Solution of some differential equations of reversible and irreversible thermodynamics

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***Abstract*—**There are presented the analytical solutions of differential equations of transfer laws in the body with *n* binding degrees of freedom of thermodynamics describing stationary and non-stationary processes in this paper. It is suggested that potential fields are one-, two- and three-dimensional. Laplace's differential equations are analysed in Cartesian, cylindrical and spherical coordinates taking into account various boundary conditions. The solutions considerably facilitate the numerically methods put into solving of some Laplace's differential equations and increase the possibility of employing these equations in thermodynamics of stationary and non-stationary processes.

***Keywords*—**Thermodynamics, stationary and non-stationary processes Laplace's equations, degree of freedom.

I. Introduction

It is known that the potential $u = u(x,y,z)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

if u is a temperature potential, the potential of the stationary electromagnetic field, a material filtration potential, the potential of the speed of non-vortex non-compressible liquid flow, the potential of the gravitational force in all space points not being in the masses created space, the potential of the electrical charges interaction in all points of charge-free region of space, the potential of the definition of castings quality, and so on.

Therefore the solutions of Laplace's equations with the corresponding boundary conditions attract attention of many researchers [1]-[8]. In this paper, it is presented a method of the solution of Laplace's differential equations system expressed in the form:

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2}+\frac{\partial^2 P_j}{\partial y^2}+\frac{\partial^2 P_j}{\partial z^2}\right)=0, \qquad (1)$$

$$i=1,2,\dots,n$$

or corresponding form in cylindrical and spherical coordinates under different boundary conditions.

The system of equations (1) describes the law of transfer for a nonequilibrium system (or body) with n by connected degrees of inner freedom and three-dimensional fields of potentials $P_j = P_j(x,y,z)$, where $P_j = \frac{\partial U}{\partial E_j}$ is a generalized potential; $U = f(E_1, E_2, \dots, E_n)$ is an internal energy of a system, J; L_{ij} − a coefficient of transfer and $L_{ij} = L_{ji}$. The coefficient L_{ii} is called a principal coefficient of transfer. It characterises conductivity of a thermodynamic system in relation to a charge integrated with potential P_i. Coefficient L_{ij} when $i \neq j$ is called a cross-coefficient. It characterises influence of j-th charge on potential P_i integrated with it [9].

II. Method of the Solution

A system of equations (1) after some transformations:

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2}+\frac{\partial^2 P_j}{\partial y^2}+\frac{\partial^2 P_j}{\partial z^2}\right)=$$

$$=\sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial x^2}+\sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial y^2}+\sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial z^2}=$$

$$=\frac{\partial^2}{\partial x^2}\left(\sum_{j=1}^{n} L_{ij}P_j\right)+\frac{\partial^2}{\partial y^2}\left(\sum_{j=1}^{n} L_{ij}P_j\right)+\frac{\partial^2}{\partial z^2}\left(\sum_{j=1}^{n} L_{ij}P_j\right)$$

and

$$u_i=\sum_{j=1}^{n} L_{ij}P_j,\ i=1,2,3,\dots n$$

is expressed as

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2}+\frac{\partial^2 P_j}{\partial y^2}+\frac{\partial^2 P_j}{\partial z^2}\right)=$$

$$=\frac{\partial^2 u_i}{\partial x^2}+\frac{\partial^2 u_i}{\partial y^2}+\frac{\partial^2 u_i}{\partial z^2}.$$

So equation (1) in Cartesian coordinates can be written in the form:

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$$\frac{\partial^2 u_i}{\partial x^2}+\frac{\partial^2 u_i}{\partial y^2}+\frac{\partial^2 u_i}{\partial z^2}=0\,; \qquad (2)$$

in cylindrical coordinates - in the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_i}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_i}{\partial \varphi^2}+\frac{\partial^2 u_i}{\partial z^2}=0 \qquad (3)$$

or in spherical coordinates - in the form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_i}{\partial r}\right)+\frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial u_i}{\partial\Theta}\right)+ \\ +\frac{1}{r^2\sin^2\Theta}\frac{\partial^2 u_i}{\partial\varphi^2}=0, \qquad (4)$$

where

$$u_i=\sum_{j=1}^{n}L_{ij}P_j\,, \\ i=1,2,...,n\,. \qquad (5)$$

Thus, the procedure of theoretical solutions of transfer differential equations of a thermodynamic system with n binding degrees of freedom is the following:

1. Find the solution u_i of Laplace's equations of a kind (2), (3) or (4) under appropriate boundary conditions. The well-known formulas indicated in works on equations of mathematical physics can be used as the basis for this purpose.

2. After determination of the free members (functions u_i) find generalized potentials P_j of a thermodynamic system. The system of linear (concerning potentials P_j) equations can be solved by various ways, for example, Cramer's rule can be applied [10]:

$$P_j=\frac{\Delta_{P_j}}{\Delta},\; j=1,2,...,n\,.$$

III. Solving Procedures

In one dimension, the equations (2), (3) and (4) have the forms:

$$\sum_{j=1}^{n}L_{ij}\frac{d^2P_j}{dx^2}=0 \quad \text{(in rectangular coordinates)},$$

$$\sum_{j=1}^{n}L_{ij}\frac{d}{dr}\left(r\frac{dP_j}{dr}\right)=0 \quad \text{(in cylindrical coordinates)},$$

$$\sum_{j=1}^{n}L_{ij}\frac{d}{dr}\left(r^2\frac{dP_j}{dr}\right)=0 \quad \text{(in spherical coordinates)}.$$

In two dimensions, the thermodynamic potentials are defined by the following systems of differential equations:

in Cartesian coordinates:

$$\sum_{j=1}^{n}L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2}+\frac{\partial^2 P_j}{\partial y^2}\right)=0\,, \qquad (6)$$

in cylindrical coordinates:

$$\sum_{j=1}^{n}L_{ij}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_j}{\partial r}\right)+\frac{\partial^2 P_j}{\partial z^2}\right]=0 \qquad (7)$$

or

$$\sum_{j=1}^{n}L_{ij}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_j}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 P_j}{\partial\varphi^2}\right]=0\,, \qquad (8)$$

and in spherical coordinates:

$$\sum_{j=1}^{n}L_{ij}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P_j}{\partial r}\right)+ \\ +\frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial P_j}{\partial\Theta}\right)\right]=0. \qquad (9)$$

The equation (7), noting (5), can be expressed as

$$\frac{\partial^2 u_i}{\partial x^2}+\frac{\partial^2 u_i}{\partial y^2}=0\,, \\ i=1,2,...,n\,. \qquad (10)$$

The equations (8) and (9), noting (5), can be expressed analogously. Consequently, the calculation of two-dimensional thermodynamic potentials $P_j = P_j(x,y)$ consists of two steps. In the first step, Laplace's equation (10) is solved using respective boundary conditions. In the second step, the system of equations (5) is solved.

One problem solved using the recommendations of the work [11] are presented below.

The three-dimensional system is given as $0 \le x \le a$, $-b \le y \le b$, $-c \le z \le c$. The potentials $P_j = P_j(x,y,z)$ satisfy the following boundary conditions:

$$P_j=P_{j0}=const,\; x=0\,;$$

$$\frac{\partial P_j}{\partial x}+h_iP_j=0,\; x=a\,;$$

$$\frac{\partial P_j}{\partial y}-h_iP_j=0,\; y=-b\,;$$

$$\frac{\partial P_j}{\partial y} + h_i P_j = 0, y = b;$$
$$\frac{\partial P_j}{\partial z} - h_i P_j = 0, z = -c;$$
$$\frac{\partial P_j}{\partial z} + h_i P_j = 0, z = c;$$
$$i = 1,2,\ldots,n.$$

In this case the functions u_i are calculated from the equations (2) under following conditions:

$$u_i = \sum_{j=1}^{n} L_{ij} P_{j0} = V_{i0} = const, x = 0;$$
$$\frac{\partial u_i}{\partial x} + h_i u_i = 0, x = a;$$
$$\frac{\partial u_i}{\partial y} - h_i u_i = 0, y = -b;$$
$$\frac{\partial u_i}{\partial y} + h_i u_i = 0, y = b;$$
$$\frac{\partial u_i}{\partial z} - h_i u_i = 0, z = -c;$$
$$\frac{\partial u_i}{\partial z} + h_i u_i = 0, z = c;$$
$$i = 1,2,\ldots,n.$$

The solution is expressed as follows:

$$u_i = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{4h_i^2 V_{i0}\cos(\alpha_{ir} y)\cos(\beta_{is} z)\varphi_i(x)}{K_{ir}\left[\left(\beta_{is}^2 + h_i^2\right)c + h_i\right]\cos(\alpha_{ir} b)\cos(\beta_{is} c)},$$

where

$$\varphi_i(x) = \frac{h_i sh[l_i(a-x)] + l_i ch[l_i(a-x)]}{h_i sh(al_i) + l_i ch(al_i)},$$
$$K_{ir} = \left(\alpha_{ir}^2 + h_i^2\right)b + h_i,$$
$$l_i^2 = \alpha_{ir}^2 + \beta_{ir}^2,$$

α_{ir} and β_{is} are positive roots of equations:

$$\alpha_i tg(b\alpha_i) = h_i,$$
$$\beta_i tg(c\beta_i) = h_i,$$
$$i = 1,2,\ldots,n.$$

IV. Solution of Differential Equations of Thermodynamics Describing Non-Stationary Processes

In non-stationary thermodynamically system (or body) having n degrees of inner freedom potentials $P = P(x,y,z,t)$ satisfy the following equations:

$$\frac{\partial P_i}{\partial t} = \sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2} + \frac{\partial^2 P_j}{\partial y^2} + \frac{\partial^2 P_j}{\partial z^2}\right), \quad (11)$$

where $i = 1, 2, \ldots, n$; L_{ij} − constant numbers; t − time; x, y, z − coordinates.

Having marked

$$\sum_{j=1}^{n} L_{ij} P_j = u_i \quad (12)$$

the equation (1) we write as follows:

$$\frac{\partial P_i}{\partial t} = \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 u_i}{\partial z^2}, \quad (13)$$

where $u_i = u_i(x,y,z,t)$; $i = 1, 2, \ldots, n$.

Potential P_j we express trough the function u_i. For this purpose we use the function (12). We have:

$$P_j = \frac{1}{|A|}\sum_{i=1}^{n} A_{ij} u_i, \quad (14)$$

where $j = 1, 2, \ldots, n$; $|A| = \begin{vmatrix} L_{11}L_{12}\ldots L_{1n} \\ L_{21}L_{22}\ldots L_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ L_{n1}L_{n2}\ldots L_{nn} \end{vmatrix}$ is the determinant of system (12); $A_{ij} = (-1)^{i+j}M_{ij}$ - adjuncts of the determinant; M_{ij} - minors of the determinant.

Having put (14) into (13) we get a new system of equations:

$$\left.\begin{array}{l}
\frac{1}{|A|}\sum_{i=1}^{n} A_{i1}\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}, \\
\frac{1}{|A|}\sum_{i=1}^{n} A_{i2}\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{1}{|A|}\sum_{i=1}^{n} A_{in}\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial^2 u_n}{\partial z^2}
\end{array}\right\} \quad (15)$$

System of equations (15) can be put as follows:

$$\sum_{i=1}^{n} A_{ik}\frac{\partial u_i}{\partial t} = |A|\left(\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial y^2} + \frac{\partial^2 u_k}{\partial z^2}\right); \quad (16)$$
$$k = 1,2,\ldots,n.$$

If the potentials $P = P(r,\varphi,z,t)$ are presented by functions in cylindrical coordinates, then the equations (11) of a non-stationary thermodynamic system after replacement (12) can be written as

$$\sum_{i=1}^{n} A_{ik} \frac{\partial u_i}{\partial t} = |A| \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_k}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_k}{\partial \varphi^2} + \frac{\partial^2 u_k}{\partial z^2} \right],$$
$$k = 1,2,\ldots n.$$

In spherical coordinate [$P = P(r,\Theta,\varphi,t)$] system of equations (11) can be expressed as follows:

$$\sum_{i=1}^{n} A_{ik} \frac{\partial u_i}{\partial t} =$$
$$= \frac{|A|}{r^2} \left[\begin{array}{l} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u_k}{\partial r} \right) + \\ + \dfrac{1}{\sin\Theta} \dfrac{\partial}{\partial \Theta}\left(\sin\Theta \dfrac{\partial u_k}{\partial \Theta} \right) + \dfrac{1}{\sin^2\Theta} \dfrac{\partial^2 u_k}{\partial \varphi^2} \end{array} \right],$$
$$k = 1,\, 2,\, \ldots\, n.$$

So the given system (11) was changed to the equivalent system (16). This system is solved numerically. Having solved the latter system of equations under corresponding conditions, we get functions u_i. Using the equation (4) we find the solutions $P_j = P_j(x,y,z,t)$ which satisfy the system of equations (11).

When the field of potentials $P = P(x,t)$ the equation (16) looks as follows:

$$\sum_{i=1}^{n} A_{ik} \frac{\partial u_i}{\partial t} = |A| \frac{\partial^2 u_k}{\partial x^2};$$
$$k = 1,2,\ldots,n.$$

If thermodynamically system having the field of potentials $P = P(x,t)$ and two degrees of freedom ($n = 2$), then $k = 1, 2$. In this case the system of equations (16) will be as follows:

$$\left. \begin{array}{l} A_{11} \dfrac{\partial u_1}{\partial t} + A_{21} \dfrac{\partial u_2}{\partial t} = |A| \dfrac{\partial^2 u_1}{\partial x^2}, \\ A_{12} \dfrac{\partial u_1}{\partial t} + A_{22} \dfrac{\partial u_2}{\partial t} = |A| \dfrac{\partial^2 u_2}{\partial x^2} \end{array} \right\}.$$

Potentials $P = P(x,y,t)$ in the thermodynamically system having two degrees of freedom satisfy the following equations:

$$\left. \begin{array}{l} A_{11} \dfrac{\partial u_1}{\partial t} + A_{21} \dfrac{\partial u_2}{\partial t} = |A| \left(\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} \right), \\ A_{12} \dfrac{\partial u_1}{\partial t} + A_{22} \dfrac{\partial u_2}{\partial t} = |A| \left(\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} \right) \end{array} \right\}$$

If potentials $P = P(x,y.t)$ in the thermodynamically system have three degrees of freedom then the system of equations (16) will be as follows:

$$A_{11} \frac{\partial u_1}{\partial t} + A_{21} \frac{\partial u_2}{\partial t} + A_{31} \frac{\partial u_3}{\partial t} = |A| \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right),$$
$$A_{12} \frac{\partial u_1}{\partial t} + A_{22} \frac{\partial u_2}{\partial t} + A_{32} \frac{\partial u_3}{\partial t} = |A| \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right),$$
$$A_{13} \frac{\partial u_1}{\partial t} + A_{23} \frac{\partial u_2}{\partial t} + A_{33} \frac{\partial u_3}{\partial t} = |A| \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} \right).$$

The course of solution of differential equations of a thermodynamically system is as follows:

1. The determinant I*A*I of the equations system (12) and adjuncts A_{ij} (there are n^2) are found.
2. The system of equations (16) is solved, i.e. auxiliary functions u_i are found
3. The potentials P_j are found from the equation (14).

The solutions of differential equations of stationary and non-stationary processes considerably facilitate the numerical methods [12-16] put into solutions of the thermodynamics systems with n binding degrees of freedom and increase the possibility of employing these systems in practice.

V. CONCLUSION

The solution of the systems differential equations of transfer laws in the body with n binding degrees of freedom of thermodynamics describing stationary and non-stationary processes has been presented. The one-, two- and three-dimensional potential fields have been analyzed. Laplace's differential equations have been analysed in Cartesian, cylindrical and spherical coordinates accounting various boundary conditions. The solutions considerably increase the possibility of employing Laplace's equations in thermodynamics. The proposed methods have been demonstrated for the three-dimensional system.

REFERENCES

[1] Wang, Q.X., Yeo, K.S., Kho, B.C., Lam, K.Y. Vortex ring modelling of toroidal bubbles. *Theoretical and Computational Fluid Dynamics.* Vol. 19, No. 5, 2005, pp. 303-317.

[2] Sheng, Y.Y., Irons, G.A., Tisdale, D.G. Transport phenomena in electric smelting of nickel matte: Part I. Electric potential distribution. *Metallurgical and Materials Transactions, B. Process Metallurgy and Materials Processing Science.* Vol. 29, No. 1, 1998, pp. 77-83.

[3] Madsen, P.A., Bingham H.B., Hua Liu. A new Boussinesq method for fully nonlinear waves from shalow to deep water. *Journal of Fluid Mechanics.* Vol. 462, 2002, pp. 1-30.

[4] Averbuch, A., Izraeli, M., Nathan, M., Ravve, I. Surface evolution in bare bamboo-type metal lines under diffusion and electric field effects. *Journal of Computational Physics.* Vol. 188, No. 2, 2003, pp. 640-677.

[5] Nishikawa, H. Multigrid third-order least-squares solution of Cauchy-Riemann equations on unstructured triangular grids. *International Journal for Numerical Methods in Fluids.* Vol. 53, 2007, pp. 443-454.

[6] Baratchart, L., Ben Abda, A., Ben Hassen, F., Leblond, J. Recovery of pointwise sources or small inclusions in 2D domains and rational approximation. *Inverse Problems*. Vol. 21, 2005, pp. 51-74.

[7] Lo, V.C., Dam, X.T. Simulation of electromigration failure by variable resistance model. *Modelling and Simulation in Materials Science and Engineering*. Vol. 5, No. 6, 1997, pp. 563-577.

[8] Mukhamediev, S.A., Galybin, A.N. Determination of stresses from the stress trajectory pattern in a plane elastic domain. *Mathematics and Mechanics of Solids.* Vol. 12, 2007, pp. 75-106.

[9] Veinik, A.I. *Thermodynamics*. Minsk: Vysheishaja shkola, 1968 (in Russian).

[10] Muir, T. The Theory of Determinants in the Historical Order of development, Vol. 1. New York: Dover, 1960.

[11]. Carslaw, H.S., Jaeger, J.C. *Conduction of Heat in Solids*. London: Oxford Univ. Press, 1993.

[12] Mastorakis, N., Martin, O. About the numerical solution of a stationary transport equation. *Proceedings of the* 5*th WSEAS Int. Conf. on SIMULATION, MODELLING AND OPTIMIZATION, Corfu, Greece, August* 17-19, 2005, pp. 419-426.

[13] Wang, G., Huang, P.G., Zhang, Y.M. Numerical analysis of metal transfer in gas metal arc welding under modified pulsed current conditions. *Metallurgical and materials transactions B.* Vol. 35B, 2004, pp. 857-866.

[14] Afshar, M. Numerical Simulations of turbulent, two-phase flows in complex geometries. *Proceedings of the* 3*rd IASME/WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT, Corfu, Greece, August* 20-22, 2005, pp. 338-343.

[15] Koeijer, G.M., Kjelstrup, S. Application of irreversible thermodynamics to distillation. *International Journal of Thermodynamics.* Vol. 7, No. 3, 2004, pp. 107-114.

[16] Awrejcewicz, J., Krysko, A. Nonlinear coupled problemsin dynamics of shells. *International Journal of Engineering Science.* Vol. 41, No. 6, 2003, pp. 587-607.