Solutions of Beams, Frames and 3D Structures on Elastic Foundation Using FEM

Karel FRYDRÝŠEK, Roland JANČO, Horst GONDEK

Abstract—This paper contains numerical methods and approaches used in the solution of plane beams and frames and 3D structures on an elastic foundation. In the first case, the solution uses beam element BEAM54 in the program ANSYS and the derivation of the stiffness matrix for this element is presented. The second approach uses a beam element in a combination with a contact element with the description of the derivative of the stiffness matrix applied for the frame on elastic foundation. Both solutions are compared with theoretical solution. The influence of the number of divisions for the beam element on the accuracy of the solution is shown. There are also presented some other application of structures on elastic foundation (biomechanics & traumatology – external fixators for treatment of complicated bone fractures, mining industry - pressure distributions in the contact between mining supports and foot-wall, rack-railway and drop-in test as a problem of 3D body on elastic foundation).

Keywords—Elastic foundation, Finite Element Method, beam element, structures, contact element, theory, applications, biomechanics, traumatology, external fixators, mining, mining supports, rack-railway, drop-in test

I. Introduction

SOLUTION of frames and beams on elastic foundation often occur in many practical cases for example, solution of building frames and constructions, buried gas pipeline systems and in design of railway tracks for railway transport, etc., see Fig.1.

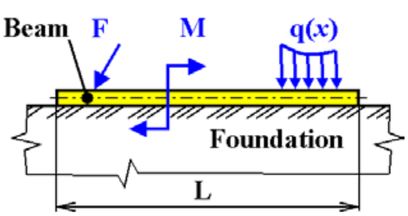

Fig. 1 Example of a beam resting on an elastic foundation. This beam is loaded by force F, couple M and distributed loading q.

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Solution of beam on elastic foundation is a statically indeterminate problem of mechanics. In this case, we have the beam with elastic foundation along the whole length and width or only over some part of the length or width. Detailed explanation of theoretical solution can be found in [2], [3] [4] and [10]. Not all problems can be solved by theoretical approach (i.e. sometimes, the theoretical solution is very complicated). In solution of these complicated problems, the Finite Elements Method (FEM) can be applied. In this paper, FEM is applied for the solution of 2D and 3D beams, frames structures on elastic Winkler's foundation including theory and practice.

II. Theoretical Background for 2D Beam on Elastic Foundation

The Winkler's foundation model is easy to formulate using energy concepts. The analysis of bending of beams on an elastic foundation (Winkler's model) is developed on the assumption that:

- The strains are small.
- The resisting pressure $p_R = K v$ /Nm^{-2}/ in the foundation are proportional at every point to the deflection $v = v(x)$ /m/ normal to its surface at that point. Displacement and resting pressure etc. can be expressed as functions of variable x /m/. The parameter $K = K(x)$ /Nm^{-3}/ is the modulus of the foundation.
- The surrounding foundation is utterly unaffected, see Fig.2a.

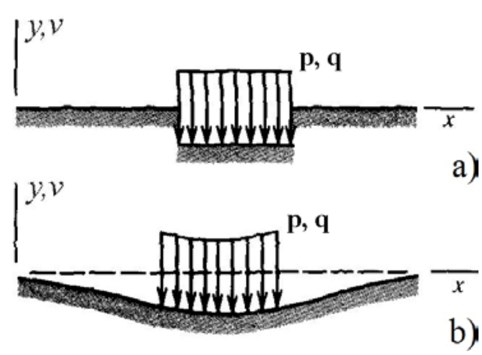

Fig. 2 Deflection of structure on elastic foundation under pressure p or distributed loading q,
(a) Winkler foundation, (b) elastic solid foundation

- The general problem of the beam on elastic foundation (Winkler's theory) is described by ordinary differential equation. In the most situations, the influences of normal force, shear force, distributed moment and temperature can be neglected (or the beam is not exposed to them). Hence

$$\frac{d^4v}{dx^4}+\frac{k}{EJ}v=\frac{\mathrm{q}}{EI}, \tag{1}$$

where $k=k(x)$ /Nm^{-2}/ is the foundation stiffness and EI /Nm^2/ is the bending stiffness.

An area dA /m^2/ of the foundation surface acts like a linear spring of stiffness k. Hence

$$k=\mathrm{p_R}\; dA/v=K\,v\,dA/v=K\,dA. \tag{2}$$

According to the theory of elasticity, the strain energy $\mathrm{U_R}$ /N/ in a linear spring is given by eq.

$$\mathrm{U_R}=k\,v^2/2. \tag{3}$$

III. 2D Beam on Elastic Foundation (First Method)

Now considering a structural element, perhaps a plate bending element or one face of a 3D solid element, which has an area A in a contact with the foundation. Lateral deflection of area A normal to the foundation, is $v=[\mathbf{N_f}]\{\mathbf{d_f}\}$ /m/, where $\{\mathbf{d_f}\}$ /m/ contains D.O.F. of element nodes in contact with foundation. Strain energy U /N/ in foundation over area is

$$U=\frac{1}{2}\int Kv^2dA=\frac{1}{2}\{\mathbf{d_f}\}^T[\mathbf{k_f}]\{\mathbf{d_f}\}, \tag{4}$$

in which the Winkler's foundation stiffness matrix for the element is

$$[\mathbf{K_f}]=\int K[\mathbf{N_f}]^T[\mathbf{N_f}]dA. \tag{5}$$

For example, if the problem deals with a beam on Winkler's foundation, $[\mathbf{N_f}]$ is identical to the shape function matrix $[\mathbf{N}]$ of the beam, where individual functions N_i are

$$N_1=1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3},\; N_2=x-\frac{2x^2}{L}+\frac{x^3}{L^2} \tag{6}$$

$$N_3=\frac{3x^2}{L^2}-\frac{2x^3}{L^3},\; N_4=-\frac{x^2}{L}+\frac{x^3}{L^2}, \tag{7}$$

where $dA=b\,dx$ /m^2/ and b /m/ is the width of the beam face in a contact with the foundation and L /m/ is length of the beam. We input equations (6) and (7) into (5), and get foundation stiffness matrix

$$[\mathbf{K_f}]=\begin{bmatrix}\frac{13bLK}{35} & \frac{11bL^2K}{210} & \frac{9bLK}{70} & -\frac{13bL^2K}{420}\\ \frac{11bL^2K}{210} & \frac{bL^3K}{105} & \frac{13bL^2K}{420} & -\frac{bL^3K}{140}\\ \frac{9bLK}{70} & \frac{13bL^2K}{420} & \frac{13bLK}{35} & -\frac{11bL^2K}{210}\\ -\frac{13bL^2K}{420} & -\frac{bL^3K}{140} & -\frac{11bL^2K}{210} & \frac{bL^3K}{105}\end{bmatrix}. \tag{8}$$

The stiffness matrix of beam without shear deformation can obtain the formal approach using equation

$$[\mathbf{K_b}]=\int \mathbf{B^T}EI\;\mathbf{B}\;dx, \tag{9}$$

where $\mathbf{B}$ is the strain-displacement matrix, which is defined for beam by $\mathbf{B}=\frac{d^2\mathbf{N}}{dx^2}$.

After mathematical solution of equation (9) using equations (6) and (7), we obtain the stiffness matrix for beam considering only bending moment and transversal load at the nodes

$$[\mathbf{K_b}]=\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}, \tag{10}$$

where E /Nm^{-2}/ is modulus of elasticity.

IV. 2D Beam on Elastic Foundation (Second Method)

The second method of solving beams and frames is by using beam and contact element. Mechanical contact will be simulated by spring element between rigid ground and beam as shown in the Fig. 3.

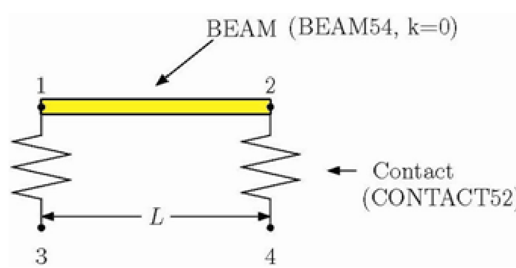

Fig. 3 Beam and contact element, see [10]

Stiffness matrix for spring element is as follows

$$[\mathbf{K}_{\text{spring}}]=\begin{bmatrix}C & -C\\ -C & C\end{bmatrix}, \tag{11}$$

where C /Nm^{-1}/is spring stiffness.

Global stiffness matrix for beam and spring element is given by combining eqn. (10) and eqn. (11), which is

$$[\mathbf{K_{bc}}]=\frac{EI}{L^3}\begin{bmatrix} 12+C_M & 6L & -12 & 6L & -C_M & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+C_M & -6L & 0 & -C_M \\ 6L & 2L^2 & -6L & 4L^2 & 0 & 0 \\ -C_M & 0 & 0 & 0 & C_M & 0 \\ 0 & 0 & -C_M & 0 & 0 & C_M \end{bmatrix}, \tag{12}$$

where C_M /1/ is the modified stiffness of spring solved by the following equation

$$C_M=\frac{L^3}{EI}C\,. \tag{13}$$

V. Program implementation (Second method)

For numerical solution we use the program environment ANSYS, which includes a special 2D element. This element is BEAM54. Properties and the characteristics of a cross-sectional area are entered in the real constants. In Fig. 4, the FEM model shows the numbering of nodes and elements.

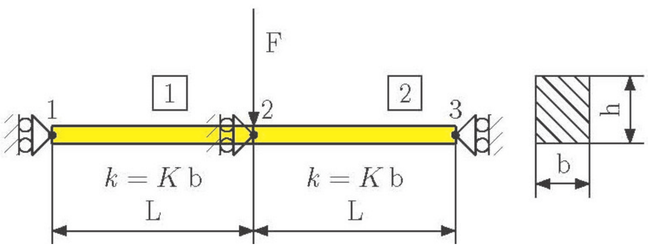

Fig. 4 FE model using BEAM54, see [10]

The theoretical solution of this model using the principle of FEM can be entered using the eqn. (8), eqn. (10) and boundary conditions. Boundary conditions are as follows: the displacement of all nodes in the direction of x is equal to zero. At point number two the force F in the y direction is applied. Theoretical solution written in matrix form is as follows

$$\begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & 0 & 0 \\ K_{1,2} & K_{2,2} & -K_{1,4} & K_{2,4} & 0 & 0 \\ K_{1,3} & -K_{1,4} & 2K_{1,1} & 0 & K_{1,3} & K_{1,4} \\ K_{1,4} & K_{2,4} & 0 & 2K_{2,2} & -K_{1,4} & K_{2,4} \\ 0 & 0 & K_{1,3} & -K_{1,4} & K_{1,1} & -K_{1,2} \\ 0 & 0 & K_{1,4} & K_{2,4} & -K_{1,2} & K_{2,2} \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ -F \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{14}$$

where

$$K_{1,1}=\frac{12EI}{L^3}+\frac{13bLK}{35}\,,\quad K_{1,2}=\frac{6EI}{L^2}+\frac{11bL^2K}{210}\,,$$

$$K_{1,3}=-\frac{12EI}{L^3}+\frac{9bLK}{70}\,,\quad K_{1,4}=\frac{6EI}{L^2}-\frac{13bL^2K}{420}\,,$$

$$K_{2,2}=\frac{4EI}{L}+\frac{bL^3K}{105}\,,\quad K_{2,4}=\frac{2EI}{L}-\frac{bL^3K}{140}\quad \text{and}\quad K=\frac{k}{b}\,.$$

VI. Verification of numerical solution

According to the foregoing chapters, let us consider the beam on elastic foundation shown in Fig. 5, where the length of one-half of beam is L =1.8 m. Beam is made of steel, which has the Young's modulus $E=2\text{x}10^5$ MPa with rectangular cross-section area by parameters $b=200$ mm and $h=400$ mm, see Fig. 4, and foundation modulus K = 10^8 Nm^{-3}.

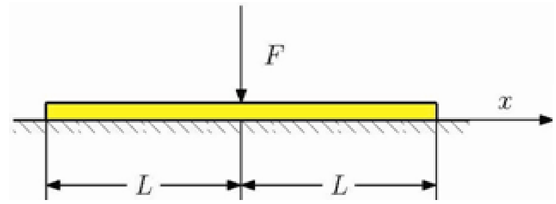

Fig. 5 Beam on elastic foundation (solved example)

There are two approaches for the numerical solution of this beam. The first approach is using the BEAM54 element in ANSYS sw, see reference [5]. This approach can be used when the foundation is without compression resistance.

If we consider compression resistance, we have to apply the approach using the contact element, for example CONTACT52, where compression resistance is prescribed by a gap. In our example, the gap is equal to zero. Because the program ANSYS contains beam elements with shear deformation, only BEAM54 is without shear deformation. For verification of mechanical contact, the element BEAM54 was considered and stiffness of elastic foundation is equal to zero.

Of course the accuracy of the result for FEM is influenced by the number of elements over the length of beam. The verification examples used only one element over the length of beam L. Influences of the number of divisions in both approaches are illustrated in the Fig. 6 (deflection) and Fig. 7 (bending moment).

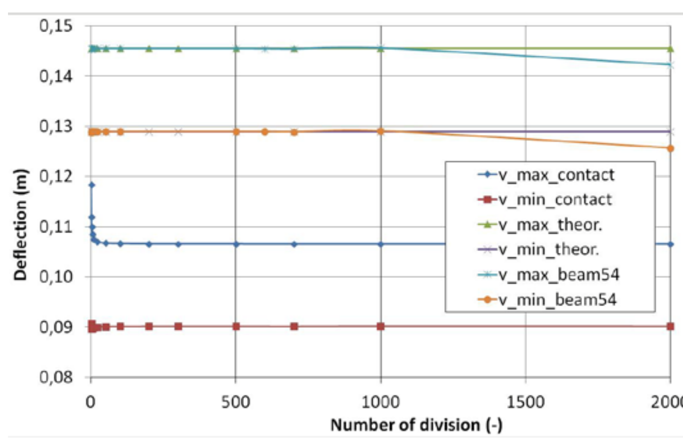

Fig. 6 The influence of number of element divisions along the length L on the minimum and maximum deflection, see [10]

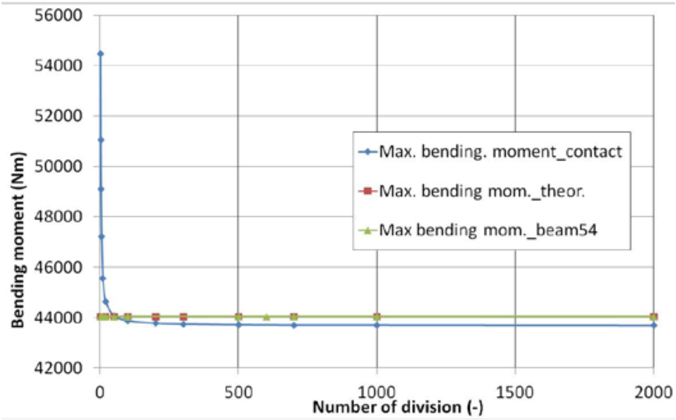

Fig. 7 The influence of the number of elements division along the length L on the bending moment, see [10]

The theoretical solution of beam on elastic foundation in Fig. 4. is described in details in reference [2, chapter 9] or [8]. From the given parameters theoretical solution is following: the maximum deflection of beam is 0.00145526 m at the distance $x = 0$ m, minimum deflection is 0.00128961 m at the distance $x = \mathrm{L}$, maximum bending moment is $Mo(x = 0 \text{ m}) = 44045.7$ Nm and minimum bending moment is $Mo(x = \mathrm{L} = 1.8 \text{ m}) = 0$ Nm. Maximum shear (transversal) force is $T(x = 0 \text{ m}) = -50000$ N. Minimum shear force is $T(x = \mathrm{L} = 1.8 \text{ m}) = 0$ N.

VII. Applications of 2D Structures on Elastic Foundation – Special Cases

In Fig. 8, see [3], [4] and [17], there is solved beam of length L /m/ with free ends. The beam is exposed to one vertical force F /N/. Modulus of the foundation is given by linear function $K(x) = K_0 + K_1 x$.

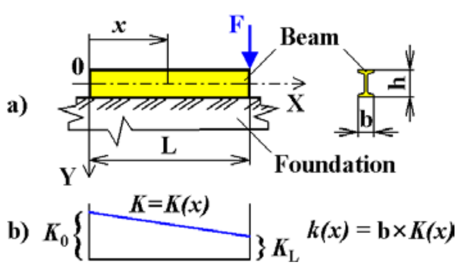

Fig. 8 Solved beam on elastic variable foundation

The approximate solution $v = v(x)$ can be found in the form of polynomial function of 6th order. Hence, the approximate results (i.e. functions of displacement v, slope, bending moment and shear force of the beam) can be derived.

This example is solved via probabilistic approach by Simulation-Based Reliability Assessment (SBRA) Method (stochastic mechanics, direct Monte Carlo approach, i.e. all inputs are given by bounded histograms, AntHill software, for example see Fig. 9) which is the modern and new trend of the solution in mechanics, see [3], [4], [9], [11], [12], [16] and [17].

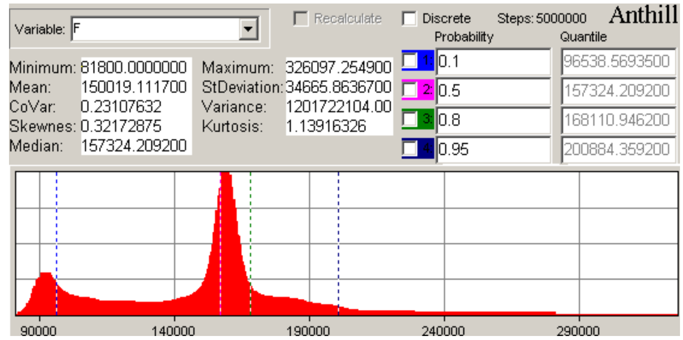

Fig. 9 Histogram of input parameter $\mathrm{F} = 157324.2^{+168773.1}_{-75524.2}$ N

Results parameters (i.e. stiffness of the foundation $k(x)$, see Fig. 10, displacement $v(x)$, maximal bending stress σ_{MAX}, see Fig. 11, factor of safety $F_S = R_e - \sigma_{\mathrm{MAX}}$ etc.) were calculated for 5×10^6 Monte Carlo simulations.

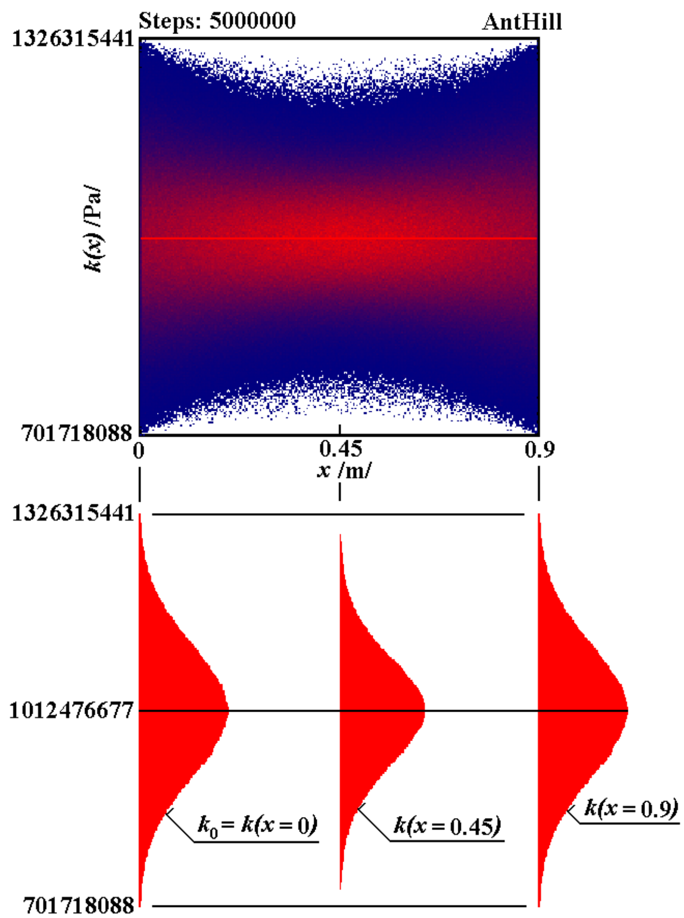

Fig. 10 2D histogram and its sections for output parameter *k(x)* /Pa/

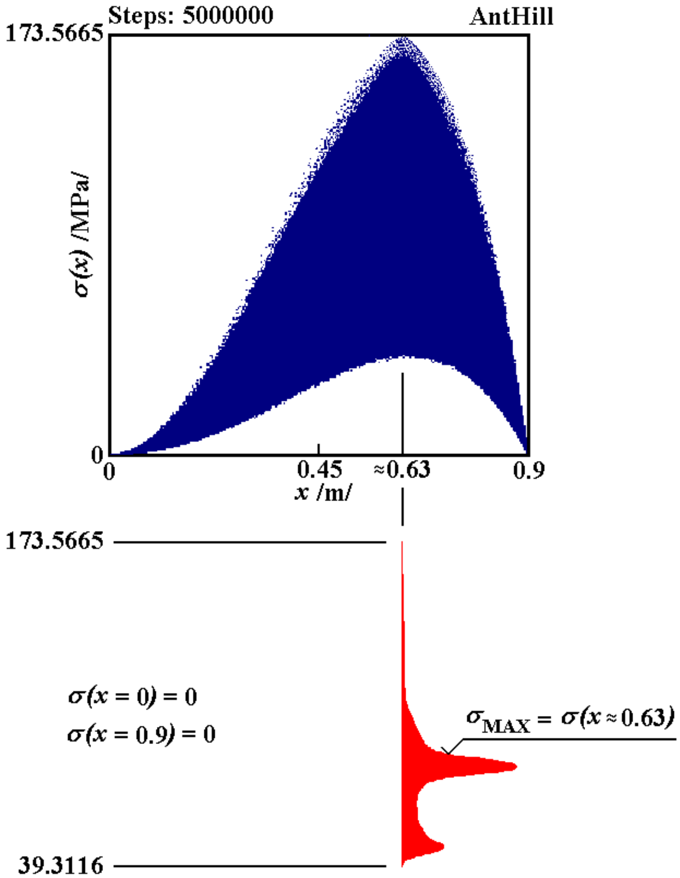

Fig. 11 2D histogram and its section for output parameter $\sigma(x)$ /MPa/

Some main results of stochastic quantity are plotted by histogram in Fig. 12 (distribution of yield stress R_e versus maximum stress σ_{MAX}). Hence, the probability that the plastic deformations occurs in the beam is 0.094%. For more information see [3] and [17].

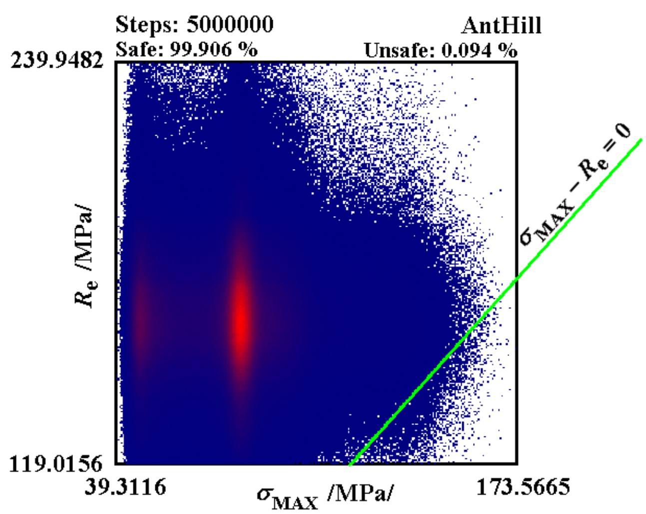

Fig. 12 2D histogram of output parameters (calculation of F_S - probabilistic reliability assessment)

For more applications, examples and information, see [2], [3], [4], [10], [12], [16] and [17].

VIII. Application of 3D Structures on Elastic Foundation – Special Cases

There are a lot of applications of 3D structures rested on elastic foundation, for example see references [2], [3], [4], [8] and [10]:

- Applications in biomechanics & traumatology (i.e. FE solutions and design of new external fixators for treatment of complicated fractures of pelvis and its acetabulum), see Fig. 13, 14, 15 and references [6], [7], [14] and [15].

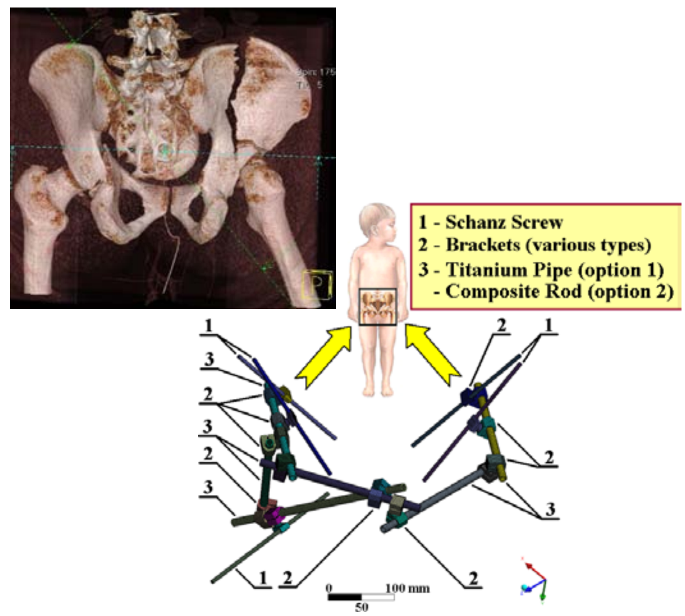

Fig. 13 Fracture of pelvis (a) anteroposterior radiograph - transverse with posterior wall acetabular fracture, (b) application of external fixator for treatment - two designs “Option 1” and “Option 2”

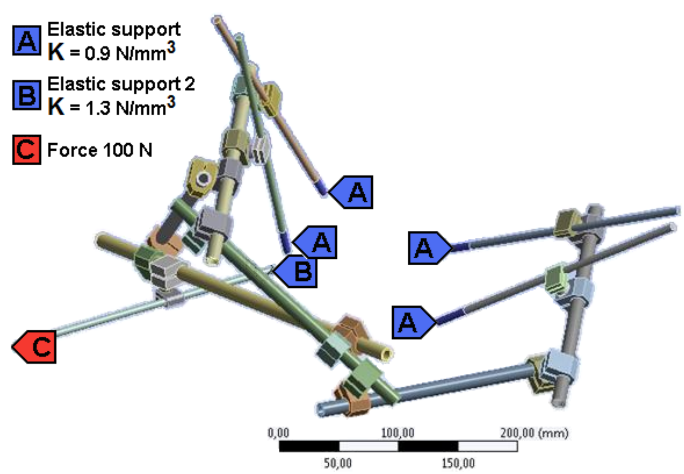

Fig. 14 External fixator for treatment of pelvis and its acetabulum - FE model, boundary conditions (A, B is applied elastic foundation, Ansys sw)

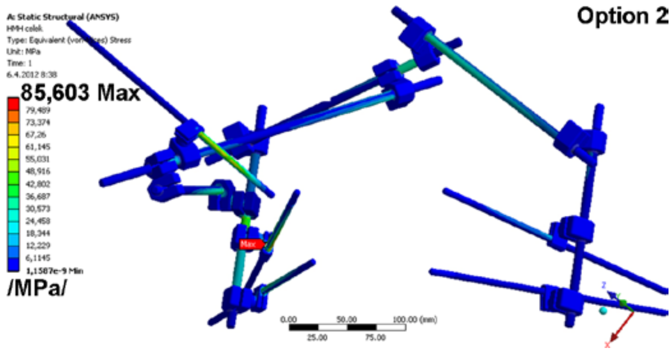

Fig. 15 "Option 2" – Finite Element modelling of the external fixator for treatment of pelvis and its acetabulum (equivalent von Mises stresses /MPa/ for tensile loading 100 N)

- Applications in mining (i.e. FE solutions for pressure distributions in the mechanical contact between mining supports and foot-wall as a problem of 3D body on elastic foundation), see Fig. 16 to 20 and references [4], [10] and [16].

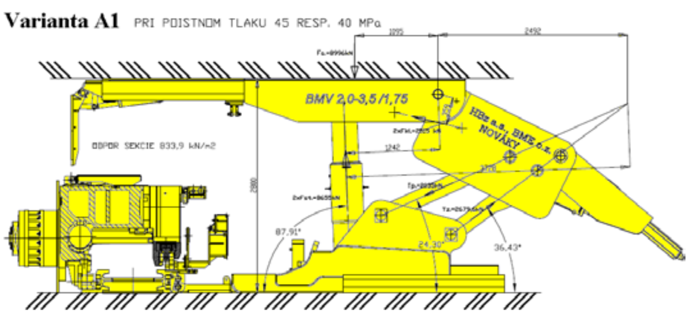

Fig. 16 Mechanical contact between mining supports and foot-wall approximated via elastic foundation

Fig. 17 Mechanical contact between mining supports and foot-wall approximated via elastic foundation (total displacement, MSC.MARC/MENTAT sw)

Fig. 18 Rack-railway track in mines

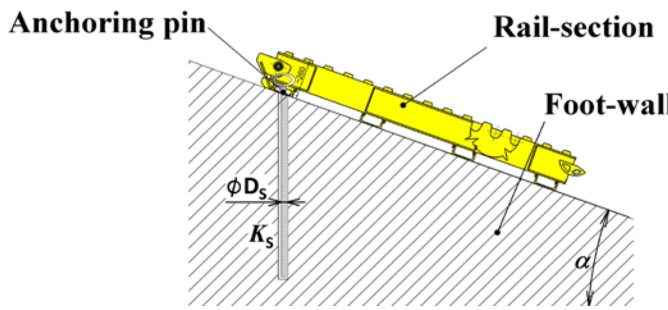

Fig. 19 Problem - Rack-railway track in mines

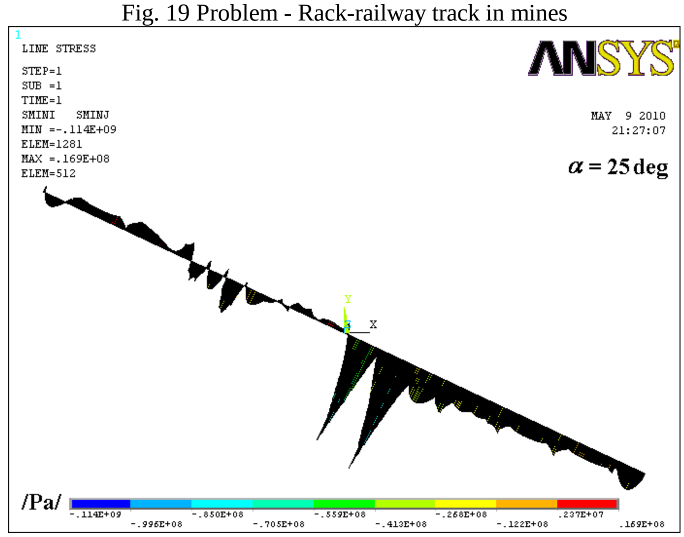

Fig. 20 Stresses in the rails and anchor pins of rack-railway track in mines (ANSYS sw)

- Other applications, see references [2], [3], [4], [8] to [11] and for example Fig. 21.

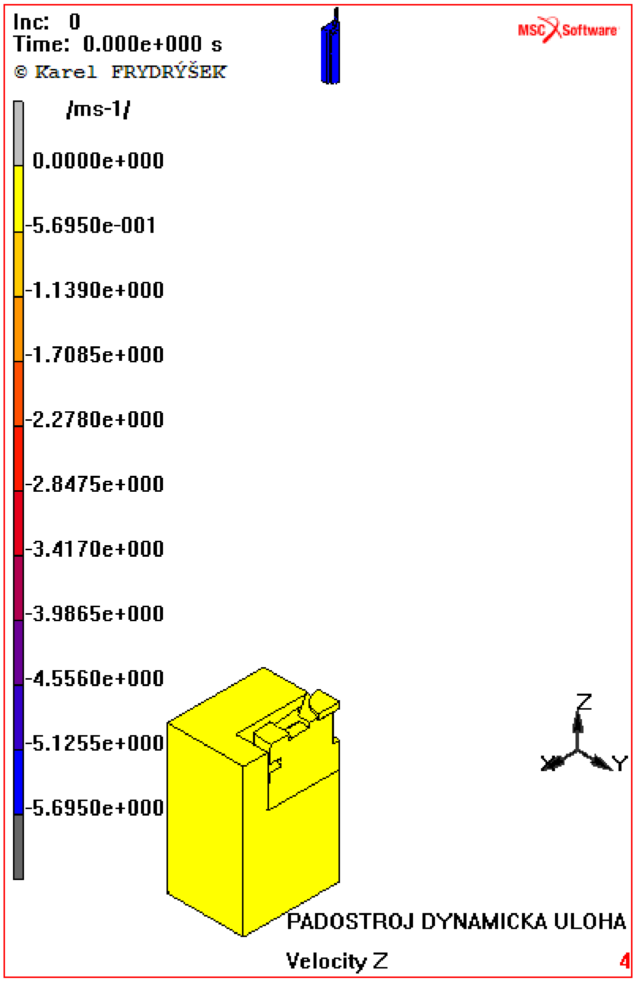

Fig. 21 Calculations of dynamic forces in the foundations of a massive drop-tester during impact tests (3D body on elastic foundation, MSC.MARC/MENTAT sw)

IX. Conclusion

General solutions of FEM applications (theory and practice) for the plane beam structures on elastic (Winkler's) foundations were derived, tested and discussed (two ways). The authors put emphasis on derivation of matrices used in FEM.

Other own examples (reports), such as applications in mining and biomechanics, dynamics, drop-in tester etc. and references are mentioned.

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