static load (cant/ superelevation deficiency at curves, which results in non-compensated lateral acceleration), (c) the load from the Non-Suspended Masses of the vehicle (the masses that are not damped by any suspension, because they are under the primary suspension of the vehicle) and (d) the load from the Suspended Masses of the vehicle, that is a damped force component of the total action on the railway track.

The static and semi-static components of the Load could be easily calculated through simple mathematical operations since they are based in simple Statics.

The calculation of the dynamic component of the Load is approached through a probabilistic analysis since it is based on a random phenomenon analysis. The dynamic component consists of (a) one part due to the Non Suspended (Unsprung) Masses of the vehicle situated under the primary suspension of the vehicle, (b) one part due to a section of Track Mass participating in the motion of the Non Suspended Masses of the vehicle and (c) one part due to the Suspended Masses of the Vehicle situated over the primary suspension of the vehicle.

In this paper the second order differential equation of motion of the Unsprung (Non Suspended) Masses of a railway vehicle rolling on the track is investigated. The investigation is performed through the Fourier transform and the solution is verified with findings from a research program performed by the Greek railways in collaboration with the French railways. Finally an equation for the calculation of the dynamic component of the Load due to the Non Suspended Masses of the vehicle and due to the section of the Track Mass that participates in the motion, is proposed.

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