Analysis of seepage with nonlinear permeability using least square finite element and finite difference methods

M. M. Toufigh, M. H. Bagheripur, A. Bazrafshan M.

***Abstract*—** Free-surface seepage problems have been attracting interests of many engineers and mathematicians due to the strong non-linearity as well as the importance in designing the hydraulic structures, such as embankments, canals, and earth and rock-fill dams. Free-surface seepage flow is governed by an elliptic partial differential equation when steady state flow conditions have been considered. In most geotechnical analyses, soil properties are assumed to be spatially and temporally invariant and thus, average property values are used. In reality, however, these soil parameters usually vary from point to point (heterogeneous) and even at one point they may have different values in various measured directions (anisotropy). Moreover, these parameters may vary in time while a geotechnical process is occurring due to an external influence such as surface pressure or due to the change of chemical compositions. Therefore in this research, the coefficients of permeability are assumed to vary in terms of geometry, external load influences and the effect of head variation in the system and the resulted nonlinear seepage problem is solved using Least Square Finite Element Method and Finite Difference Method. The seepage Problem is analyzed for two cases of variable and constant coefficients of permeability. The effect of a variable coefficient of permeability may not be significant on small dams, but as the height of the dam increases, the effect becomes more considerable. It is believed that a variable permeability analysis such as the one described in this paper should be taken into account.

***Keywords*—** Anisotropy, Consolidation, Nonlinearity, Seepage.

I. Introduction

Free-surface seepage problems have been attracting interests of many engineers and mathematicians due to the strong non-linearity as well as the importance in designing the hydraulic structures, such as embankments, canals, and earth and rock-fill dams. Determination of free-surface profile, velocity, and pressure distributions is may be main purpose of free-surface seepage analysis. Free-surface seepage flow is governed by an elliptic partial differential equation when steady state flow conditions have been considered. Solution of this elliptic partial differential equation may be carried out analytically and numerically. Analytical solutions of this equation require several assumptions such as ideal solution domains and homogeneous material properties. On the other hand, numerical solutions have to be used if the solution domain has complicated geometry and/or inhomogeneous material properties. Among the numerical solution techniques finite element method and finite difference method are perhaps the most popular. In most geotechnical analyses, soil properties are assumed to be spatially and temporally invariant and thus, average property values are used. In reality, however, these soil parameters usually vary from point to point (heterogeneous) and even at one point they may have different values in various measured directions (anisotropy). Moreover, these parameters may vary in time while a geotechnical process is occurring due to an external influence such as surface pressure or due to the change of chemical compositions. For computations in flow problems using numerical techniques usually homogeneous conditions are assumed for the coefficient of permeabilities and then anisotropic conditions are assumed throughout. In this research, the coefficients of permeability are supposed to vary in term of geometry, external load influences such as those causing consolidation effects, and the effect of head variation in the system where seepage is taking place. In order to define these variations, two conditions are presented in this paper. The first condition can be explained by, for instance, an embankment load over a confined saturated fine grain soil layer. This load would begin to consolidate underlying materials. At the end of consolidation process, the permeability of the materials are changed and can be described by a governing differential equation, which can then be solved. In addition to the first case, a second case can be defined in which variations of the head can also have an effect on the consolidation process resulting in permeability variations. This effect can be seen in Terzaghi's effective stress equation. The influence of head variation is introduced by a defined function, which can be solved numerically. This changes the governing differential equation to a nonlinear one, where one of the parameters (head), which define the

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coefficients of the governing differential equation, is unknown. A numerical solution is required in such cases.

In the present study, in order to solve the nonlinear governing differential equation, the Least Square Finite Element Formulation (LSFEF) and the finite difference formulation is utilized.

The finite element (FE) method is a very powerful tool to solve many sophisticated engineering problems. FE analysis has been implemented in a number of areas in engineering such as solid mechanics, heat transfer and hydrodynamics as well as geotechnical interests such as Desai and Christian [1] for general geotechnical uses, Beacher and Ingra [2] and Righetti and Harrop-Williams [3] for stress analysis and Finn [4] and Smith and Freeze [5], [6] and Griffiths and Fenton [7] for seepage analysis.

General finite element formulations, such as the Variational or Weighted residual processes methods used by Zienkiewicz [8] cannot be employed to solve non-linear equations such as Navier Stokes, Burgers or Laplace equations. The Galerkin and least square methods are an extension of the Weighted Residual method. Using Galerkin method for the solution of the Navier Stokes equation has many associated difficulties as (a) the coefficient matrix is not symmetric and in the pressure variation direction in the continuity equation would perform as ill-conditioned and (b) convergence of this system in non-linear problems is very slow and sometimes may come up With some difficulties with iterations. This method has recently been used by researchers such as Zienkiewicz et al. [9], Lynn and Arya [10], and Winterscheidt and Surana [11], [12] in many areas such as solution of partial and hyperbolic differential equations or boundary layer flow, gas dynamics, and compressible fluid and gas problems. LSFEF method was used based on the minimizing of the error function in differential equations with non-linear partial differentiation.

Finite difference analysis has been implemented in a number of areas in engineering such as heat transfer [13] and hydrodynamics [14] ,[15] as well as geotechnical interests such as seepage [16]-[20] and consolidation [21], [22]. It also works well with anisotropic materials [23].

The objective of the finite difference method for solving a partial differential equation (PDE) is to transform a calculus problem into an algebraic problem by:

1. Discretizing the continuous physical domain into a discrete finite difference grid.
2. Approximating the exact derivatives in the initial-value PDE by algebraic finite difference approximations (FDAs).
3. Substituting the FDAs into the PDE to obtain an algebraic finite difference equation (FDE).
4. Solving the resulting algebraic FDE [24].

The development of computer technology may ease solving the partial differential equations (PDE). One is the spreadsheet modeling. The popularity of spreadsheets in the solution of engineering problems has been recently increasing since setup of spreadsheets well fits into the finite-difference grid schemes. By utilizing this feature of spreadsheets several studies have been carried out in different fields of engineering problems [16].

II. VARIABILITY OF THE COEFFICIENT OF PERMEABILITY

In flow problems, both the magnitude and direction of governing fluid flows are highly sensitive to the coefficient of the permeability. For simplicity, this parameter is usually assumed to be a constant in space and time. In this study, the coefficient of permeability is assumed to be spatially variable. The variation of coefficient of permeability was defined for different cases, and then the resulted governing differential equation was solved. In order to define a function for the variation of the permeability two conditions were proposed.

First Condition: This is a simple condition where the coefficient of the permeability is a function of material properties and geometrical conditions. From most classical soil mechanics literature it is well known that coefficient of permeability is directly proportional to the void ratio of the soil. As the void ratio increases or decreases, so does the coefficient of permeability, Lambe and Whitman [25]. Only confined flow was considered here. As an example, one can consider construction of an embankment dam over a saturated fine grain soil. As the construction starts, the consolidation of the material beneath the embankment will begin. Due to non-uniformity of the applied load, the consolidation of the materials under the embankments will vary, which will result in void ratios that vary in space and time. This would therefore introduce variation of coefficient of permeability at different locations and directions under the embankments. Generally, these coefficients of permeability are a minimum at the centerline of the embankments and increase as the distance from the centerline increases. These coefficients of permeability would also be time dependent as long as the consolidation process is occurring. In order to define good estimates for coefficient of permeability in flow problems for any given point, mainly dependent upon soil type, fabric and structure, and consolidation stage one should undertake laboratory testing to define the equations for k_x and k_y, the coefficients of permeability in x and y direction, respectively. The variation of k_x in horizontal direction can be simply expressed by any order binominal equation, which in this study was considered to be second order.

$$k_x = a_x x^2 + b_x x + c_x \qquad (1)$$

Where a_x, b_x and c_x, are the coefficients that can be determined from a curve fitting procedure based on the results from laboratory and field-testing. Similarly k_y the coefficient of permeability in vertical direction can be expressed by a similar second order binominal equation of the form:

$$k_y = a_y x^2 + b_y x + c_y \qquad (2)$$

where a_y, b_y and c_y are the coefficient which can be determined from curve fitting procedure based on the results from laboratory testing. Generally k_y in vertical direction can vary by either the effect of overburden pressure of the natural soils or the influence of excess stresses due to an embankment load. For the first case, as the overburden pressure increases with depth, there would be a tendency for the material to become more compacted, therefore reducing k_y with depth. For the second case, as the depth increases the effect of embankment load decreases i.e. less consolidation, and thus k_y increases. The effect of the second imposed condition is opposite to the first case, and these physical effects with depth should be superimposed in order to define (2) for every starting point at interface of embankment and natural soil in the vertical direction.

Second Condition: In this condition, the coefficient of permeability, in addition to the first case, can be affected by the variation of heads in the upstream, downstream, or in the soil. In the next section the relationship between hydraulic head and the coefficient of permeability is described.

Relationship between Effective Stress and Soil Void Ratio

In the soil consolidation process, the relationship between effective stress and void ratio can be demonstrated in *e* vs. *log p* space, as an example Fig. 1, Leroueil et al. [26]. The first portion of the curve with lower slope, which is due to unloading of the sample, is not considered here. Only the second portion with slope of c_c, which is mainly due to loading, is considered. The void ratio "*e*" of the material at any stage of the consolidation can be determined by:

$$e = c_c \log \frac{\sigma'}{\sigma_1'} + e_1 \qquad (3)$$

where σ' is the applied effective stresses (head) corresponding to e and σ_1' is the known effective stress corresponding to e_1. Equation (3) can be written as:

$$e = c_c \log \sigma' - c_c \log \sigma_1' + e_1 \qquad (4)$$

or

$$e = a \log \sigma' + b \qquad (5)$$

where $a = c_c$ and $b = -c_c \log \sigma_1' + e_1$.

Relationship between Void Ratio and Coefficient of Permeability

It can be observed from previous research of Lambe and Whitman [25], Leroueil [26] and Cedergren [27] that the relationship between void ratio and logarithm of coefficient of permeability is linear, Fig. 1. Similar to previous case, "*e*" void ratio of the material at any stage can be determined by:

$$e = c_k \log \frac{k}{k_1} + e_1 \qquad (6)$$

where c_k is the slope of the curve, k is the unknown coefficient of permeability corresponding to *e*, and k_1 is the known coefficient of permeability corresponding to e_1. By rearranging (6), the coefficient of permeability can be found as follows:

$$\log k = \frac{e}{c_k} - \frac{e_1}{c_k} + \log k_1 \qquad (7)$$

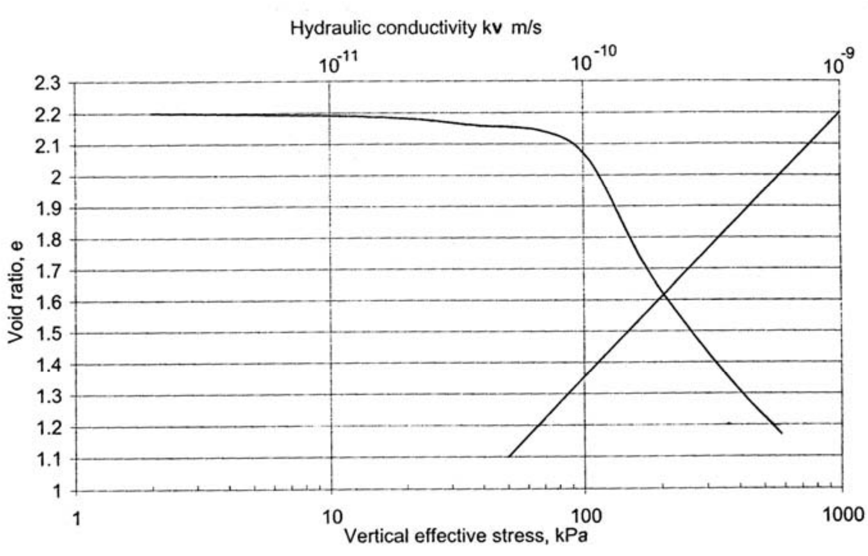

Fig. 1 Typical e against $\log \sigma'$ and e against $\log k$ curve (after Leroueil et al.)

Since $\frac{-e_1}{c_k} + \log k_1$ is a constant value assumed to be equal to d, and $c = \frac{1}{c_k}$, therefore:

$$\log k = c\, e + d \qquad (8)$$

And finally k can be written as:

$$k = 10^{(c\, e + d)} \qquad (9)$$

And with substitution of (5) into (9), it can be written as:

$$k = 10^{(\alpha \log \sigma' + \beta)} \qquad (10)$$

where α is equal to $c\, a$ and β is equal to $c\, b + d$,which all of the parameters *a*, *b*, *c*, and *d* are constant and can be determined from laboratory or in-situ testing.

Relationship between Heads (Total or Pressure) and Coefficient of Permeability

From the effective stress Terzaghi's Equation and from the information in Figure 2, the effective stress at any point can be written as:

$$\sigma' = (-y\ \gamma_{sat} + \gamma_w H) - (h + y)\gamma_w \qquad (11)$$

where γ_{sat} is the saturated density of the soil, *h* is the total head, *h* + *y* is the pressure head, *H* is the upstream water height and γ_w is water density.

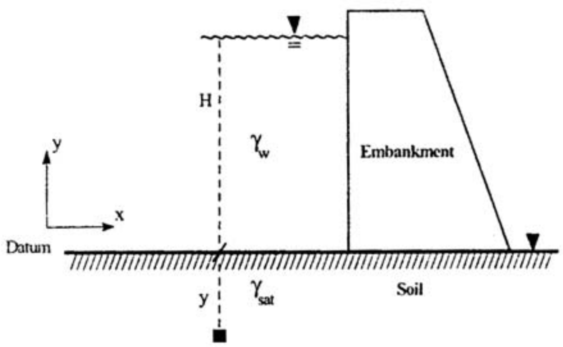

Fig. 2 Schematic diagram of an embankment dam.

By substituting (11) into (10), it can be written as:

$$k = 10^{\alpha \log[-y\ \gamma_{sat} + (H-h-y)\gamma_w] + \beta} \qquad (12)$$

Equation (12) can be simplified to

$$k = 10^{\beta}\,[-y\ \gamma_{sat} + (H-h-y)\gamma_w]^{\alpha} \qquad (13)$$

In the above equation α, β, γ_{sat} and γ_w are constants that depend on material properties and can be determined from laboratory or in-situ testing. The value of total head *h* depends on the geometry of the considered point and is an unknown value, *H* is the height of water at upstream and y is the depth of the considered point from datum. It can be concluded from the above equation that at any point within the confined flow the coefficient of permeability can be defined as a function of total head *h* which will directly influence the solution of the governing differential equation.

III. Non-linear Governing Seepage Equation

The 2-D governing equation of water flow in porous media under laminar conditions, where Darcy's law is applicable is given by:

$$\frac{\partial}{\partial x}(\gamma_w k_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(\gamma_w k_y \frac{\partial h}{\partial y}) = 0 \qquad (14)$$

The above equation can be simplified by assuming γ_w, water density, to stay constant at all times, and therefore:

$$\frac{\partial}{\partial x}(k_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial h}{\partial y}) = 0 \qquad (15)$$

Under conditions of homogeneity, k_x and k_y are assumed to be constants which do not vary in space. In addition, applying anisotropy conditions requires $k_x \neq k_y$. Generally for simplicity k_x and k_y are assumed to be constant and for more simplicity, they are assumed to be equal and constant. However, in this research these coefficients are assumed to be variable which would change the differential equation to a non-linear one. Equation (15) can be expressed as follows:

$$\frac{dk_x}{dx}\frac{dh}{dx} + k_x\frac{d^2h}{dx^2} + \frac{dk_y}{dy}\frac{dh}{dy} + k_y\frac{d^2h}{dy^2} = 0 \qquad (16)$$

k_x and k_y can now be expressed by (1) and (2) and (13) and can be substituted in (15). Formulation of the least square finite element method requires first order differential equations. This can be adopted by assigning hydraulic gradients in the x and y directions as follow:

$$P_x = \frac{dh}{dx} = I_x \qquad P_y = \frac{dh}{dy} = I_y \qquad (17)$$

Equation (17) was used in least square finite element formulation. In finite difference formulation hydraulic gradient is evaluated by:

$$i_x = \frac{\Delta h}{\Delta x} \qquad (18)$$

Secondary Solutions: In seepage problems, in addition to evaluation and calculation of heads at various locations in the system, three other parameters are important to be evaluated. These are total discharge rate, exit hydraulic gradient, and uplift pressure. These parameters are known as the secondary solutions. Total discharge rate can be calculated on the bases of discharge for each element at any section, and the summation of these discharge rates would be the total discharge rate of the system. In finite element formulation it will be:

$$Q = -\sum_{i=1}^{N} d_i\,[k_{xi}][I_{xi}] \qquad (19)$$

where d_i is the width of the element *I*, with the value I_x as the average of the eight node hydraulic gradient for each element. In finite difference formulation, the discharge rate will be:

$$Q = -\sum_{i=1}^{N} d_i\ k_{xi}\ i_{xi} \qquad (20)$$

where *d*, is the distance between two adjacent nodes, with the value i_{xi} as the hydraulic gradient for that nodes. The exit hydraulic gradient would be known at the downstream section of the system. Uplift pressure can be calculated on the bases of Bernoulli's equation by knowing total head (*h*) from analysis and evaluation of the concerning point from geometry assuming, that *v2/2g =0*.

IV. Numerical Examples

In this section two examples are provided for the proposed types of variations on k_x and k_y.

Example 1: To illustrate the proposed methods, consider Example 18.2 of Lambe and Whitman [25]. A schematic diagram of a concrete dam for use in LSFEM and FDM is given in Fig. 3. This system consists of two sheet piles of 21 meters height at upstream and downstream of the dam. In order to analyses the problem in LSFEM, the permeable section of the system was divided into 18 elements with 77 nodes (Fig. 3a). As for the FDM, the permeable section of the system was divided into 67 nodes (Fig. 3b). Sheet piles were considered as impermeable boundaries, where $\partial h/\partial x = 0$ and other impermeable boundaries where $\partial h/\partial y = 0$ are at the bottom of the 64 meter thick permeable layer and dam itself. A thirty-meter distance away from the system (sheet piles) was chosen as a limit for numerical analysis where it assumed there is no flow taking place away from these limits in the permeable layer. Variable heads at upstream and downstream locations were chosen in order to examine the effect of the proposed solution. In order to apply (13), the following values were used based on Effati [28].

$$\begin{aligned} &\alpha = -0.034049 \\ &\beta = -1.0 \\ &\gamma_{sat} = 22\ kN/m^3 \\ &\gamma_w = 10\ kN/m^3 \end{aligned} \tag{24}$$

Results for head, the coefficient of permeability k and discharge rate were obtained based on the above values.

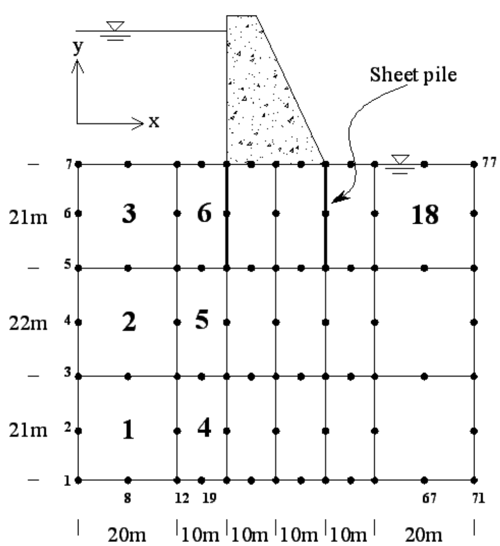

Fig. 3a Schematic diagram of the LEFEM mesh.

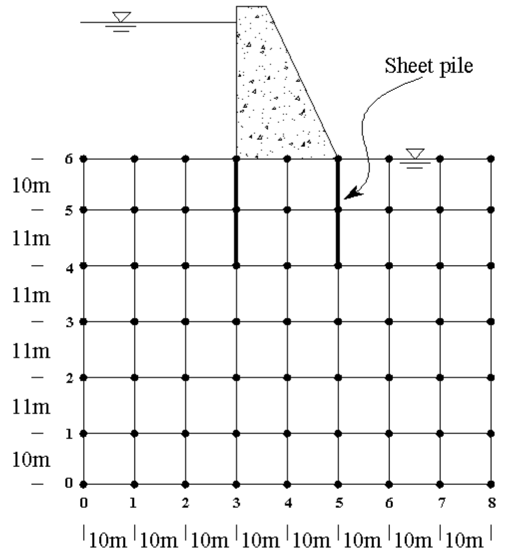

Fig. 3b Schematic diagram of the FDM mesh.

Comparing the results of the heads obtained here with flow nets in Lambe and Whitman [25] shows only very small differences. Any conclusions based on head results and flow nets alone may not be justified due to the accuracy of the results. A flow net drawing is based on a trial and error procedure and is not affected by upstream or downstream heads. In Fig. 4 variation of coefficient of permeability k is shown against head (water height) in the upstream. It is clear from this figure that, (i) when the head at any node varies, it would influence the permeability of that node, (ii) as the head increases the values of permeability decrease, and (iii) when the head at any point increases, consolidation of the material occurs resulting in reduced the permeability. The variation of k against head is non-linear because the proposed function for k in (13) is non-linear.

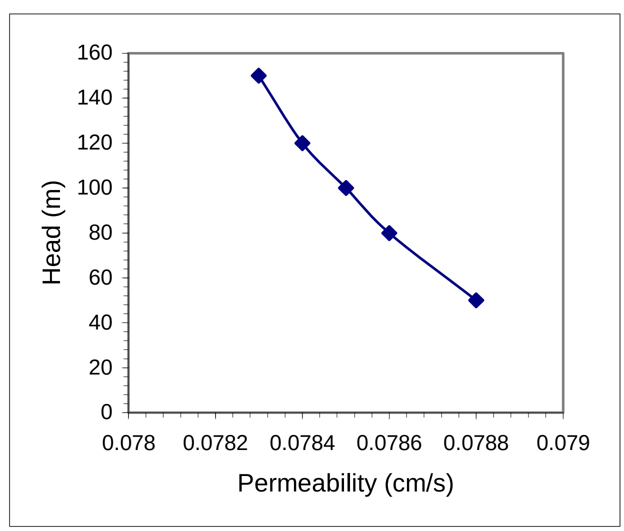

Fig. 4 Variation of coefficient of permeability against head.

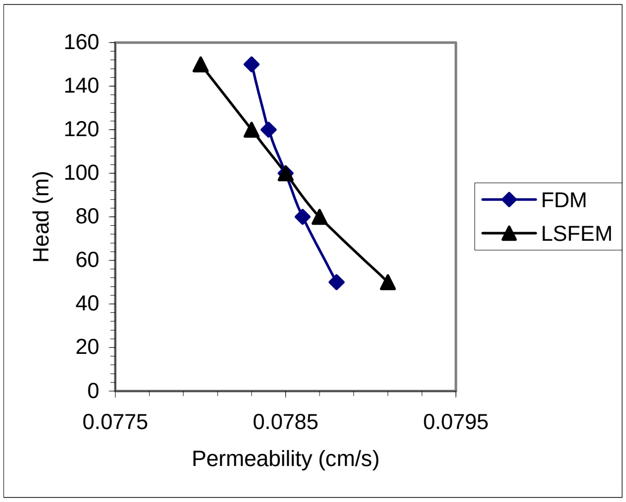

Fig. 5 Comparison between FDM and LSFEM for variable *k*.

Fig. 5 shows the comparison between FD method and Least Square FE method for variable *k*. As it is seen, both methods show decrease of k with increase of head and the differences are in an acceptable range.

Fig. 6 shows the variation of the discharge rate under the dam against upstream head. Two types of curves are shown in this figure, one with constant permeability and the other with variable permeability (proposed method). In the one with constant permeability, similar to most classical seepage problems, permeability is assumed constant throughout the analysis and the system, and if it varies, it is not due to the effect of upstream head. In this case *k* was assumed to be 0.080 cm/s. But in the other one variable *k* refers to the influence of head on discharge rate. It can be seen from Fig. 6 that for both cases when upstream head increases, the discharge rate also increases. It should be clarified, however, that in the actual case, head effects influenced permeability. The discharge rate is different from that of the constant permeability case. The difference would be higher for longer values of upstream head, i.e. for h = 120 m the effect of the head on discharge rate is about 2%. This is mainly due to the effect of upstream head on permeability.

In fig. 7, finite difference method and least square finite element method are compared for variable *k*. It is observed that FD method gives lower discharge than LSFE method and both methods show the same pattern.

Example 2: In this example, variations of k_x and k_y are not effected by a direct influence of head but they are based on other effects using (1) and (2).

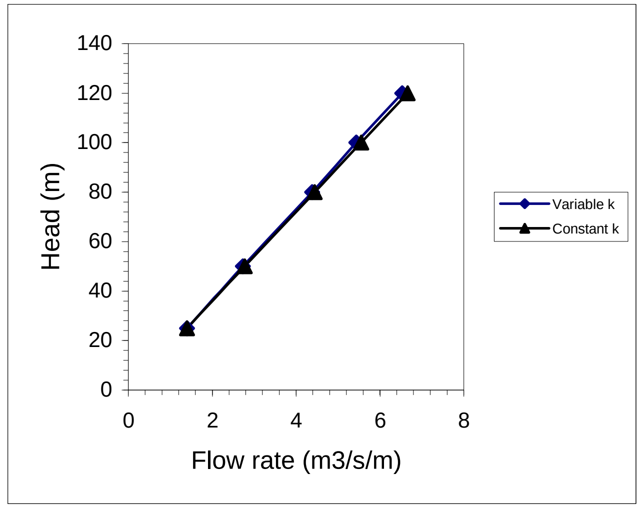

Fig. 6 Discharge rate variation under dam vs head for constant and variable coefficient of permeability.

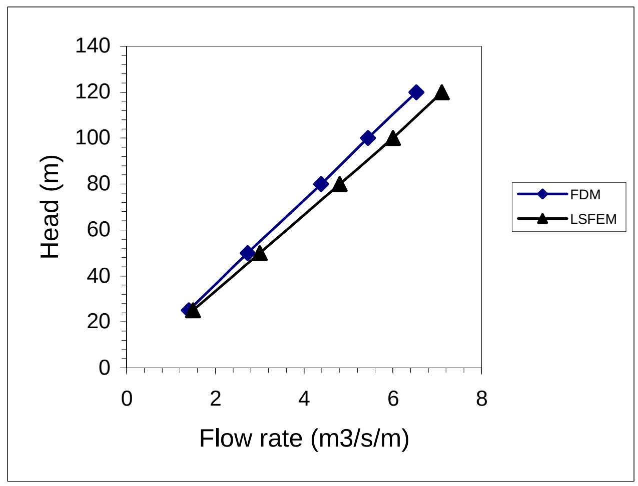

Fig. 7 Comparison between FDM and LSFEM for variable *k*

The FD mesh and LEFE mesh of a concrete dam are given in Fig. 8. The permeable section of the system was divided into 105 nodes for FDM and 20 elements and 85 nodes for LSFEM. The top and bottom portion of the permeable section with thickness of 40 meters were considered as impermeable boundaries, where $\partial h / \partial y = 0$. Sixty meters from the toe and heel of the dam were chosen as a limit for numerical analysis where no flow was assumed to take place away from these limits in the permeable layer. The proposed variations for k_x and k_y are based on (1) and (2) and Effati [28].

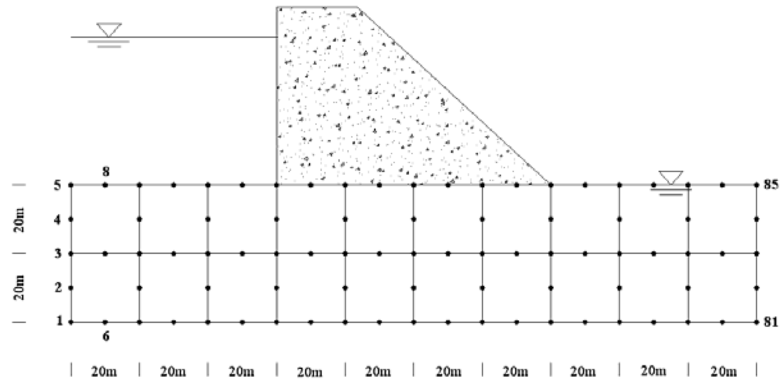

Fig. 8a Schematic diagram of the system with LSFE mesh.

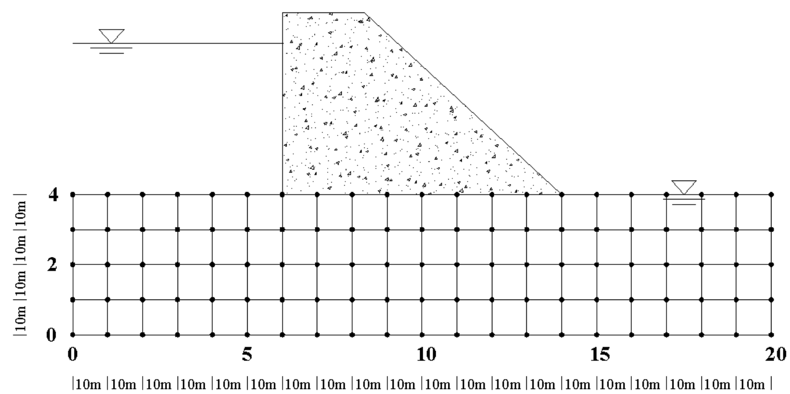

Fig 8b Schematic diagram of the system with FD mesh.

$$
\begin{aligned}
k_x &= 0.375\times10^{-3}\, x^2 - 0.375\times10^{-2}\, x + 10 \\
k_y &= 0.255\times10^{-2}\, y^2 - 0.375\times10^{-2}\, y + 2.5
\end{aligned}
\qquad (25)
$$

In this example results for exit gradient and uplift pressure are presented based on above values for k_x and k_y .

Fig. 9 shows the exit gradient in vertical direction against upstream head for constant and variable permeability based on data in this example. For low upstream head the difference between constant and variable permeability conditions is sometimes negligible, but as the upstream head increases, in large dams the difference becomes more significant which might influence the design of the whole system. For the upstream height of 180 meters, the exit gradient difference is about 24%, which would reduce the factor of safety against piping to a low point, which, in turn, may result in changing the geometry of the dam. It should be noted that constant values of permeability considered in the computation would result in higher values for exit gradient, which would be on the safer side. Fig. 10 shows the comparison of FD method and LSFE method for variable *k*. As it is seen, both methods show increase of exit gradient (absolute value) with head and the results are in good agreement.

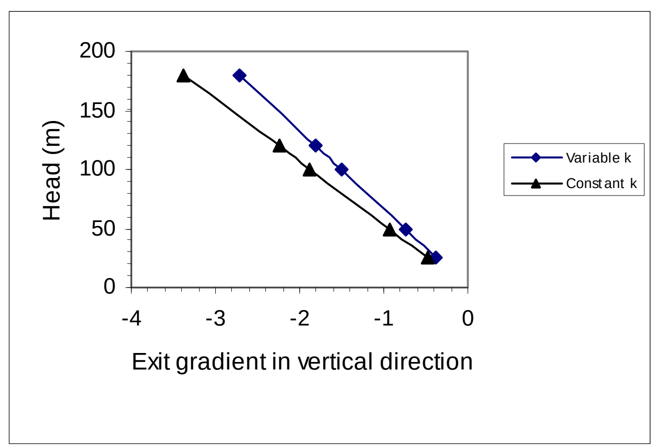

Fig. 9 Variation of exit hydraulic gradient against head for constant and variable permeability.

Fig. 11 shows the uplift pressure at the bottom of the dam against upstream head for constant and variable permeability based on data in this example. As the head in the upstream of the dam will increase, obviously the uplift pressure under dam increases, but as it can be seen, there is a difference between constant or variable permeability conditions. This difference would be around 8% for an upstream head of 150 meters.

Again, it should be noted that with constant permeability the value of uplift pressure is higher than that with the variable permeability, which would also be on the safe side.

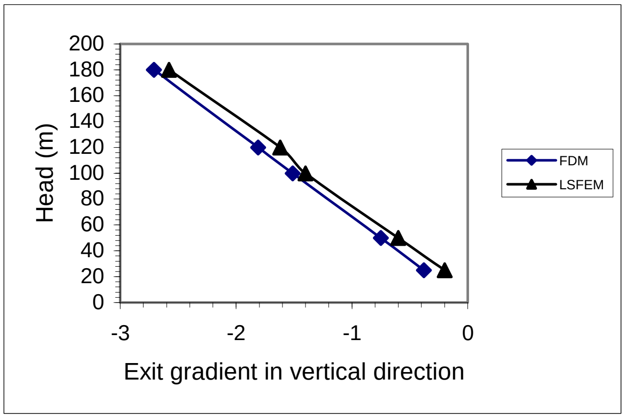

Fig. 10 Comparison between FDM and LSFEM for variable *k*.

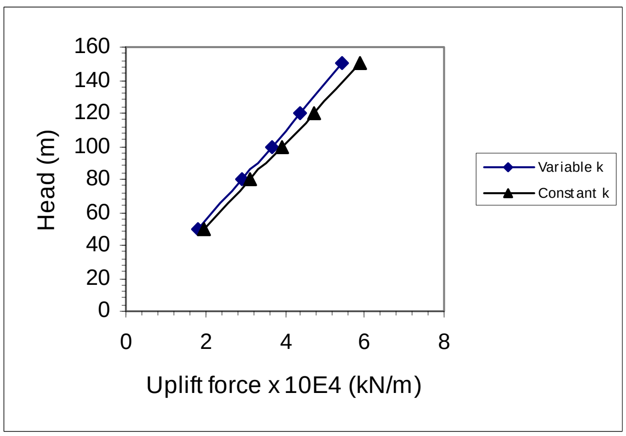

Fig. 11 Comparison of the uplift pressure against head for constant and variable *k*.

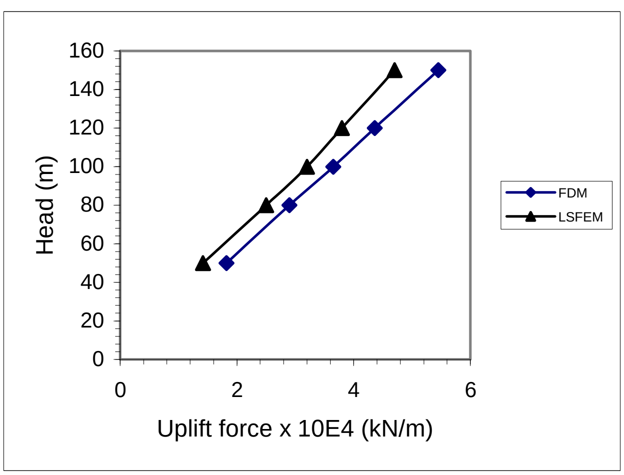

Fig. 12 Comparison between FDM and LSFEM for variable *k*.

Fig. 12 also shows the uplift pressure against upstream head for FD method and LSFE method. It is observed that FD method gives higher uplift pressure than the LSFE method.

V. Conclusions

This paper presents a non-linear governing differential equation for a confined seepage problem under non-homogeneous and anisotropic conditions. This non-linear performance is introduced by the governing equation based on actual material behavior and solving the resulting non-linear differential equation numerically using the least square finite element formulation and finite difference formulation. These methods were used to solve several seepage problems to examine the accuracy of the results. The solutions show good accuracy and convergence. The advantage of these methods is their capability to solve nonlinear problems compared to routine methods with constant coefficients in order to increase the accuracy of the solution. The results of both methods are also compared with each other and they show a very good agreement with each other. Some clear conclusions can be drawn from this study as follows:

a) Generally, results of head changes (i.e., flow net analyses) either by first or second conditions for variable permeability conditions compares favorably to the case when the permeability is assumed to be constant and very little difference is observed.

b) Comparison of the results for discharge rate between constant and variable permeability conditions shows little effect of low head on discharge rate results. However, as upstream head increases, the effect of variable permeabilities becomes more significant. Usually the difference in discharge rate between variable and constant permeability for a typical head is not more than 8%.

c) Results of exit gradient for a critical condition show that the values are less affected for low head, but the effect increases for higher head. The results, assuming variable permeability conditions, would give a lower safety factor regarding piping, etc.

d) In terms of uplift pressure, as the head increases the uplift pressure also increases, but there is only a slight difference between uplift pressure under constant or variable permeability conditions for any given head. This result is consistent with part (a).

e) In general, the effect of variable coefficient of permeability may not be significant on small dams, but as the height of the dam increases, the effect becomes more considerable. It is believed that this would influence the geometry and design of the dam and that variable permeability analysis such as the one described in this paper should be conducted.

f) Finite difference method and least square finite element method results were in good agreement and the differences between the results are negligible. These methods, because of their more simplicity and less resource consuming, are preferable methods for dealing with (non-linear) seepage problems.

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