Monomiality principle and related operational techniques for orthogonal polynomials and special functions

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***Abstract*—** The concepts and the related aspects of the monomiality principle are presented in this paper to explore different approaches for some classes of orthogonal polynomials. The associated operational calculus introduced by the monomiality principle allows us to reformulate the theory of Hermite, Laguerre and Legendre polynomials from a unified point of view. They are indeed shown to be particular cases of more general polynomials, whose usefulness in purely mathematical and applied context is discussed. The powerful tool represented by the Hermite and Laguerre polynomials allows us to derive classes of isospectral problems in applied mathematics and economics.

***Keywords*—** Orthogonal Polynomials, Hermite, Laguerre, Legendre, monomiality principle, generating functions.

I. Monomiality principle

MANY properties of conventional and generalized orthogonal polynomials have been shown to be derivable, in a straightforward way, within an operational framework, which is a consequence of the *monomiality principle* [1]. The concepts of *quasi-monomiality* are often exploited to derive classes of isospectral problems. By *quasi-monomial* we mean any expression characterized by an integer *n*, satisfying the relations:

$$\begin{aligned}\hat{M}\, f_n &= f_{n+1} \\ \hat{P}\, f_n &= n f_{n-1}\end{aligned}, \tag{1}$$

where $\hat{M}$ and $\hat{P}$ play the role of multiplicative and derivative operators. An example of quasi-monomial is provided by:

$$_{\delta}x_n = \prod_{m=0}^{n}(x - m\delta) \tag{2}$$

whose associated multiplication and derivative operators, read:

$$\begin{aligned}\hat{M} &= x e^{\delta \frac{d}{dx}} \\ \hat{P} &= \frac{e^{\delta \frac{d}{dx}} - 1}{\delta}\end{aligned}. \tag{3}$$

It is worth noting that, when $\delta = 0$, then:

$$_{\delta}x_n = x^n \tag{4}$$

and:

$$\begin{aligned}\hat{M} &= x \\ \hat{P} &= \frac{d}{dx}\end{aligned}. \tag{5}$$

More in general, a given polynomial $p_n(x)$, $n \in \mathbb{N}$, $x \in \mathbb{C}$ can be considered a *quasi-monomial* if two operators $\hat{M}$ and $\hat{P}$, called multiplicative and derivatives operator respectively, can be defined in such a way that:

$$\begin{aligned}\hat{M}\, p_n(x) &= p_{n+1}(x) \\ \hat{P}\, p_n(x) &= n p_{n-1}(x)\end{aligned} \tag{6}$$

with:

$$\left[\hat{M}, \hat{P}\right] = \hat{M}\hat{P} - \hat{P}\hat{M} = \hat{1} \tag{7}$$

that is $\hat{M}$, $\hat{P}$ and $\hat{1}$ satisfy a Weyl group [2] structure with respect commutation operation. The *"rules"* we have just established can be exploited to completely characterize the family of polynomials $p_n(x)$, we note indeed that:

If $\hat{M}$ and $\hat{P}$ have a differential realization, the polynomial $p_n(x)$ satisfy the differential equation:

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$$\hat{M}\,\hat{P}\,p_n(x) = p_n(x)\,. \tag{8}$$

If $p_0(x)=1$, then $p_n(x)$ can be explicitly constructed as:

$$\hat{M}^{\,n}(1) = p_n(x)\,. \tag{9}$$

If $p_0(x)=1$, then the generating function of $p_n(x)$ can always be cast in the form:

$$e^{t\hat{M}}(1) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} p_n(x)\,,\ t \in \mathbb{R}\,,\ |t| < +\infty\,. \tag{10}$$

II. Hermite and Laguerre polynomials

The Hermite [3,4] and Laguerre [1] polynomials are two examples of *quasi-monomial*. It is therefore possible to show that their properties can be derived by using the *monomiality principle*. To make the discussion more complete we will consider a more general case, by analyzing the two-variable extension of Hermite polynomials, defined by:

$$H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{11}$$

and linked to the ordinary case, by:

$$H_n\left(x,-\frac{1}{2}\right) = He_n(x)\,. \tag{12}$$

Moreover, it is evident that:

$$H_n(x,0) = x^n\,. \tag{13}$$

The polynomials $H_n(x,y)$ have been shown as to be *quasi-monomial* under the action of the operators:

$$\begin{aligned} \hat{M} &= x + 2y\frac{\partial}{\partial x} \\ \hat{P} &= \frac{\partial}{\partial x} \end{aligned} \tag{14}$$

and then, according with the previous statements, we obtain:

- Differential equation

$$\left(2y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x}\right) H_n(x,y) = nH_{n-1}(x,y)\,. \tag{15}$$

- Generating function

$$e^{t\left(x+2y\frac{\partial}{\partial x}\right)}(1) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x,y)\,,\ t \in \mathbb{R}\,,\ |t| < +\infty\,. \tag{16}$$

From the above relations, a fairly straightforward conclusion is the proof that the generalized Hermite polynomials of two variables satisfies the heat equation:

$$\frac{\partial}{\partial y} H_n(x,y) = \frac{\partial^2}{\partial x^2} H_n(x,y)\,. \tag{17}$$

The proof is just a consequence of the structure of the generating function itself. By keeping, indeed the derivatives of both sides of (16) with respect to *t* and then equating the *t*-like powers, we find:

$$\frac{\partial}{\partial y} H_n(x,y) = n(n-1)H_{n-2}(x,y) \tag{18}$$

$$\frac{\partial}{\partial x} H_n(x,y) = nH_{n-1}(x,y) \tag{19}$$

for which the heat equation follows. This statement allows a further important result, indeed by regarding it as an ordinary first order equation in the variable *y* and by treating the differential operators as an ordinary number, we can write the polynomials $H_n(x,y)$ in terms of the following operational definition:

$$H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}} H_n(x,0) = e^{y\frac{\partial^2}{\partial x^2}} x^n\,. \tag{20}$$

In the same way it is possible to prove that the generalized Laguerre polynomials can be treated as *quasi-monomial* by using the operators:

$$\begin{aligned} \hat{M} &= y - \hat{D}_x^{-1} \\ \hat{P} &= -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \end{aligned} \tag{21}$$

where $\hat{D}_x^{-1}$ denotes the inverse of the derivative operator [5-7]. Being essentially an integral operator, it will be specified by the operational rule:

$$\hat{D}_x^{-1}(1) = \frac{x^n}{n!}\,. \tag{22}$$

(15)

According to the previous prescriptions, the generalized Laguerre polynomials are defined by the series:

$$L_n(x,y) = \left(y - \hat{D}_x^{-1}\right)^n = n!\sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(r!)^2 (n-r)!}\,, \tag{23}$$

by the differential equation:

$$\left[yx\frac{\partial^2}{\partial x^2}+(y-x)\frac{\partial}{\partial x}+n\right]L_n(x,y)=0\,, \tag{23}$$

and by the generating function:

$$e^{t\left(y-\hat{D}_x^{-1}\right)}(1)=e^{yt}e^{-t\hat{D}_x^{-1}}(1)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}L_n(x,y)\,. \tag{24}$$

By expanding the exponential containing the negative derivative operator, we find:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}L_n(x,y)=e^{yt}\sum_{r=0}^{+\infty}\frac{(-1)^r t^r}{r!}\hat{D}_x^{-r}(1)=e^{yt}\sum_{r=0}^{+\infty}\frac{(-1)^r t^r x^r}{(r!)^2}. \tag{25}$$

Regarding the series on the r.h.s. of the previous relation, we note that it can be recognized in terms of known functions, as:

$$\sum_{r=0}^{+\infty}\frac{(-1)^r x^r}{(r!)^2}=J_0(2\sqrt{x}) \tag{26}$$

where $J_0(x)$ is the *0^th^* order cylindrical Bessel function, so that:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}L_n(x,y)=e^{yt}J_0(2\sqrt{xt})\,. \tag{27}$$

Regarding the identification of the function on r.h.s. of the (25), we note that, the functions:

$$C_n(x)=\sum_{r=0}^{+\infty}\frac{(-1)^r x^r}{r!(n+r)!}=x^{-\frac{n}{2}}J_0(2\sqrt{x})\,, \tag{28}$$

specified by the generating function:

$$\sum_{n=-\infty}^{+\infty}t^n C_n(x)=e^{t-\frac{x}{t}} \tag{29}$$

are known as Tricomi functions.

III. Legendre polynomials

The *monomiality Principle* provides the possibility of deriving most of the properties of many polynomials of conventional or generalized nature. So far we have not stated a general procedure to find the *quasi-monomiality* operators for any polynomials, but we can use, however, an intuitive approach to develop to apply the method to different families of polynomials.

In the previous section we have seen that the generalized Hermite polynomials $H_n(x,y)$ are *quasi-monomial* as well as the two-variable generalized Laguerre polynomials $L_n(x,y)$ under the action, respectively, of the operator stated in (14) and (21), which can be combined to form:

$$\begin{aligned}\hat{M}&=y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\\ \hat{P}&=\frac{\partial}{\partial y}\end{aligned} \tag{30}$$

The above operators satisfy the Weyl [2] group structure and they therefore good candidates as monomiality operators. By exploiting the rule written in equation (9) of the first section, we can construct the polynomials associated with the above operators, just starting from the multiplication operator, namely:

$$\hat{M}^n(1)=\left(y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)^n(1)=n!\sum_{r=0}^{[n/2]}\frac{y^{n-2r}x^r}{(r!)^2(n-2r)!}. \tag{31}$$

As expected, the polynomials on r.h.s. of previous equation, which we will denote by [1]:

$${}_2L_n(x,y)\,,$$

exhibit a structure in between Laguerre and Hermite polynomials, and for the operational point of view, they can also be written as:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}\,{}_2L_n(x,y)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\hat{M}^n(1)=e^{t\left(y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)}(1) \tag{32}$$

By defining the operators:

$$\begin{aligned}\hat{A}&=ty\\ \hat{B}&=2t\hat{D}_x^{-1}\frac{\partial}{\partial y}\end{aligned} \tag{33}$$

we find, from the Weyl decoupling rule:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}\,{}_2L_n(x,y)=e^{ty}e^{t^2\hat{D}_x^{-1}}(1)=e^{ty}C_0(-xt^2)\,. \tag{34}$$

The above relation has the following important consequence: the polynomials ${}_2L_n(x,y)$ satisfies the differential equation:

$$\frac{\partial^2}{\partial y^2}\,{}_2L_n(x,y)=\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right){}_2L_n(x,y) \tag{35}$$

$${}_2L_n(0,y)=y^n\,, \tag{36}$$

which implies that these polynomials can be derived from the operational relation:

$${}_2L_n(x,y)=C_0\left(-x\frac{\partial^2}{\partial y^2}\right)y^n\,. \tag{37}$$

These polynomials provides also a generalization of the ordinary Legendre polynomials, we find, indeed:

$$\begin{aligned}&{}_2L_n\left(-\frac{1}{4}(1-y^2),y\right)=P_n(y)=\\&=n!\sum_{r=0}^{[n/2]}\frac{(-1)^{n-2r}(1-y^2)^{n-2r}\,y^r}{2^{2n-4r}(r!)^2(n-2r)!}\end{aligned} \tag{38}$$

and the properties of the Legendre polynomials can be therefore studied by using the formalism and the concepts of the *monomiality principle*; that is the Legendre polynomials can be treated as *quasi-monomial*.

IV. GENERALIZED LEGENDRE POLYNOMIALS

In the section III we have presented the Legendre polynomials by a different point of view, that is by combining the results of the *monomiality principle* stated in the cases of the Hermite and Laguerre polynomials. This procedure can be also extended by using different manipulations on the operators which we use to recognize the families of polynomials as quasi-monomials.

We can finally expose a different example to explore a further family of Legendre-like polynomials. Let, in fact, the following monomiality operators:

$$\begin{aligned}&\hat{M}=-\hat{D}_x^{-1}+\hat{D}_y^{-1}\\&\hat{P}_x=-\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\\&\hat{P}_y=\frac{\partial}{\partial y}y\frac{\partial}{\partial y}\end{aligned} \tag{39}$$

according to the so far developed formalism the associated polynomials will be explicit given by:

$$R_n(x,y)=n!\left(-\hat{D}_x^{-1}+\hat{D}_y^{-1}\right)^n(1)\,, \tag{40}$$

which yields:

$$R_n(x,y)=n!\sum_{r=0}^{n}\frac{(-1)^{n-r}x^{n-r}y^r}{(r!)^2\left[(n-r)!\right]^2}\,. \tag{41}$$

A fairly natural consequence of the above relations is that the polynomials $R_n(x,y)$ can be expressed as a discrete binomial convolution of Laguerre polynomials. We note indeed that (see eq. (40)):

$$\begin{aligned}&R_n(x,y)=n!\left[\left(1-\hat{D}_x^{-1}\right)-\left(1-\hat{D}_y^{-1}\right)\right]^n=\\&=n!\sum_{r=0}^{n}(-1)^r\binom{n}{r}L_{n-r}(x)L_r(y).\end{aligned} \tag{42}$$

The generating function satisfied by this polynomials can be evaluated by using the same tools shown in the previous sections, thus getting:

$$\begin{aligned}&\sum_{n=0}^{+\infty}\frac{t^n}{(n!)^2}R_n(x,y)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\left(-\hat{D}_x^{-1}+\hat{D}_y^{-1}\right)^n=\\&=C_0(-yt)C_0(xt).\end{aligned} \tag{43}$$

According to the structure of the monomiality operators and of this last identity, it is straightforward to prove that the polynomials $R_n(x,y)$ satisfies the following partial differential equation:

$$-\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)R_n(x,y)=\left(\frac{\partial}{\partial y}y\frac{\partial}{\partial y}\right)R_n(x,y)\,, \tag{44}$$

and, by noting that:

$$R_n(x,0)=\frac{(-x)^n}{n!}\,,$$

yields the operational definition:

$$R_n(x,y)=C_0\left[y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right]\frac{(-x)^n}{n!}\,. \tag{45}$$

This last family of polynomials too can be reduced to ordinary polynomials; by noting indeed that the Legendre polynomials $P_n(x)$ can be alternatively expressed in the form:

$$P_n(x)=(n!)^2\sum_{r=0}^{n}\frac{(-1)^{n-r}\left(\frac{1-x}{2}\right)^{n-r}\left(\frac{1+x}{2}\right)^r}{(r!)^2\left[(n-2r)!\right]^2}\,, \tag{46}$$

we obtain that $R_n(x,y)$ reduces to $P_n(x)$ when:

$$P_n(x) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right). \tag{47}$$

This last result is interesting by itself and because it allows to conclude that the Legendre polynomials can be viewed as a binomial convolution of Laguerre polynomials.

In the previous section we have written the generating function of the generalized two-variable Laguerre polynomials of the type:

$${}_2L_n(x,y)$$

by using the zero order Tricomi function (see eq. (28)), we can also derive a further generating function as:

$$\sum_{n=0}^{+\infty} t^n \, {}_2L_n(x,y) = \frac{1}{\sqrt{1-2yt+(y^2-4x)t^2}}. \tag{48}$$

Since the Legendre polynomials $P_n(x)$ are linked to the above polynomials by the relation stated in equation (38), we can immediately write the related generating function:

$$\sum_{n=0}^{+\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}. \tag{49}$$

The relation in the equation (47), linking the Legendre polynomials with the polynomials $R_n(x,y)$, can be recasting in the form:

$$R_n(x,y) = (x+y)^n P_n\left(\frac{y-x}{y+x}\right), \tag{50}$$

and can be used to derive the generating functions of the polynomials $R_n(x,y)$. By using the relation in the equation (49), we find, in fact:

$$\sum_{n=0}^{+\infty} t^n R_n(x,y) = \frac{1}{\sqrt{1-2(x-y)t+(x+y)^2t^2}}. \tag{51}$$

Otherwise, according to equation (50), we can immediately write:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} R_n(x,y) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}(x+y)^n P_n\left(\frac{y-x}{y+x}\right). \tag{52}$$

This class of polynomials can be using to state many different relations involving generalizations of Legendre and Laguerre polynomials, useful to provide a powerful tools to simplified many aspects of computation, with application in statistics [8], probability theory, electromagnetics [9-12], industrial engineering and economics [13-17].

V. *BASED*-BESSEL FUNCTIONS

The considerations presented in the previous sections, confirm that the most of the properties of families of polynomials, recognized as quasi-monomial, can be deduced, quite straightforwardly, by using operational rules associated with the relevant multiplication and derivative operators. Furthermore, they suggest that we can introduce or *"define"* families of isospectral problems [18-22] by exploiting the correspondence:

$$\hat{M} \to x$$
$$\hat{P} \to \frac{\partial}{\partial x}$$
$$p_n(x) \to x^n.$$

We can therefore use the polynomials $p_n(x)$ as a basis to introduce *"new"* functions with eigenvalues corresponding to the ordinary case. The most useful example is provided by a *p-based* Bessel function, defined as:

$${}_pJ_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r p_{n+2r}}{2^{n+2r} r!(n+r)!}, \tag{53}$$

which is easily shown to satisfy the equation:

$$\left[\hat{M}\hat{P}\hat{M}\hat{P} - \left(\hat{M}^2 - n^2\right)\right] = {}_pJ_n(x) = 0, \tag{54}$$

which provides an isospectral problem to the ordinary Bessel equation.

Since the generating function of the ordinary cylinder Bessel function is:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x), \tag{55}$$

we can cast the relevant to *p-based* Bessel function as:

$$\exp\left[\frac{\hat{M}}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n \, {}_pJ_n(x). \tag{56}$$

So that, we can introduce different *p-based* Bessel function, by using the polynomials presented in the previous sections, since we have proved that they satisfied the rules of the monomiality principle. In fact, in the case of *Hermite-based* Bessel function, we can immediately obtain:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{4}\left(t-\frac{1}{t}\right)^2\right]=\sum_{n=-\infty}^{+\infty} t^n \, {}_H J_n(x,y)\,, \qquad (57)$$

which is a trivial consequence of the structure of the multiplicative operator related to the Hermite polynomials and of the Weyl decoupling rule. This last identity can be exploited to derive the series expansion definition:

$$ {}_H J_n(x,y)=\sum_{r=0}^{+\infty}\frac{(-1)^r H_{n+2r}(x,y)}{2^{n+2r}\, r!(n+r)!}\,, \qquad (58)$$

and the link with the two-variable Bessel function is given by:

$$ {}_H J_n(x,y)=e^y\sum_{r=0}^{+\infty} J_{n+2r}(x,2y)\frac{(-y)^2}{r!}\,. \qquad (59)$$

An important result, stressing the relevance of the Hermite-Bessel functions in applications, is the related Jacobi-Anger expansion, which is obtained by performing the substitution:

$$t\to e^{i\theta}\,,\ \theta\in(0,2\pi)$$

in the equation (57), we have:

$$e^{ix\sin\theta-y(\sin\theta)^2}=\sum_{n=-\infty}^{+\infty} e^{in\theta}\, {}_H J_n(x,y)\,, \qquad (60)$$

which can be further exploited to derive the integral representation:

$$ {}_H J_n(x,y)=\frac{1}{\pi}\int_0^{\pi} e^{-y(\sin\theta)^2}\cos(x\sin\theta-n\theta)d\theta\,.$$

These functions have considerable application in the theory of the emission of electromagnetic radiation by charged particles moving in magnetic fields. In a future paper we will investigate other possible generalization of the *p-based* Bessel functions.

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