The Stability of Equilibriums of a Fifth Order Ordinary Differential Equation

Chung-Hsien Tsai, Shy-Jen Guo, and Jeng-Chi Yan

***Abstract*—**The objective of this paper is to study the stability of equilibrium points of a model equation, which governs two-dimensional steady capillary-gravity waves of an ideal fluid flow with Bond number near 1/3 and Froude number close to one. Nine cases in the parameter plane (τ_1, F_2), we found that the equilibrium point (0,0,0,0) is Liapunov stable in Case 3 except when $w_1/w_2 = 2$, almost stable in Case 4, and Liapunov unstable otherwise.

***Keywords*—**Arnold's stability theorem, Birkhoff normal form, Steady capillary-gravity wave,

I. INTRODUCTION

PROGRESSIVE capillary-gravity waves on an irrotional incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [17]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [11] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

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It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden-Broeck's work [9], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2\epsilon^2$, $\tau = 1/3 + \tau_1\epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = 0 \tag{1}$$

Equation (1) has been studied extensively by many authors [1-7,9] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-7,9], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_1 < 0$ and $F_2 > 0$ [1,9], etc.

II. PROBLEM FORMULATION

We add a bump $y = b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = b \tag{2}$$

Equation (2) has been studied extensively by Tsai and Guo [21-26] and several types of solutions have been found.

We follow Zufiria [27] to construct a Hamiltonian associated to (2).

When $\mathbf{b} = 0$, we rewrite (2) as

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta + \frac{135}{2}\eta^2 = 0. \tag{3}$$

We multiply $-\eta_x$ to (3) and integrate the resulting equation, then equation (3) has first integral as

$$H = 45F_2\eta^2 + \frac{1}{2}\eta_{xx}^2 - \eta_{xxx}\eta_x + \frac{45}{2}\tau_1\eta_x^2 - \frac{45}{2}\eta^3, \tag{4}$$

where H is a constant. Introducing the change of variables

$$\left.\begin{aligned} q_1 &= \eta, \quad p_1 = \eta_{xxx} - 45\tau_1\eta_{x,} \\ q_2 &= \eta_{xx}, \quad p_2 = \eta_{x,} \end{aligned}\right\}$$

then (4) becomes

$$H\left(q_1, q_2, p_1, p_2\right) = 45F_2 q_1^2 + \frac{1}{2}q_2^2 - p_1 p_2 - \frac{45}{2}\tau_1 p_2^2 - \frac{45}{2}q_1^3, \qquad (5)$$

and we have

$$\frac{dz}{dx} = J\nabla_z H(z) = Az + g(z) \equiv f(z, \mu), \qquad (6)$$

where $\mu = (\tau_1, F_2) \in \mathbf{R}^2$,

$$z = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbf{R}^4, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad (7)$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -45\tau_1 \\ -90F_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, g(z) = \begin{pmatrix} 0 \\ 0 \\ \frac{135}{2}q_1^2 \\ 0 \end{pmatrix}. \qquad (8)$$

Therefore (5) is a two degree of freedom Hamiltonian with two parameters τ_1 and F_2. Because different parameters (τ_1, F_2) in (5) give rise to different eigenvalues λ for the linearized system of (6) at the origin, we divide the parameter plane (τ_1, F_2) into following nine cases

Case 0 $(\tau_1 = 0, F_2 = 0)$: $\lambda = 0,0,0,0$.

Case 1 $(\tau_1 \in \mathbf{R}, F_2 > 0)$: $\lambda = \pm r, \pm wi;\ r, w > 0$.

Case 2 $(\tau_1 < 0, F_2 = 0)$: $\lambda = 0, 0, \pm wi;\ w > 0$.

Case 3 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm w_1 i, \pm w_2 i;\ w_1 > w_2 > 0$.

Case 4 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm wi, \pm wi;\ w > 0$

Case 5 $(\tau_1 \in \mathbf{R}, F_2 < 0, (45\tau_1)^2 + 360F_2 < 0)$:
$\lambda = \pm a \pm bi;\ a, b > 0$

Case 6 $(\tau_1 > 0, F_2 < 0, (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm r, \pm r;\ r > 0$

Case 7 $(\tau_1 > 0, F_2 < 0, (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm r_1, \pm r_2;\ r_1 > r_2 > 0$

Case 8 $(\tau_1 > 0, F_2 = 0)$: $\lambda = 0, 0, \pm r;\ r > 0$.

We rewrite (2) as follows,

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = -45(\mathbf{b}(x)) + \frac{3}{2}\eta^2) \equiv f, \qquad (9)$$

III. Stable cases for zero solution

For Case 3 with $w_1/w_2 \neq 2$ and Case 4, we'll utilize the results by Meyer [15], Markeev [12], and Sokol'skii [18] to show that the origin is stable, except that Case 4 is almost stable.

3.1 Case 3, with $w_1/w_2 \neq 2$

In Case 3, there are two pairs of pure imaginary eigenvalues, $\pm w_1 i$ and $\pm w_2 i$. Arnold's Stability Theorem [15] (p.236) will be used to prove the equilibrium point $z = 0$ is stable for $w_1/w_2 \neq 2, 3$, while Markeev's results in [12] will show that the origin is unstable for $w_1/w_2 = 2$ and stable for $w_1/w_2 = 3$.

3.1.1 Arnold's stability theorem for $w_1/w_2 \neq 2, 3$

We state the method cited from [15] as follows. Consider an analytic Hamiltonian, H, which has an equilibrium point at the origin in R^{2n}, and assume that the Hamiltonian is zero at the origin. Then H has a Taylor series expansion of the form

$$H(\tilde{z}) = \sum_{i=0}^{\infty} H_i(\tilde{z}), \qquad (10)$$

where H_i is a homogeneous polynominal in z of deg- ree i + 2.

Theorem 1 [15] (p.184) (Birkhoff normal form) Assume that the quadratic part of (10) is of the form

$$H_0(\tilde{z}) = \sum_{i=1}^{\infty} \lambda_i \tilde{q}_i \tilde{p}_i, \qquad (11)$$

where $\tilde{z} = (\tilde{q}_1, \ldots, \tilde{q}_n, \tilde{p}_1, \ldots, \tilde{p}_n)$ and the λ_i's are independent over the integers, i.e., there is no nontrivial relation of the form

$$\sum_{i=1}^{\infty} K_i \lambda_i = 0,$$

where the k_i 's are integers. Then there exists a formal, symplectic change of variables $\tilde{z} = Q(y) = y + \ldots$ which transforms the Hamiltonian (8) to the Hamiltonian,

$$K(y) = \sum_{i=1}^{\infty} K_i(y),$$

where $y = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $K_i(y)$ is a homogenous polynomial of degree i + 2 in the n products $x_1 y_1, \ldots, x_n y_n$. So, $K(x_1, \ldots, x_n, y_1, \ldots, y_n) = K(x_1 y_1, \ldots, x_n y_n)$ where K is a function of n variables. Moreover, in this case, the normal form is unique.

In Birkhoff normal form, $K_{2i+1} = 0$ for i = 0, 1, 2 … since $K(x_1, \ldots, x_n, y_1, \ldots, y_n) = K(x_1 y_1, \ldots, x_n y_n)$. Consider two-degree of freedom Hamiltonian for simplicity and assume the Hamiltonian has been transformed to Birkhoff normal form upto K_{2N}

$$K = K_0 + K_2 + \ldots + K_{2N} + K^+. \qquad (12)$$

By "action" variables $I_1 = ix_1 y_1$ and $I_2 = ix_2 y_2$, then $\tilde{K}_{2j}(I_1, I_2) \equiv K_{2j}(x_1 y_1, x_2 y_2)$.

Arnold proved the following theorem which is based on the existence of invariant tori in KAM theory and gives sufficient conditions for stability of nonresonant systems in terms of the transformed Hamiltonian $\tilde{K}_{2j}(I_1, I_2)$ put in Birkhoof normal form.

Theorem 2 [15] (p.236) (Arnold's Stability Theorem) The origin is stable for the system whose Hamiltonian is (12), provided for some j, $0 \le j \le N$, $D_{2j} = \tilde{K}_{2j}$, $(w_2, w_1) \neq 0.$

First we find a symplectic transformation T_3 to transform the quadratic part of Hamiltonian (5) to the form (11), where

$$T_3 = \begin{pmatrix} \frac{-w_{i_2}}{w_{r_1}w_1} & \frac{w_{i_2}}{w_{r_2}w_2} & \frac{w_{i_2}}{w_{r_1}w_1} & \frac{-w_{i_2}}{w_{r_2}w_2} \\ \frac{w_{i_2}w_1}{w_{r_1}} & \frac{-w_{i_2}w_2}{w_{r_2}} & \frac{-w_{i_2}w_1}{w_{r_1}} & \frac{w_{i_2}w_2}{w_{r_2}} \\ \frac{w_{i_1}w_2^2}{w_{r_1}} & \frac{w_{i_1}w_1^2}{w_{r_2}} & \frac{w_{i_1}w_2^2}{w_{r_1}} & \frac{w_{i_1}w_1^2}{w_{r_2}} \\ \frac{w_{i_1}}{w_{r_1}} & \frac{w_{i_1}}{w_{r_2}} & \frac{w_{i_1}}{w_{r_1}} & \frac{w_{i_1}}{w_{r_2}} \end{pmatrix}$$

w_1, w_2 are imaginary parts of eigenvalues in Case 3 and

$$w_{i_1} = \frac{1+i}{2}, w_{i_2} = \frac{1-i}{2}, w_{r_1} = \sqrt{\frac{w_1^{\ 2} - w_2^{\ 2}}{w_1}}, w_{r_2} = \sqrt{\frac{w_1^{\ 2} - w_2^{\ 2}}{w_2}} \quad (13)$$

with the properties that $T_3^T J T_3 = J, z = T_3\tilde{z}$, where $\tilde{z} = (\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2)$, J as in (7). Then (5) becomes

$$H = iw_1\tilde{q}_1\tilde{p}_1 - iw_2\tilde{q}_2\tilde{p}_2 - \frac{45}{2}w_{i_2}$$
$$\left(\frac{w_{r_2}w_2(\tilde{p}_1 - \tilde{q}_1) + w_{r_1}w_1(\tilde{q}_2 - \tilde{p}_2)}{w_{r_1}w_{r_2}w_1w_2}\right)^3, \quad (14)$$

and the quadratic part is $H_0(\tilde{z}) = iw_1\tilde{q}_1\tilde{p}_1 - iw_2\tilde{q}_2\tilde{p}_2$

Next, we apply Lie transform [7] to transform (14) to Birkhoff normal form up to fourth order K_2 as in (12). Here we assume that $w_1/w_2 \neq 2, 3$ since we can not remove the resonant terms such that the transformation yields Birkhoff normal form. Then we have

$$D_2 = \frac{6075(20 - 53w^2 - 276w^4 - 53w^6 + 20w^8)}{16w^4(-1+w^2)^2(4 - 17w^2 + 4w^4)w_2^8}$$

where $w = w_1/w_2 > 1$

Thus, by Arnold's stability theorem and $w_1/w_2 \neq 2, 3, z = 0$ is stable except when

$$w = w^* = \frac{1}{4}\sqrt{\frac{1}{5}(53 + 3\sqrt{3121} + \sqrt{24498 + 318\sqrt{3121}})} \approx 2.308.$$

In this case $D_2 = 0$. Stability is assured if $D_4 \neq 0$, and so on. We did not get the value for D_4 at $w = w^*$ because of " Out of memory" by using the software Mathematica. But the origin is stable at $w = w^*$ from our numerical experience.

3.1.2 Markeev's results for $w_1/w_2 = 3$

In [12], Markeev considered the resonance situation for $w = w_1/w_2 = 3$ with a Hamiltonian in the follow- ing form

$$H = \frac{1}{2}(\tilde{p}_1^2 + w_1^2\tilde{q}_1^2) - \frac{1}{2}(\tilde{p}_2^2 + w_2^2\tilde{q}_2^2) + \sum_{v=3}\tilde{g}_{v_1v_2v_3v_4}$$
$$\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4} + \sum_{v=4}\tilde{h}_{v_1v_2v_3v_4}\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4} + O(|\tilde{z}|^5) \quad (15)$$

where

$$|\tilde{z}| = \sqrt{\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{p}_1^2 + \tilde{p}_2^2}, \quad v = v_1 + v_2 + v_3 + v_4.$$

We transform Hamiltonian (5) to (15) by a symplectic transformation $z = T_{33}\tilde{z}$, where

$$T_{33} = \begin{pmatrix} 0 & 0 & \frac{1}{w_{r_1}w_1^{3/2}} & -\frac{1}{w_{r_2}w_2^{3/2}} \\ 0 & 0 & -\frac{w_1^{1/2}}{w_{r_1}} & \frac{w_2^{1/2}}{w_{r_2}} \\ \frac{w_1^{1/2}w_2^2}{w_{r_1}} & \frac{w_2^{1/2}w_1^2}{w_{r_2}} & 0 & 0 \\ \frac{w_1^{1/2}}{w_{r_1}} & \frac{w_2^{1/2}}{w_{r_2}} & 0 & 0 \end{pmatrix},$$

then (5) becomes

$$H = \frac{1}{2}(\tilde{p}_1^2 + w_1^2\tilde{q}_1^2) - \frac{1}{2}(\tilde{p}_2^2 + w_2^2\tilde{q}_2^2) - \frac{45\tilde{p}_1^3}{2w_{r_1}w_1^{9/2}} +$$
$$\frac{45\tilde{p}_2^3}{2w_{r_2}w_2^{9/2}} - \frac{135\tilde{p}_1\tilde{p}_2^2}{2w_{r_1}w_{r_2}^2w_1^{3/2}w_2^3} + \frac{135\tilde{p}_1^2\tilde{p}_2}{2w_{r_1}^2w_{r_2}w_1^3w_2^{3/2}} \quad (16)$$

In [12], with the assumption

$$x_{1003}^2 + y_{1003}^2 \neq 0, \quad (17)$$

(see Appendix for x_{1003} , y_{1003}), Markeev utilized several canoncial transformations and applied Birkhoff transformation to remove all third order terms. Out of the fourth degree terms, only the resonant ones and those containing $\tilde{q}_i$ and $\tilde{p}_i$ in the same degree will remain. With assumption (17), Markeev proved the following results by Liapunov's theorem [13] (p.32).

Theorem 3 If the inequality

$$x_{1003}^2 + y_{1003}^2 \neq 0, \; 3w_2\sqrt{x_{1003}^2 + y_{1003}^2} < |l_{2020} - 3l_{1111} + 9l_{0202}|,$$

holds and the Hamiltonian (15) contains no terms of the order higher than the fourth, then the equilibrium is stable. (see Appendix for l_{2020}, l_{1111}, and l_{0202})

Note that Markeev [12] provided numerical formulas for $x_{1003}, y_{1003}, l_{2020}, l_{1111}, l_{0202}$ in terms of the cofficients in (15), i.e., $\tilde{g}_{v_1v_2v_3v_4}$ and $\tilde{h}_{v_1v_2v_3v_4}$. By comparing Hamiltonians (15) and (16) term by term, we have

$$\left.\begin{aligned} \tilde{g}_{v_1v_2v_3v_4} &= 0, \quad for \quad v_1 + v_2 + v_3 + v_4 = 3, \\ \tilde{h}_{v_1v_2v_3v_4} &= 0, \quad for \quad v_1 + v_2 + v_3 + v_4 = 4, \end{aligned}\right\} \quad (18)$$

except

$$\tilde{g}_{0012} = -\tfrac{15\sqrt{3}}{4w_2^6}, \tilde{g}_{0021} = \tfrac{15\sqrt{3}}{8w_2^6}, \tilde{g}_{0030} = -\tfrac{15\sqrt{3}}{16w_2^6}, \; \tilde{g}_{0003} = \tfrac{5\sqrt{3}}{2w_2^6}, \quad (19)$$

By (18) and (19) and following Markeev's nummerical formulas, we have

$$x_{1003} = -\tfrac{1215}{256w_2^{11}}, \; y_{1003} = 0,$$
$$l_{2020} = -\tfrac{3165}{7168w_2^{10}}, \quad l_{1111} = \tfrac{2295}{1792w_2^{10}}, \quad l_{0202} = -\tfrac{2835}{1024w_2^{10}},$$

and then obtain $x_{1003}^2+y_{1003}^2\neq 0$ and $\{(3w_2\sqrt{x_{1003}^2+y_{1003}^2})-|l_{2020}-3l_{1111}+9l_{0202}|\}=-53625/(3584w_2^{10})<0$. Thus, by Theorem 3, the equilibrium $z=0$ is stable when $w_1/w_2=3$.

3.2 Case 4

In this case, there are two pairs of double eigenvalues $\pm iw$ with two two-dimensional Jordan blocks. Sokol'skii [18] deals with this situation. Let the Hamilton function of the problem be represented in the form

$$H=\frac{1}{2}(\tilde{q}_1^2+\tilde{q}_2^2)+w(\tilde{q}_1\tilde{p}_2-\tilde{q}_2\tilde{p}_1)+\sum_{v=3}^{\infty}\tilde{h}_{v_1v_2v_3v_4}\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4}=H_2+H_3+H_4+\ldots+H_m+\ldots \quad (20)$$

where H_m are mth-degree polynomials in the coordinates q_i and momenta p_i, $i=1,2$. The form H_3 in (20) can be annulled completely and the form H_4 simplified by the Birkhoff transformation. The Ha- miltonian (20) then can be reduced to

$$H=\frac{1}{2}(q_1^{*2}+q_2^{*2})+w(q_1^*p_2^*-q_2^*p_1^*)+(p_1^{*2}+p_2^{*2})$$
$$\left\{A_4(p_1^{*2}+p_2^{*2})+B_4(q_1^*p_2^*-q_2^*p_1^*)+C_4(q_1^{*2}+p_2^{*2})\right\}+H_5+\ldots \quad (21)$$

With (21), Sokol'skii [8] got the following theorem

Theorem 4 The equilibrum position is almost stable if $A_4>0$ and is Liapunov unstable if $A_4<0$ in (19).

After transforming (5) to (20) by a symplectic transformation, $(q_1,q_2,p_1,p_2)^t=T_4(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)^t$, where

$$T_4=\begin{pmatrix} \frac{1}{2\sqrt{2}w^2} & 0 & 0 & -\frac{1}{2\sqrt{2}w} \\ \frac{3}{2\sqrt{2}} & 0 & 0 & \frac{w}{\sqrt{2}} \\ 0 & \frac{3w}{2\sqrt{2}} & \frac{w^2}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}\omega} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad (22)$$

we have

$$H=\frac{1}{2}(\tilde{q}_1^2+\tilde{q}_2^2)+w(\tilde{q}_1\tilde{p}_2-\tilde{q}_2\tilde{p}_1)$$
$$-\frac{45\tilde{q}_1^3}{32\sqrt{2}w^6}+\frac{135\tilde{q}_1^2\tilde{p}_2}{16\sqrt{2}w^5}-\frac{135\tilde{q}_1\tilde{p}_2^2}{8\sqrt{2}w^4}+\frac{45\tilde{p}_2^3}{4\sqrt{2}w^3} \quad (23)$$

Comparing (20) and (23), we have

$$\tilde{h}_{3000}=-\tfrac{45}{32\sqrt{2}w^6},\ \tilde{h}_{2001}=\tfrac{135}{16\sqrt{2}w^5},\ \tilde{h}_{1002}=-\tfrac{135}{8\sqrt{2}w^4},\ \tilde{h}_{0003}=\tfrac{45}{4\sqrt{2}w^3}, \quad (24)$$

and $\tilde{h}_{v_1v_2v_3v_4}=0$ for other $v_1v_2v_3v_4$. Sokol'skii [18] also provided numerical formulas in terms of $h_{v_1v_2v_3v_4}$ to compute A_4 in (21) and then we obtain

$$A_4=\frac{38475}{256w^8}$$

By Theorem 4, the equilibrium at the origin is almost stable which as defined in Moser [16]

IV. Unstable cases for zero solution

For Case 0, 2 and Case 3 with $w_1/w_2=2$, we'll utilize the results by Sokol'skii [19], [20], and Markeev [12] respectively to show that the origin is unstable. For Case 1, 5, 6, 7, 8, we use a theorem by Liapunov to prove the instability for these five cases.

4.1 Case 0

In this case, all four eigenvalues are zero. Sokol'skii [20] deals with this situation. Let the Hamilton function of the problem be represented in the form

$$H=H_2+H_3+H_4+\ldots+H_m+\ldots \quad (25)$$

where the H_m are mth-degree homogeneous polynomials in the generalized coordinates q_k and momenta p_k (k = 1, 2):

$$H_m=\sum_{v_1+v_2+v_3+v_4=m}h_{v_1v_2v_3v_4}q_1^{v_1}q_2^{v_2}p_1^{v_3}p_2^{v_4} \quad (26)$$

According to Sokol'skii [20], if the rank of the second derivative of H in (25) at the equilibrium zero is 3, then there is a symplectic transformation T_0, with $(q_1,q_2,p_1,p_2)^t=T_0(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)^t$, to transform H in (25) to a simpler form

$$H=\frac{1}{2}\delta\tilde{p}_1^2-\tilde{q}_1\tilde{q}_2+\sum_{m=3}^{\infty}\tilde{h}_{v_1v_2v_3v_4}\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4},\ (\delta=\pm 1) \quad (27)$$

With (27) and by Chetaev theorem [13] (p.43), Sokol'skii [20] obtained the following result,

Theorem 5 In (27), if $\tilde{h}_{0003}\neq 0$, then the equilibrum position is unstable.

From (9), the rank of the second derivative of H in (5) at the equilibrium zero is 3 when $\tau_1=0$ and $F_2=0$. Following Sokol'skii [20], we found a slmplectic transformation T_0,

$$T_0=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (28)$$

which transform (5) to (27) by $(q_1,q_2,p_1,p_2)^t=T_0(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)^t$ and then we have

$$H=\frac{1}{2}\tilde{p}_1^2-\tilde{q}_1\tilde{q}_2-\frac{45}{2}\tilde{p}_2^3. \quad (29)$$

From (29), we obtain

$$\tilde{h}_{0003}=-\frac{45}{2}\neq 0.$$

By Theorem 5, the equilibrium at $z=0$ is unstable.

4.2 Case 1, Case 5, Case 6, Case 7 and Case 8

Lyapunov proved the following theorem:

Theorem 6 [8] Suppose $f(z^*)=0$ and z^* is a stable equilibrium point of the nonlinear system $dz/dx=f(z)$, then no eigenvalue of $Df(z^*)$ has a positive real part.

For Cases 1, 5, 6, 7, 8, the equilibrium point $z=0$ is unstable since there is at least one eigenv- alue with positive real part in each case.

4.3 Case 2

In this case, two eigenvalues are zero and the other two are pure imagimary, $\pm wi$. Sokol'skii [19] deals with this situation. Let the Hamilton function of the problem be represented in the form same as (25) and (26). If there is a symplectic transformation T_2, with $(q_1,q_2,p_1,p_2)^t = T_2(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)^t$, to transform H in (25) to

$$H = \frac{1}{2}\delta_1\tilde{q}_1^2 + \frac{1}{2}\delta_2 w(\tilde{q}_2^2+\tilde{p}_2^2)$$

$$+\sum_{m=3}^{\infty}\tilde{h}_{v_1v_2v_3v_4}\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4}, \quad (\delta_1=\pm 1, \quad \delta_2=\pm 1), \qquad (30)$$

then with (30) and by Liapunov's instability theorem [4] (p.45), Sokol'skii [19] proved the following results:

Theorem 7 In (30), if $\tilde{h}_{0030}\neq 0$, then the equilibrium position is unstable.

Following Sokol'skii [19], we found a slmplectic transformation T_2,

$$T_2 = \begin{pmatrix} 0 & \frac{1}{w^{3/2}} & -\frac{1}{w} & 0 \\ 0 & -w^{1/2} & 0 & 0 \\ w & 0 & 0 & 0 \\ \frac{1}{w} & 0 & 0 & -\frac{1}{w^{1/2}} \end{pmatrix}, \qquad (31)$$

which transforms (5) to (30) by

$$(q_1,q_2,p_1,p_2)^t = T_2\,(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)^t$$

and then we have

$$H = -\frac{1}{2}\tilde{q}_1^2 + \frac{1}{2}w(\tilde{q}_2^2+\tilde{p}_2^2) - \frac{45}{2w^{9/2}}\tilde{q}_2^3 .$$

$$+\frac{135}{2w^4}\tilde{q}_2^2\tilde{p}_1 - \frac{135}{2w^{7/2}}\tilde{q}_2\tilde{p}_1^2 + \frac{45}{2w^3}\tilde{p}_1^3 \qquad (32)$$

From (32), we obtain

$$\tilde{h}_{0300} = -\frac{45}{2w^{9/2}} \neq 0 .$$

By Theorem 7, the equilibrium at z = 0 is unstable in this case.

4.4 Case 3, with $w_1/w_2 = 2$

In [12], with Hamiltonian (13) and the assumption

$$x_{1002}^2 + y_{1002}^2 \neq 0 \qquad (33)$$

where

$$x_{1002} = -\frac{w_1}{2w_2}\tilde{g}_{0111} - \frac{1}{2}\tilde{g}_{1002} + \frac{1}{2w_2^2}\tilde{g}_{1200},$$

$$y_{1002} = -\frac{w_1}{2}\tilde{g}_{0012} + \frac{w_1}{2w_2^2}\tilde{g}_{0210} + \frac{1}{2w_2}\tilde{g}_{1101} \qquad (34)$$

Markeev utilized several canoncial transformations and applied Birkhoff transformation to remove all third order terms except the resonant ones. Then the Hamiltonian (15) becomes, in the new variables q_i^* and p_i^*,

$$H = w_2(q_1^{*2}+p_1^{*2}) - \frac{1}{2}w_2(q_2^{*2}+p_2^{*2}) - \frac{1}{2}$$

$$\sqrt{2w_2(x_{1002}^2+y_{1002}^2)}\left(\frac{1}{2}q_1^*(p_2^{*2}-q_2^{*2}) + p_1^*q_2^*p_2^*\right) + O(|z^*|^4) \qquad (35)$$

With Hamiltonian (35) and assumption (33), Markeev proved the following result by means of Chetaev theorem [13] (p.43).

Theorem 8 If the inequality $x_{1002}^2 + y_{1002}^2 \neq 0$ holds for the Hamiltonian of a perturbed montion, then the equilibrium is unstable.

Put (18) and (19) in (34) to obtain

$$x_{1002}^2 + y_{1002}^2 = \frac{675}{16w_2^{10}} \neq 0 .$$

Thus, by Theorem 8, the equilibrium z = 0 is unstable when $w_1/w_2 = 2$.

At this stage, we are ready to summarize our disscussion above as follows:

Theorem 9 The zero solution of equation (3) is Liapunov stable in Case 3 except when $w_1/w_2 = 2$, almost stable in Case 4, and unstable otherwise.

Fig. 1 illustrates Theorem 9 in the parameter plane (τ_1, F_2).

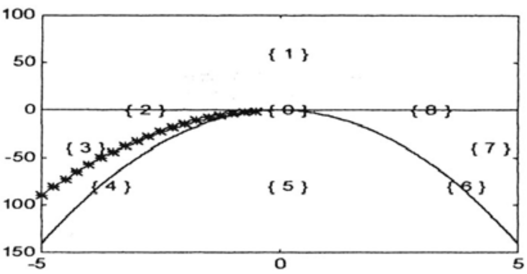

Figure 1: Nine cases in the parameter plane (τ_1, F_2) for equilibrium (0,0,0,0). The equilibrium is Liapunov stable in Case 3 except when $w_1/w_2 = 2$, i.e., $F_2 = -\frac{18}{5}\tau_1^2$ with $\tau_1 < 0$, the "***" curve, almost stable in Case 4, and Liapunov unstable otherwise.

V.5 CASES FOR NONZERO SOLUTION

In order to study the motion near the other equilibrium point $(\frac{4}{3}F_2, 0, 0, 0)$ and utilize the results above, we transform (6) to new coorinates by

$$\tilde{q}_1 = q_1 - \frac{4}{3}F_2,\ \tilde{q}_2 = q_2,\ \tilde{q}_3 = q_3,\ \tilde{q}_4 = q_4 .$$

This transformation to the new coordinates $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)$ is obviously symplectic. So we can perform this change of coordinates in the Hamiltonian (6) and preserve its structure. Expanding in the new variables, we obtain

$$H(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2) = -45F_2\tilde{q}_1^2 + \frac{1}{2}\tilde{q}_2^2 - \tilde{p}_1\tilde{p}_2 - \frac{45}{2}\tau_1\tilde{p}_2^2 - \frac{45}{2}\tilde{q}_1^3 \quad (36)$$

There are no linear terms because the expansion is performed near an equilibrum and the constant term has been omitted because it contributes nothing in forming the corresponding systems of differential equations.

We see that the only difference between Hamil- tonian (5) and (36) is the cofficients of q_1^2 and $\tilde{q}_1^2$, which are $45F_2$ and $-45F_2$ respectively, i.e., Hamiltonian (5) and (36) are symmetric with respect to $F_2 = 0$. Therefore, by considering the symmetry with respect to $F_2 = 0$, the results in Theorem 9 for $(q_1,q_2,q_3,q_4) = (0,0,0,0)$, the equilibrium of Hamilton equations of (5), can be carried over for $(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4) = (0,0,0,0)$, the equilibrium of Hamilton equations of (36),i.e., for $(q_1,q_2,q_3,q_4) = (\frac{4}{3}F_2,0,0,0)$, the equilibrium of Hamilton equations of (5).

As before, we divide the parameter plane (τ_1,F_2) into nine cases as follows, where λ's are the eigenva- lues of the second derivative of the Hamiltonian (36):

Case 10: $\tau_1 = 0, F_2 = 0$ implies λ = 0, 0, 0, 0.

Case 11: $\tau_1 \in \mathbf{R}, F_2 < 0$ implies $\lambda = \pm$ r11, $\pm iw_{11}; r_{11}, w_{11} > 0$.

Case 12: $\tau_1 < 0, F_2 = 0$ implies $\lambda = 0,0,\pm iw_{12};\ w_{12} > 0$.

Case 13: $\tau_1 < 0, F_2 > 0, (45\tau_1)^2 - 360F_2 > 0$
implies $\lambda = \pm iw_{131}; \pm iw_{132};\ w_{131} > w_{132} > 0$.

Case 14: $\tau_1 < 0,\ F_2 > 0,\ (45\tau_1)^2 - 360F_2 = 0$
implies $\lambda = \pm iw_{14}, \pm iw_{14};\ w_{14} > 0$.

Case 15: $\tau_1 \in \mathbf{R}, F_2 > 0, (45\tau_1)^2 - 360F_2 < 0$
implies $\lambda = \pm a_{15} + ib_{15};\ a_{15}, b_{15} > 0$.

Case 16: $\tau_1 > 0,\ F_2 > 0,\ (45\tau_1)^2 - 360F_2 = 0$
implies $\lambda = \pm r_{16}, \pm r_{16};\ r_{16} > 0$.

Case 17: $\tau_1 > 0,\ F_2 > 0,\ (45\tau_1)^2 - 360F_2 > 0$
implies $\lambda = \pm r_{171}, \pm r_{172};\ r_{171} > r_{172} > 0$.

Case 18: $\tau_1 > 0, F_2 = 0$ implies $\lambda = 0,0,\pm r_{18};\ > 0$.

Where the eigenvalues in Case 1j have the same expressions as the correspond Case j (for j = 0, 1,…,8) except the change of F_2 to $-F_2$ and then we have the following corollary:

Corollary 10 The equilibrium $(\frac{4}{3}F_2,0,0,0)$ of equation (3) is Liapunov stable in Case 13 except when $w_{131}/w_{132} = 2$, almost stable in Case 14, and unstable otherwise.

Fig. 2 illustrates Corollary 10 in the parameter plane (τ_1,F_2).

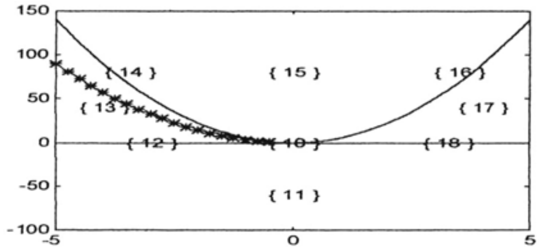

Figure 2: Nine cases in the parameter plan (τ_1,F_2) for equilibrium $(\frac{4}{3}F_2,0,0,0)$. The equilibrium is Liapunov stable in Case 13 except when $w_{131}/w_{132} = 2$, i.e., $F_2 = -\frac{18}{5}\tau_1^2$ with $\tau_1 < 0$, the "***" curve, almost stable in Case 14, and Liapunov unstable otherwise.

VI. Numerical results

For verifying the stability results, we assume that the bump b(x) of equation (2) has a compact support on [–1, 1] and solve (2) numerically as an initial value problem starting from x = –2 on the range $-2 \le x \le 500$ by using the classical fourth-order Runge- Kutta method, which will be called Scheme 1. We adjust the size of $|\mathbf{b}|$ to change the values of $d\eta^n/dx^n$ at x = 1, n = 0, 1, 2, 3. The numerical results show that when the values of $d\eta^n/dx^n$ at x = 1, n = 0, 1, 2, 3, are sufficiently close to (0, 0, 0, 0) then $\eta(x)$ is bounded in a neighborhood of (0, 0, 0, 0) for $-2 \le x \le 500$ if the zero solution is stable. For most of the unstable cases, $\eta(x)$ becomes unbounded as x increases even if the values of $d\eta^n/dx^n$ at x = 1, n = 0, 1, 2 , 3, are sufficiently close to (0, 0, 0, 0). This is because the condition that the initial values are sufficiently close to zero is not sufficient to find a bounded solution if the equilibrium point zero is unstable. Another reason is that if the values of $d\eta^n/dx^n$ at x = 1, n = 0, 1, 2 , 3 obtained from our numerical scheme do not match the bounded solutions at x = 1 in these unstable cases, the solutions will be unbounded for x > 1. However, there are some exceptions in Case 1 and Case 5 where bounded solutions exist on $-2 \le x \le 500$.

VII. Conclusion

Nine cases in the parameter plane (τ_1, F_2) for equilibrium (0,0,0,0). The equilibrium is Liapunov stable in Case 3 except when $w_1/w_2 = 2$, i.e., $F_2 = -\frac{18}{5}\tau_1^2$ with $\tau_1 < 0$, the "***" curve, almost stable in Case 4, and Liapunov unstable otherwise.

Nine cases in the parameter plan (τ_1,F_2) for equilibrium $(\frac{4}{3}F_2,0,0,0)$. The equilibrium is Liapunov stable in Case 13 except when $w_{131}/w_{132} = 2$, i.e., $F_2 = -\frac{18}{5}\tau_1^2$ with $\tau_1 < 0$, the "***" curve, almost stable in Case 14, and Liapunov unstable otherwise.

VIII. Appendix

Numerical formulas for Theorem 3 and 8

In [12], Markeev considered the resonance situation for $w = w_1/w_2 = 2,3$ with a Hamiltonian in the form

$$H = \frac{1}{2}(\tilde{p}_1^2 + w_1^2\tilde{q}_1^2) - \frac{1}{2}(\tilde{p}_2^2 + w_2^2\tilde{q}_2^2) + \sum_{v=3}\tilde{g}_{v_1v_2v_3v_4}$$
$$\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4} + \sum_{v=4}\tilde{h}_{v_1v_2v_3v_4}\tilde{q}_1^{v_1}\tilde{q}_2^{v_2}\tilde{p}_1^{v_3}\tilde{p}_2^{v_4} + O(|\tilde{z}|^5)$$

where

$$|\tilde{z}| = \sqrt{\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{p}_1^2 + \tilde{p}_2^2}, \quad v = v_1 + v_2 + v_3 + v_4.$$

When $w = w_1/w_2 = 2$, Markeev [12] proved Theorem 8 and provided numerical formulas for computing x_{1002}, y_{1002} in (31) as follows:

$$x_{1002} = -\frac{w_1}{2w_2}\tilde{g}_{0111} - \frac{1}{2}\tilde{g}_{1002} + \frac{1}{2w_2^2}\tilde{g}_{1200},$$
$$y_{1002} = -\frac{w_1}{2}\tilde{g}_{0012} + \frac{w_1}{2w_2^2}\tilde{g}_{0210} + \frac{1}{2w_2}\tilde{g}_{1101}.$$

When $w = w_1/w_2 = 3$, Markeev [12] proved Theorem 3 and also provided numerical formulas for computing x_{1003}, y_{1003}, l_{2020}, l_{1111}, and l_{0202} in (15) as follows:

$$x_{1003} = u_{1003} - \frac{9}{5}(x_{0120}x_{0012} + y_{0120}y_{0012}) - w_2^{-1}(x_{1002}y_{1011}$$
$$+x_{1011}y_{1002}) + 4w_2^{-2}(x_{1002}x_{0201} + y_{1002}y_{0201})$$
$$+\frac{3}{2}(x_{0003}x_{0111} + y_{0003}y_{0111}),$$
$$y_{1003} = v_{1003} - \frac{9}{5}(x_{0120}y_{0012} - x_{0012}y_{0120}) - w_2^{-1}(y_{1002}y_{1011}$$
$$-x_{1011}x_{1002}) + 4w_2^{-2}(y_{1002}x_{0201} - x_{1002}y_{0201})$$
$$+\frac{3}{2}(y_{0003}x_{0111} - x_{0003}y_{0111}),$$
$$l_{2020} = h'_{2020} + \frac{27}{8}w_2^2(x_{0030}^2 + y_{0030}^2) + \frac{3}{2}(x_{1020}^2 + y_{1020}^2)$$
$$+\frac{9}{10}(x_{0120}^2 + y_{0120}^2) - \frac{1}{2}(x_{1011}^2 + y_{1011}^2)$$
$$-\frac{9}{56}w_2^2(x_{0021}^2 + y_{0021}^2),$$
$$l_{1111} = h'_{1111} - \frac{2}{3}(x_{1002}^2 + y_{1002}^2) + \frac{3}{10}w_2^2(x_{0012}^2 + y_{0012}^2)$$
$$-\frac{9}{14}w_2^2(x_{0021}^2 + y_{0021}^2) - \frac{18}{5}(x_{0120}^2 + y_{0120}^2) - 2(x_{0111}x_{0120}$$
$$+y_{0111}y_{0120}) - 4w_2^{-1}(x_{0201}y_{1011} + x_{1011}y_{0201}),$$
$$l_{0202} = h'_{0202} - \frac{3}{8}w_2^2(x_{0003}^2 + y_{0003}^2) - 6w_2^{-2}(x_{0201}^2 + y_{0201}^2) +$$
$$\frac{1}{6}(x_{1002}^2 + y_{1002}^2) + \frac{1}{2}(x_{0111}^2 + y_{0111}^2) + \frac{3}{40}w_2^2(x_{0012}^2 + y_{0012}^2)$$

where

$$u_{1003} = \tfrac{1}{2}w_1\tilde{h}_{0013} + \tfrac{1}{2}w_2^{-3}\tilde{h}_{1300} - \tfrac{1}{2}w_2^{-1}\tilde{h}_{1102} - \tfrac{1}{2}w_1w_2^{-2}\tilde{h}_{0211},$$
$$v_{1003} = -\tfrac{1}{2}w_1w_2^{-1}\tilde{h}_{0112} - \tfrac{1}{2}\tilde{h}_{1003} + \tfrac{1}{2}w_2^{-2}\tilde{h}_{1201} + \tfrac{1}{2}w_1w_2^{-3}\tilde{h}_{0310},$$
$$h'_{2020} = -\tfrac{3}{2}w_1^2\tilde{h}_{0040} - \tfrac{3}{2}w_1^{-2}\tilde{h}_{4000} - \tfrac{1}{2}\tilde{h}_{2020},$$
$$h'_{1111} = w_1w_2\tilde{h}_{0022} + w_1^{-1}w_2^{-1}\tilde{h}_{2200} + w_1w_2^{-1}\tilde{h}_{0220} + w_1^{-1}w_2\tilde{h}_{2002},$$
$$h'_{0202} = -\tfrac{3}{2}w_2^2\tilde{h}_{0004} - \tfrac{3}{2}w_2^{-2}\tilde{h}_{0400} - \tfrac{1}{2}\tilde{h}_{0202},$$
$$x_{0030} = \tilde{g}_{0030} - w_1^{-2}\tilde{g}_{2010},$$
$$y_{0030} = w_1^{-1}\tilde{g}_{1020} - w_1^{-3}\tilde{g}_{3000},$$
$$x_{1020} = -\tfrac{1}{2}\tilde{g}_{1020} - \tfrac{3}{2}w_1^{-1}\tilde{g}_{3000},$$
$$y_{1020} = \tfrac{3}{2}w_1\tilde{g}_{0030} + \tfrac{1}{2}w_1^{-1}\tilde{g}_{2010},$$
$$x_{1011} = -w_1\tilde{g}_{0021} - w_1^{-1}\tilde{g}_{2001},\ y_{1011} = w_1w_2^{-1}\tilde{g}_{0120} + w_1^{-1}w_2^{-1}\tilde{g}_{2100},$$
$$x_{0111} = w_1^{-1}w_2\tilde{g}_{1002} + w_1^{-1}w_2^{-1}\tilde{g}_{1200},\ y_{0111} = -w_2\tilde{g}_{0012} - w_2^{-1}\tilde{g}_{0210},$$
$$x_{0201} = -\tfrac{1}{4}w_2\tilde{g}_{0102} - \tfrac{3}{4}w_2^{-1}\tilde{g}_{0300},$$
$$y_{0201} = \tfrac{3}{4}w_2^2\tilde{g}_{0003} - \tfrac{1}{4}\tilde{g}_{0201},$$
$$x_{0003} = -w_2^{-1}\tilde{g}_{0102} + w_2^{-3}\tilde{g}_{0300},$$
$$y_{0003} = -\tilde{g}_{0003} + w_2^{-2}\tilde{g}_{0201},$$
$$x_{0120} = -\tfrac{1}{2}w_2\tilde{g}_{0021} + \tfrac{1}{2}w_1^{-1}\tilde{g}_{1110} + \tfrac{1}{2}w_1^{-2}w_2\tilde{g}_{2001},$$
$$y_{0120} = -\tfrac{1}{2}w_2\tilde{g}_{0120} - \tfrac{1}{2}w_1^{-1}w_2\tilde{g}_{1011} + \tfrac{1}{2}w_1^{-2}\tilde{g}_{2100},$$
$$x_{0021} = w_2^{-1}\tilde{g}_{0120} - w_1^{-1}\tilde{g}_{1011} - w_1^{-2}w_2^{-1}\tilde{g}_{2100},$$
$$y_{0021} = \tilde{g}_{0021} - w_1^{-1}w_2^{-1}\tilde{g}_{1110} - w_1^{-2}\tilde{g}_{2001},$$
$$x_{1002} = -\tfrac{1}{2}w_1w_2^{-1}\tilde{g}_{0111} - \tfrac{1}{2}\tilde{g}_{1002} + \tfrac{1}{2}w_2^{-2}\tilde{g}_{1200},$$
$$y_{1002} = -\tfrac{1}{2}w_1\tilde{g}_{0012} + \tfrac{1}{2}w_1w_2^{-2}\tilde{g}_{0210} + \tfrac{1}{2}w_2^{-1}\tilde{g}_{1101},$$
$$x_{0012} = -\tilde{g}_{0012} + w_2^{-2}\tilde{g}_{0210} - w_1^{-1}w_2^{-1}\tilde{g}_{1101},$$
$$y_{0012} = -w_2^{-1}\tilde{g}_{0111} - w_1^{-1}\tilde{g}_{1002} + w_1^{-1}w_2^{-2}\tilde{g}_{1200},$$

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