Analytic and Numeric Solution of Nonlinear Partial Differential Equations of Fractional Order

M. A. ABDOU , M. M. EL – KOJOK & S. A. RAAD

***Abstract*—** The existence and uniqueness solution of the Cauchy problem are discussed and proved in a Banach space E due to Bielecki method and Picard method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solutions are obtained. The product Nyström method is used as a numerical method to obtain a nonlinear system of algebraic equations. Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally, an application is given and numerical results are obtained.

***Keywords*—** nonlinear partial differential equation of fractional order , semigroups, nonlinear algebraic system , Nyström method.

I. INTRODUCTION

Many authors have interested in using the semigroups methods for partial differential equations. In [1], Claeyssen and Schuchman discussed the minimal extension of the classical semigroup theory for second-order damped differential equations in Banach spaces with closed, densely defined linear operators as coefficients. In [2], Picho'r and Rudnicki proved a new theorem for asymptotic stability of Markov semigroups .In [3], Ntouyas and Tsamatos studied the global existence of solutions for semilinear evolution equations with nonlocal conditions , via a fixed point analysis approach. In [4], Li and Shaw studied a natural generalization of the above two notions to a wider class of operator families, called exponentially equicontinuous n-times integrated C-semigroups. The n-times integrated exponentially bounded semigroups of operators, $n \in \mathrm{N}$, on a Banach space, especially for $n=1$, were investigated in [5-10] and applied to abstract Cauchy problems with operators which do not generate C_0 -semigroups (see also Bcais [11]). In [12], Mijatovie and Pilipovie introduced and analyzed α - times integrated semigroups for $\alpha \in \left(\frac{1}{2},1\right)$. In [13] , El-Borai studied the Cauchy problem in a Banach space E for a linear fractional evolution equation . In his paper , the existence and uniqueness of the solution of the Cauchy problem were discussed and proved. Also, the solution was obtained in terms of some probability densities. In [14], El-Borai discussed the existence and uniqueness solution of the nonlinear Cauchy problem. Also, some properties concerning the stability of solutions were obtained. In [15] Abdou et al., improved the work of El-Borai in [13] and used the product Nyström method (PNM) to obtain numerically the solution of the Cauchy problem .

In this work , we treat the following Cauchy problem of the fractional evolution equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = Au(x,t) + F(x,t,B(t)u(x,t)),$$

with the initial condition $u(x,0)=u_0(x)$,

in a Banach space E . Here $u(x,t)$ is an E-valued function on $E\times[0,T]$, $T<\infty$, A is a linear closed operator defined on a dense set S_1 in E into E, $\{B(t), t\in[0,T]\}$ is a family of linear closed operators defined on a dense set $S_2 \supset S_1$ in E into E , F is a given abstract function defined on $E\times[0,T]$ with values in E , $u_0(x)\in E$ and $0<\alpha<1$.

II. NONLINEAR FRACTIONAL EVOLUTION EQUATION

Consider the following Cauchy problem of the fractional evolution equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = Au(x,t) + F(x,t,B(t)u(x,t)), \tag{1}$$

with the initial condition : $u(x,0)=u_0(x)$, (2)

in a Banach space E , where $u(x,t)$ is an E-valued function on $E\times[0,T]$, $T<\infty$, A is a linear closed and bounded operator defined on a dense set S_1, $\{B(t), t\in[0,T]\}$ is a family of linear closed and bounded operators defined on a dense set $S_2 \supset S_1$ in E into E , F is a given abstract function defined on $E\times[0,T]$ with values in E , $u_0(x)\in E$ and $0<\alpha<1$.

It is assumed that A generates an analytic semigroup $Q(t)$ .This condition implies :

$$\|Q(t)\| \le k \quad for \ t \ge 0 \ , \text{ and } \|AQ(t)\| \le \frac{k}{t} \quad for \ t>0 \ , \tag{3}$$

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where $\|\cdot\|$ is the norm in E and k is a positive constant (see Zaidman [16]).

Let us suppose that $B(t)g$ is uniformly Hölder continuous in $t\in[0,T]$, for every $g\in S_1$; that is

$$\| B(t_2)g - B(t_1)g \| \le k_1 (t_2-t_1)^{\beta} , \qquad (4)$$

for all $t_2>t_1$, $t_1,t_2 \in[0,T]$, where k_1 and β are positive constants , $\beta\le 1$.

We suppose also that there exists a number $\gamma\in(0,1)$, such that

$$\| B(t_2)Q(t_1)h \| \le \frac{k_2}{t_1^{\gamma}} \| h \| , \qquad (5)$$

where $t_1>0$, $t_2\in[0,T]$, $h\in E$ and k_2 is a positive constant (see[13,17,18]).

(Notice that $Q(t)h\in S_1$ for each $h\in E$ and each $t>0$) .

Also, it is assumed that, the function F satisfies the following conditions :

(i) F is uniformly Hölder continuous in $t\in[0,T]$; that is

$$\| F(x,t_2,W) - F(x,t_1,W) \| \le \ell (t_2-t_1)^{\beta} , \qquad (6)$$

for all $t_2>t_1$, $t_1,t_2\in[0,T]$ and all $x ,W\in E$, where ℓ and β are positive constants , $\beta\le 1$, $W=B(t)u(x,t)$, and $\|\cdot\|$ is the norm in E .

(ii) F satisfies Lipschitz condition

$$\left| F(x,t,W) - F(x,t,W^*) \right| \le N(x,t) \left| W-W^* \right| ; \ (\| N(x,t) \| \le \ell_1), \qquad (7)$$

for all x , $W,W^*\in E$ and all $t\in[0,T]$, where ℓ_1 is a positive constant .

(iii) $\| F(x,t,W) \| \le \ell^* \| W \|$, (ℓ^* is a constant). (8)

Following Gelfand and Shilov [19], Schneider and Wayes [20], we can define the integral of order $\alpha>0$ by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta)\, d\theta .$$

If $0<\alpha<1$, we can define the derivative of order α by

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^{\alpha}}\, d\theta , \qquad \left(f'(\theta) = \frac{d f(\theta)}{d\theta} \right) ,$$

where f is an abstract function with values in E .

Now, it is suitable to rewrite the Cauchy problem (1) , (2) , in the form

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} A u(x,\theta)\, d\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} F(x,\theta,B(\theta)u(x,\theta))\, d\theta \qquad (9)$$

Let $C_E(E\times[0,T])$ be the set of all continuous functions $u(x,t)\in E$, and define on $C_E(E\times[0,T])$ a norm by

$$\| u(x,t) \|_{C_E(E\times[0,T])} = \max_{x,t} \| u(x,t) \|_E , \text{ for all } t\in[0,T] , \ x\in E .$$

By a solution of the Cauchy problem (1), (2), we mean an abstract function $u(x,t)$ such that the following conditions are satisfied :

(a) $u(x,t)\in C_E(E\times[0,T])$ and $u(x,t)\in S_1$ for all $t\in[0,T]$, $x\in E$.

(b) $\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$ exists and continuous on $E\times[0,T]$, where $0<\alpha<1$.

(c) $u(x,t)$ satisfies Eq.(1) with the initial condition (2) on $E\times[0,T]$.

Lemma 1 :

Let E be a Banach space of elements $u(x,t)$, then (E,d_1) is a complete metric space, where

$$d_1(u(x,t),v(x,t)) = \max_{x,t} \left\{ e^{-\lambda(t+x)} \| u(x,t)-v(x,t) \|_E \right\} , \ (\lambda>0) . \qquad (10)$$

Corollary 1:

If E is a Banach space , then for the positive constants L, M and ℓ_1, $(E,\tilde{d})$ is a complete metric spaces , where

$$\tilde{d}(u(x,t),v(x,t)) = \max_{x,t} \left\{ e^{-\left(\frac{(L+\ell_1 M)^2 T^{2\alpha-1}}{2(2\alpha-1)}\right)(t+x)} \| u(x,t)-v(x,t) \|_E \right\} . \qquad (11)$$

Lemma 2 :

If $\lambda>1$ and $0<\delta<1$, then

$$\int_0^t (t-\eta)^{\delta-1} d\eta \le \left(\frac{1}{\lambda}\right)^{\delta-1} t , \qquad (12)$$

and

$$\int_0^t e^{\lambda\eta} (t-\eta)^{\delta-1} d\eta \le \left(\frac{1}{\lambda}\right)^{\delta} \left[1+\frac{1}{\delta}\right] e^{\lambda t} . \qquad (13)$$

III. The Existence and Uniqueness Solution of Nonlinear Fractional Evolution Equation

Here, the existence and uniqueness solution of Eq.(9) and consequently its equivalent Cauchy problem (1), (2), will be discussed and proved in a Banach space E by virtue of Bielecki method and Picard method .

(a) *Modified Bielecki method :*

In this method , we will prove the existence and uniqueness solution of Eq.(9) for $\frac{1}{2}<\alpha<1$, and $0<\alpha<1$.

Case (i) : For $\frac{1}{2}<\alpha<1$, we consider the following theorem .

Theorem 1:

If A and B are linear bounded operators, and F satisfies the conditions (7) and (8) , then for $\frac{1}{2}<\alpha<1$, Eq.(9) has a unique solution in a Banach space E .

Proof :

Let $\tilde{K}$ be an operator defined by

$$\tilde{K}u(x,t)=u_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}Au(x,\theta)\,d\theta+$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}F\ (x,\theta,B(\theta)u(x,\theta))\,d\theta\,. \qquad (14)$$

Hence , we have

$$\left\|\tilde{K}u(x,t)\right\|\le\left\|u_0(x)\right\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|Au(x,\theta)\right\|d\theta+$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|F\ (x,\theta,B(\theta)u(x,\theta))\right\|d\theta\,.$$

Since A and B are bounded operators , there exists positive constants L and M such that

$$\left\|Au(x,t)\right\|\le L\left\|u(x,t)\right\|,\text{and}\left\|B(t)u(x,t)\right\|\le M\left\|u(x,t)\right\|, \quad (15)$$

Using the conditions (8) and (15) , the above inequality becomes

$$\left\|\tilde{K}u(x,t)\right\|\le\left\|u_0(x)\right\|+\frac{(L+\ell^*M)}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|u(x,\theta)\right\|d\theta.$$

Squaring both sides of the above inequality , then applying Cauchy-Schwarz inequality, we obtain

$$\left\|\tilde{K}u(x,t)\right\|^2\le 2\left\|u_0(x)\right\|^2+2\,\delta_1^2\left\|u(x,\theta)\right\|^2 \quad ;$$

$$\left(\delta_1=\left(\frac{L+\ell^*M}{\Gamma(\alpha)}\right)\frac{T^{\alpha}}{\sqrt{2\alpha-1}}\right)\,. \qquad (16)$$

Inequality (16) shows that , the operator $\tilde{K}$ maps the ball $B_{r_1}\subset E$ into itself , where

$$r_1^2=\frac{2\sigma^2}{1-2\delta_1^2}\quad , \quad (\sigma=\left\|u_0(x)\right\|).$$

Since $r_1>0$, $\sigma>0$, therefore $\delta_1<1$. Also , the inequality (16) involves the boundedness of the operator $\tilde{K}$.

For the two functions $u(x,t)$ and $v(x,t)$ in E , the formula (14) leads to

$$\left\|\tilde{K}u(x,t)-\tilde{K}v(x,t)\right\|\le\frac{1}{\Gamma(\alpha)}\left\{\int_0^t (t-\theta)^{\alpha-1}\left\|A(u(x,\theta)-v(x,\theta))\right\|d\theta\right.$$

$$\left.+\int_0^t (t-\theta)^{\alpha-1}\left\|F(x,\theta,B(\theta)u(x,\theta))-F(x,\theta,B(\theta)v(x,\theta))\,d\theta\right\|\right\}\,.$$

In view of the conditions (15) and (7) , the above inequality takes the form

$$\left\|\tilde{K}u(x,t)-\tilde{K}v(x,t)\right\|\le\frac{(L+\ell_1M)}{\Gamma(\alpha)}\left\{\int_0^t (t-\theta)^{\alpha-1}\left\|u(x,\theta)-v(x,\theta)\right\|d\theta\right\}.$$

Squaring both sides of the above inequality , then applying Cauchy-Schwarz inequality, we get

$$\left\|\tilde{K}u(x,t)-\tilde{K}v(x,t)\right\|^2\le\max_{x,t}\left\{e^{-(\frac{(L+\ell_1M)^2}{2\alpha-1}T^{2\alpha-1})(t+x)}\left\|u(x,t)-v(x,t)\right\|^2\right\}$$

$$\times\left(\frac{L+\ell_1M}{\Gamma(\alpha)}\right)^2\frac{T^{2\alpha-1}}{(2\alpha-1)}\int_0^t e^{(\frac{(L+\ell_1M)^2}{2\alpha-1}T^{2\alpha-1})(\theta+x)}\,d\theta.$$

Thus , we have

$$\max_{x,t}\left\{e^{-(\frac{(L+\ell_1M)^2}{2\alpha-1}T^{2\alpha-1})(t+x)}\left\|\tilde{K}u(x,t)-\tilde{K}v(x,t)\right\|^2\right\}\le$$

$$\left(\frac{1}{\Gamma(\alpha)}\right)^2\max_{x,t}\left\{e^{-(\frac{(L+\ell_1M)^2}{2\alpha-1}T^{2\alpha-1})(t+x)}\left\|u(x,t)-v(x,t)\right\|^2\right\}. \qquad (17)$$

In view of (11) , the previous inequality (17) can be adapted in the form

$$\tilde{d}\,(\tilde{K}u(x,t),\tilde{K}v(x,t))\le\frac{1}{\Gamma(\alpha)}\tilde{d}\,(u(x,t),v(x,t)).$$

Since $\frac{1}{2}<\alpha<1$, then $\frac{1}{\Gamma(\alpha)}<1$, therefore $\tilde{d}$ is a contraction mapping . By Banach fixed point theorem , $\tilde{K}$ has a unique fixed point which is , of course , the unique solution of Eq.(9).

Case (ii) : For $0<\alpha<1$, we consider the following theorem .

Theorem 2 :

If A and B are linear bounded operators , and F satisfies the conditions (7) and (8) , then for $0<\alpha<1$, Eq.(9) has a unique solution in a Banach space E .

Proof :

From Eq.(14) , we have

$$\left\|\tilde{K}u(x,t)\right\|\le\left\|u_0(x)\right\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|Au(x,\theta)\right\|d\theta+$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|F\ (x,\theta,B(\theta)u(x,\theta))\right\|d\theta\,.$$

In view of the conditions (15) and (8) , the above inequality becomes

$$\left\|\tilde{K}u(x,t)\right\|\le\left\|u_0(x)\right\|+\frac{(L+\ell^*M)}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\left\|u(x,\theta)\right\|d\theta\,.$$

Using (12) in (17) , we get

$$\left\|\tilde{K}u(x,t)\right\|\le\left\|u_0(x)\right\|+\frac{(L+\ell^*M)}{\Gamma(\alpha)}(\frac{1}{\lambda})^{\alpha-1}T\left\|u(x,t)\right\|\,,$$

$$(T=\max_{0\le t\le T}t)\,. \qquad (18)$$

Inequality (18) shows that, the operator $\tilde{K}$ maps the ball $B_{r_2}\subset E$ into itself , where

$$r_2=\frac{\sigma}{1-\delta_2}\quad ;\quad (\sigma=\left\|u_0(x)\right\|\,,\ \delta_2=\frac{(L+\ell^*M)}{\Gamma(\alpha)}\,(\frac{1}{\lambda})^{\alpha-1}T\,)$$

.

Since $r_2>0$, $\sigma>0$, therefore $\delta_2<1$. Also , the inequality (18) involves the boundedness of the operator $\tilde{K}$.

For the two functions $u(x,t)$ and $v(x,t)$ in E, the formula (14) leads to

$$\left\| \tilde{K}u(x,t) - \tilde{K}v(x,t) \right\| \le \frac{1}{\Gamma(\alpha)} \left\{ \int_0^t (t-\theta)^{\alpha-1} \tilde{F}\, d\theta \right.$$

$$\left. + \int_0^t (t-\theta)^{\alpha-1} \left\| A(u(x,\theta) - v(x,\theta)) \right\| d\theta \right\},$$

where $\tilde{F} = \| F(x,\theta, B(\theta)u(x,\theta)) - F(x,\theta,B(\theta)v(x,\theta)) \|$.

Using the conditions (15) and (7) , the above inequality becomes

$$\left\| \tilde{K}u(x,t) - \tilde{K}v(x,t) \right\| \le$$

$$\frac{(L+\ell_1 M)}{\Gamma(\alpha)} \left\{ \int_0^t (t-\theta)^{\alpha-1} \| u(x,\theta) - v(x,\theta) \| d\theta \right\}.$$

Hence for $\lambda > 1$, we have

$$\left\| \tilde{K}u(x,t) - \tilde{K}v(x,t) \right\| \le \frac{(L+\ell_1 M)}{\Gamma(\alpha)}$$

$$\times \max_{x,t} \left\{ e^{-\lambda(t+x)} \| u(x,t) - v(x,t) \| \right\} \int_0^t (t-\theta)^{\alpha-1} e^{\lambda(\theta+x)}\, d\theta.$$

In view of (13) , the above inequality takes the form

$$\max_{x,t} \left\{ e^{-\lambda(t+x)} \left\| \tilde{K}u(x,t) - \tilde{K}v(x,t) \right\| \right\} \le$$

$$\sigma_1 \max_{x,t} \left\{ e^{-\lambda(t+x)} \| u(x,t) - v(x,t) \| \right\}, \qquad (19)$$

where $\sigma_1 = \dfrac{(L+\ell_1 M)}{\Gamma(\alpha)} \left(\dfrac{1}{\lambda}\right)^{\alpha} \left[1 + \dfrac{1}{\alpha}\right]$.

The last inequality (19) can be adapted in the form

$d_1(\tilde{K}\, u(x,t), \tilde{K}\, v(x,t)) \le \sigma_1\, d_1(u(x,t), v(x,t))$.

If we choose λ sufficiently large , then we have $\sigma_1 < 1$; therefore d_1 is a contraction mapping . By Banach fixed point theorem , $\tilde{K}$ has a unique fixed point which is , of course, the unique solution of Eq.(9) .

This completes the proof of the theorem .

(b) *Picard's method :*

The formula (9) is equivalent to the following integral equation (see[13])

$$u(x,t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0(x)\, d\theta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) Q((t-\eta)^\alpha \theta)\, \tilde{w}(x,\eta)\, d\theta\, d\eta\ , \qquad (20)$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $[0,\infty)$,

$$\tilde{w}(x,t) = F(x,t,B(t)u(x,t)) = F(x,t,W(x,t)), \qquad (21)$$

and

$$W(x,t) = \int_0^\infty \zeta_\alpha(\theta) B(t) Q(t^\alpha \theta) u_0(x)\, d\theta +$$

$$\alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) B(t)\, Q((t-\eta)^\alpha \theta)\, \tilde{w}(x,\eta)\, d\theta\, d\eta\ . \qquad (22)$$

Theorem 2:

The Cauchy problem (1), (2), has a unique solution in the Banach space $C_E(E \times [0,T])$.

The proof of this theorem comes as a result of the following lemmas .

Lemma 3 :

Under the conditions (5) and (7) , the integral equation (20) has a solution in $C_E(E \times [0,T])$.

Proof :

Using the method of successive approximations , the formulas (21) and (22) , lead to

$$\tilde{w}_{n+1}(x,t) = F(x,t, \int_0^\infty \zeta_\alpha(\theta) B(t) Q(t^\alpha \theta) u_0(x)\, d\theta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) B(t) Q((t-\eta)^\alpha \theta))\, \tilde{w}_n(x,\eta)\, d\theta\, d\eta).$$

Hence , in view of the condition (7) , we get

$$\| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \| \le \alpha\, \ell_1 \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) \| BQ\, \tilde{w}_n \| d\theta\, d\eta,$$

where $\| BQ\, \tilde{w}_n \| = \left\| B(t) Q((t-\eta)^\alpha \theta)) [\tilde{w}_n(x,\eta) - \tilde{w}_{n-1}(x,\eta)] \right\|$.

Using the condition (5) , we obtain

$$\| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \| \le M^* \int_0^t e^{\lambda(\eta+x)} (t-\eta)^{\nu-1} d\eta$$

$$\times \max_{x,t} \left[e^{-\lambda(t+x)} \| \tilde{w}_n(x,t) - \tilde{w}_{n-1}(x,t) \| \right], \qquad (23)$$

where ,

$$\nu = \alpha(1-\gamma), M^* = \alpha\, \ell_1 \int_0^\infty \theta^{1-\gamma} \zeta_\alpha(\theta)\, d\theta \ \text{ and } \ \lambda > 1. \qquad (24)$$

Introducing (13) in (24) ,we have

$$\max_{x,t} \left[e^{-\lambda(t+x)} \| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \| \right] \le$$

$$M^* \left(\frac{1}{\lambda}\right)^{\nu} \left[1 + \frac{1}{\nu}\right] \max_{x,t} \left[e^{-\lambda(t+x)} \| \tilde{w}_n(x,t) - \tilde{w}_{n-1}(x,t) \| \right].$$

We can choose λ sufficiently large such that

$$M^* \left(\frac{1}{\lambda}\right)^{\nu} \left[1 + \frac{1}{\nu}\right] = \mu < 1\,. \qquad (25)$$

Hence , the above inequality can be adapted in the form

$$\max_{x,t} [e^{-\lambda(t+x)} \| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \|] \le$$

$$\mu \max_{x,t} [e^{-\lambda(t+x)} \| \tilde{w}_n(x,t) - \tilde{w}_{n-1}(x,t) \|]\,.$$

By a successive application of the above inequality , we get

$$\max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \|]$$
$$\le \mu \max_{x,t} [\, e^{-\lambda(t+x)} \| \tilde{w}_n(x,t) - \tilde{w}_{n-1}(x,t) \|]$$
$$\le (\mu)^2 \max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_{n-1}(x,t) - \tilde{w}_{n-2}(x,t) \|]$$
$$\le ... \le (\mu)^n \max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_1(x,t) - \tilde{w}_0(x,t) \|] ,$$

where $\tilde{w}_0(x,t)$ is the zero approximation which can be taken the zero element in the space *E*. Thus, the series $\sum_{n=0}^{\infty} \| \tilde{w}_{n+1}(x,t) - \tilde{w}_n(x,t) \|$ converges uniformly in $E \times [0,T]$.

Since $\tilde{w}_{n+1}(x,t) = \sum_{i=0}^{n} (\tilde{w}_{i+1}(x,t) - \tilde{w}_i(x,t))$, it follows that the sequence $\{ \tilde{w}_n(x,t) \}$ converges uniformly in the space $C_E(E \times [0,T])$ to a continuous function $F(x,t,W(x,t))$ which satisfies Eq.(20) for all $(x,t) \in E \times [0,T]$. Consequently, $u(x,t) \in C_E(E \times [0,T])$, where

$$u(x,t) = \int_0^{\infty} \zeta(\theta) Q(t^{\alpha} \theta) u_0(x) d\theta +$$
$$\alpha \int_0^t \int_0^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_{\alpha}(\theta) Q((t-\eta)^{\alpha} \theta) F(x,\eta,W(x,\eta)) d\theta\, d\eta .$$

Lemma 4 :

Under the conditions (5) and (7) , the integral equation (20) has a unique solution in $C_E(E \times [0,T])$.

Proof :

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of Eq.(20) , then from the formulas (21) and (22) with the aid of condition (7) , we have

$$\| \tilde{w}_2(x,t) - \tilde{w}_1(x,t) \| \le \alpha\, \ell_1$$
$$\int_0^t \int_0^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_{\alpha}(\theta) \left\| B(t) Q((t-\eta)^{\alpha}\theta) [\tilde{w}_2(x,\eta) - \tilde{w}_1(x,\eta)] \right\| d\theta\, d\eta.$$

Using the same argument of lemma (3) , we get

$$\max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_2(x,t) - \tilde{w}_1(x,t) \|_E] \le \mu\, \rho ,$$

where $\rho = \max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_2(x,t) - \tilde{w}_1(x,t) \|_E]$.

Thus , from (25) we have

$$\rho = \max_{x,t}[\, e^{-\lambda(t+x)} \| \tilde{w}_2(x,t) - \tilde{w}_1(x,t) \|_E] = 0 .$$

This completes the proof of the lemma .

Lemma 5 :

Under the conditions (4) , (5) and (6) , the solution $u(x,t)$ of Eq.(20) satisfies a uniform Hölder condition .(see [13])

Proof of Theorem 3 :

By virtue of lemmas (3)-(5) , we deduce that the solution $u(x,t)$ of Eq.(20) represents the unique solution of the Cauchy problem (1), (2) in the Banach space $C_E(E \times [0,T])$, and $u(x,t) \in S_1$.

Now, we will prove the stability of the solutions of the Cauchy problem (1) , (2). In other words , we will show that the Cauchy problem (1), (2) is correctly formulated .

Theorem 4 :

Let $\{ u_n(x,t) \}$ be a sequence of functions , each of which is a solution of Eq.(1) with the initial condition $u_n(x,0) = g_n(x)$, where $g_n(x) \in S_1$ $(n = 1,2,...)$. If the sequence $\{ g_n(x) \}$ converges to an element $u_0(x) \in S_1$, the sequence $\{ A g_n(x) \}$ converges and the sequence $\{ B(t) g_n(x) \}$ converges uniformly on $E \times [0,T]$. Then, the sequence of solutions $\{ u_n(x,t) \}$ converges uniformly on $E \times [0,T]$ to a limit function $u(x,t)$, which is the solution of the Cauchy problem (1) , (2) .

Proof :

Consider the sequences $\{ f_n(x,t) \}$ and $\{ u_n^*(x,t) \}$, where

$$\frac{\partial^{\alpha} u_n^*(x,t)}{\partial t^{\alpha}} - A u_n^*(x,t) = f_n(x,t) ,$$
$$u_n^*(x,t) = u_n(x,t) - g_n(x) , \quad u_n(x,0) = g_n(x) ,$$
$$u_n^*(x,t) = \alpha \int_0^t \int_0^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_{\alpha}(\theta) Q((t-\eta)^{\alpha}\theta)\, f_n(x,\eta) d\theta\, d\eta ,$$

and

$$f_n(x,t) = F(x,t,B(t) u_n^*(x,t) + B(t) g_n(x)) + A g_n(x).$$

Thus , we get

$$\| f_n(x,t) - f_m(x,t) \| \le \| A g_n(x) - A g_m(x) \| +$$
$$\left\| F(x,t,B u_n^*(x,t)) + B g_n(x) - F(x,t,B u_m^*(x,t)) + B\, g_m(x) \right\| .$$

Using the condition (7) , we obtain

$$\| f_n(x,t) - f_m(x,t) \| \le \ell_1 \left\| B(t)[u_n^*(x,t) - u_m^*(x,t)] \right\| +$$
$$\ell_1 \| B(t) g_n(x) - B(t) g_m(x) \| + \| A g_n(x) - A g_m(x) \| .$$

Consequently ,

$$\| f_n(x,t) - f_m(x,t) \| \le \ell_1 \| B(t) g_n(x) - B(t) g_m(x) \| + \| A g_n(x) - A g_m(x) \|$$
$$+ \alpha\, \ell_1 \int_0^t \int_0^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_{\alpha}(\theta) \| B\, Q\, [\, f_n - f_m] \| d\theta\, d\eta.$$
$$\| B\, Q\, [\, f_n - f_m] \| = \left\| B(t) Q((t-\eta)^{\alpha}\theta) [\, f_n(x,\eta) - f_m(x,\eta)] \right\|$$

In view of the conditions (5) and (24) , the above inequality becomes

$$\| f_n(x,t) - f_m(x,t) \| \le M^* \int_0^t (t-\eta)^{\nu-1} \| f_n(x,\eta) - f_m(x,\eta) \| d\eta$$
$$+ \ell_1 \| B(t) g_n(x) - B(t) g_m(x) \| + \| A g_n(x) - A g_m(x) \| .$$

Given $\varepsilon > 0$, we can find a positive integer $N = N(\varepsilon)$ such that

$\| f_n(x,t) - f_m(x,t) \| \le$

$M^* \int_0^t (t-\eta)^{\nu-1} \| f_n(x,\eta) - f_m(x,\eta) \| d\eta + (1-\mu_2)\,\varepsilon$,

for all $n \ge N, m \ge N$ and $(x,t) \in E \times [0,T]$.

Using (13), the above inequality takes the form

$(1-\mu) e^{-\lambda(t+x)} \| f_n(x,t) - f_m(x,t) \| \le (1-\mu) e^{-\lambda(t+x)} \varepsilon$.

Thus , for sufficiently large λ , we get

$\max_{x,t} [e^{-\lambda(t+x)} \| f_n(x,t) - f_m(x,t) \|] \le \varepsilon$.

Since E is a complete space , it follows that the sequence $\{ f_n(x,t) \}$ converges uniformly on $E \times [0,T]$ to a continuous function $f(x,t)$, so the sequence $\{ u_n^*(x,t) \}$ converges uniformly on $E \times [0,T]$ to a continuous function $u^*(x,t)$. It can be proved that $f(x,t)$ satisfies a uniform Hölder condition on $[0,T]$, thus $u^*(x,t) \in S_1$.

This completes the proof of the theorem .

Corollary 2 :

The integral equation (20) has a unique solution in the Banach space $C_{\Re}(\Re \times [0,T])$.

IV. The Numerical Solution of Nonlinear Fractional Evolution Equation

In this section , we will use the product Nyström method (see Linz [21] , and Dzhuraev [22]) , to obtain numerically , the solution of the Cauchy problem (1) , (2) , in the Banach space $C_{\Re}(\Re \times [0,T])$, where

$$\| u(x,t) \|_{C_{\Re}(\Re \times [0,T])} = \max_{x,t} | u(x,t) | \quad ,$$

$$\forall\, t \in [0,T] \, , \, -\infty < x < \infty \quad .$$

For this , we write Eq.(20) in the form

$$u(x,t) = f^*(x,t) + \alpha \int_0^t p(t,\eta) Q^*(t,\eta) F(x,\eta,B(\eta)u(x,\eta))\, d\eta \, , \qquad (26)$$

where ,

$$f^*(x,t) = \int_0^\infty \zeta_\alpha(\theta)\, Q(t^\alpha \theta) u_0(x)\, d\theta \quad , \qquad (27)$$

$$Q^*(t,\eta) = \int_0^\infty \theta\, \zeta_\alpha(\theta)\, Q((t-\eta)^\alpha\, \theta)\, d\theta \quad , \qquad (28)$$

and the bad kernel

$$p(t,\eta) = (t-\eta)^{\alpha-1} \quad , \quad (0<\alpha<1,\ 0 \le \eta \le t \le T \ ;\ T < \infty\ ). \qquad (29)$$

Here , the unknown function $u(x,t) \in C_{\Re}(\Re \times [0,T])$, while $f^*(x,t)$, $Q^*(t,\eta)$ and $p(t,\eta)$ are known functions and satisfy the following conditions :

(1') $f^*(x,t)$ is a continuous function in $(\Re \times [0,T])$.

(2') $Q^*(t,\eta)$ with its partial derivatives are continuous function in $[0,T]$.

(3') $p(t,\eta)$ is a badly behaved function of its arguments such that :

(a) for each continuous function $u(x,t)$ and $0 \le t_1 \le t_2 \le t$, the integrals

$$\int_{t_1}^{t_2} p(t,\eta)\, Q^*(t,\eta)\, F(x,\eta,B(\eta)u(x,\eta))\, d\eta \quad ,$$

and

$$\int_0^t p(t,\eta) Q^*(t,\eta)\, F(x,\eta,B(\eta)u(x,\eta))\, d\eta \quad ,$$

are continuous functions in $(\Re \times [0,T])$.

(b) $p(t,\eta)$ is absolutely integrable with respect to η for all $0 \le t \le T$.

(4') the given function $F(x,t,B(t)u(x,t))$ is continuous in $\Re \times [0,T]$, and satisfies Lipschitz condition

$$| F(x,t,B(t)u(x,t)) - F(x,t,B(t)v(x,t)) | \le$$

$$N^*(x,t)\, | B(t)(u(x,t) - v(x,t)) | \, ; (\max_{x,t} | N^*(x,t) | \le L^*). \qquad (30)$$

for all $u(x,t), v(x,t) \in C_{\Re}(\Re \times [0,T])$, where L^* is a positive constant .

Remark 1 :

In view of Corollary (3), the integral equation (26) has a unique solution in the Banach space $C_{\Re}(\Re \times [0,T])$.

Applying the product Nyström method , then putting $t = t_i = \eta_i = x_i = x$, $t_i = ih$, $h = t_{i+1} - t_i$, ($i = 0,1,\ldots,N$, *and* N *is even*) and using the following notations

$$u(t_i,x_i) = u_{i,i} \ ,\ Q^*(t_i,\eta_i) = Q^*_{i,j} \quad ,\quad f^*(t_i,x_i) = f^*_{i,i} \ ,$$

$$F(x_i,\eta_i,B(\eta_i)u(x_i,\eta_i)) = F_{i,i}(B_i u_{i,i}) \quad ,$$

the integral equation (26) can be transformed to the following nonlinear algebraic system

$$u_{i,i} = f^*_{i,i} + \alpha \sum_{j=0}^{N} w_{i,j}\, Q^*_{i,j}\, F_{j,j}(B_j u_{j,j}), \qquad (i = 0,1,2,\ldots,N) \qquad (31)$$

where ,

$$w_{i,0} = \beta_1(t_i) \quad , \quad w_{i,2j+1} = 2\gamma_{j+1}(t_i)$$

$$w_{i,2j} = \alpha^*{}_j(t_i) + \beta_{j+1}(t_i) \quad , \quad w_{i,N} = \alpha^*{}_{\frac{N}{2}}(t_i). \qquad (32)$$

And

$$\alpha^*{}_j(t_i) = \frac{1}{2h^2} \int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta - \eta_{2j-2})(\eta - \eta_{2j-1})\, d\eta \, ,$$

$$\beta_j(t_i) = \frac{1}{2h^2} \int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta_{2j-1} - \eta)(\eta_{2j} - \eta)\, d\eta \ ,$$

$$\gamma_j(t_i)=\frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta-\eta_{2j-2})(\eta_{2j}-\eta)\,d\eta\ . \qquad (33)$$

Evaluating the integrals of Eq.(33), where $p(t,\eta)=(t-\eta)^{\alpha-1}$, and introducing the results in the values of $w's$, we get

$$w_{i,0}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)}\Big\{[2|i-2|+\alpha+2]|i-2|^{\alpha+1}-$$
$$-[2|i|^2-3(2+\alpha)|i|+2(\alpha+1)(\alpha+2)]|i|^{\alpha}\Big\}\ ,$$

$$w_{i,2j+1}=\frac{2h^{\alpha}}{\alpha(\alpha+1)(\alpha+2)}$$
$$\Big\{(\alpha+2)[|i-2j-2|^{\alpha+1}+|i-2j|^{\alpha+1}]+|i-2j-2|^{\alpha+2}-|i-2j|^{\alpha+2}\Big\}\ ,$$

$$w_{i,2j}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)}\Big\{(\alpha+2)|i-2j+2|^{\alpha+1}+(\alpha+2)\,|i-2j-2|^{\alpha+1}$$
$$+6(\alpha+2)|i-2j|^{\alpha+1}+2|i-2j-2|^{\alpha+2}-2|i-2j+2|^{\alpha+2}\Big\}\ ,$$

and

$$w_{i,N}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)}\Big\{2(\alpha+1)(\alpha+2)|i-N|^{\alpha}+3(\alpha+2)|i-N|^{\alpha+1}$$
$$+(\alpha+2)|i-N+2|^{\alpha+1}+2|i-N|^{\alpha+2}-2|i-n+2|^{\alpha+2}\Big\}. \qquad (34)$$

The nonlinear algebraic system (31) represents $(N+1)$ equations in $u_{i,i}$. Therefore, the approximate solution of $u(x,t)$ can written in the vector form

$$U=F^*+\alpha W^*F(BU). \qquad (35)$$

Or in a matrix form as

$$\begin{bmatrix}u_{0,0}\\u_{1,1}\\ \cdot\\ \cdot\\ \cdot\\ u_{N,N}\end{bmatrix}=\begin{bmatrix}f^*_{0,0}\\f^*_{1,1}\\ \cdot\\ \cdot\\ \cdot\\ f^*_{N,N}\end{bmatrix}+$$
$$\alpha\begin{bmatrix}w_{0,0}Q^*_{0,0} & w_{0,1}Q^*_{0,1} & \cdots & w_{0,N}Q^*_{0,N}\\ w_{1,0}Q^*_{1,0} & w_{1,1}Q^*_{1,1} & \cdots & w_{1,N}Q^*_{1,N}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ w_{N,0}Q^*_{N,0} & w_{N,1}Q^*_{N,1} & \cdots & w_{N,N}Q^*_{N,N}\end{bmatrix}\begin{bmatrix}F_{0,0}(B_0u_{0,0})\\ F_{1,1}(B_1u_{1,1})\\ \cdot\\ \cdot\\ \cdot\\ F_{N,N}(B_Nu_{N,N})\end{bmatrix}. \qquad (36)$$

Theorem 5 :

The algebraic system (31) has a unique solution in the Banach space ℓ^{∞}, under the following conditions :

$$\sup_i\left|f^*_{i,i}\right|\le q\quad ,\quad (q\ is\ \text{a constant})\ . \qquad (37)$$

$$\sup_i\sum_{j=0}^{N}\left|w_{i,j}Q^*_{i,j}\right|\le q^*\quad ,\quad (q^*\ is\ \text{a constant})\,. \qquad (38)$$

$$\sup_i\left|B_i u_{i,i}\right|\le M\ \sup_i\left|u_{i,i}\right|\quad ,\quad (M\ is\ \text{a constant})\,. \qquad (39)$$

The known functions $F_{i,i}(B_iu_{i,i})$, for the constants $D>D_1$, $D>D_2$, satisfy

$$\sup_i\left|F_{i,i}(B_iu_{i,i})\right|\le D_1\sup_i\left|B_iu_{i,i}\right|\ , \qquad (40)$$

$$\sup_i\left|F_{i,i}(B_iu_{i,i})-F_{i,i}(B_iv_{i,i})\right|\le D_2\sup_i\left|B_i(u_{i,i}-v_{i,i})\right|. \qquad (41)$$

Then the algebraic system (31) has a unique solution in the Banach space ℓ^{∞}.

Proof :

Let $\tilde{Y}$ be the set of all functions $U=\{u_{i,i}\}$ in ℓ^{∞} such that $\|U\|_{\ell^{\infty}}\le\gamma_1$, γ_1 is a constant . Define the operator T^* by

$$T^*U=F^*+\alpha W^*F(BU). \qquad (42)$$

From the formulas (31) and (42) , we have

$$\left|T^*u_{i,i}\right|\le\sup_i\left|f_{i,i}\right|+\alpha\sup_i\sum_{j=0}^{N}\left|w_{i,j}Q^*_{i,j}\right|\sup_i\left|F_{j,j}(B_ju_{j,j})\right|,\ \forall\ i\ .$$

Using the conditions (37- 40), we get

$$\left\|T^*U\right\|_{\ell^{\infty}}\le q+\lambda_1\|U\|_{\ell^{\infty}}\quad ,\quad (\lambda_1=\alpha q^*DM)\ . \qquad (43)$$

Inequality (43) shows that , the operator T^* maps the set $\tilde{Y}$ into itself , where $\gamma_1=\dfrac{q}{1-\lambda_1}$.

Since $\gamma_1>0$, $q>0$, therefore $\lambda_1<1$. Also , the inequality (43) involves the boundedness of the operator T^*.

For the two functions U and V in ℓ^{∞}, the formulas (31) and (42) lead to

$$\left|T^*u_{i,i}-T^*v_{i,i}\right|\le\alpha$$
$$\sup_i\sum_{j=0}^{N}\left|w_{i,j}Q^*_{i,j}\right|\sup_j\left|F_{j,j}(B_ju_{j,j})-F_{j,j}(B_jv_{j,j})\right|,\ \forall\ i$$

Using the conditions (38) , (39) and (41), we obtain

$$\left\|T^*U-T^*V\right\|_{\ell^{\infty}}\le\lambda_1\|U-V\|_{\ell^{\infty}}\,. \qquad (44)$$

The previous inequality (44) proves that , the operator T^* is continuous in ℓ^{∞}, and under the condition $\lambda_1<1$, T^* is contractive . Hence , by Banach fixed point theorem , T^* has a unique fixed point which is the unique solution of the nonlinear algebraic system in ℓ^{∞}.

Theorem 6 :

If the conditions (38) , (39) and (41) are satisfied , and the sequence of functions $\{F^*_m\}=\{(f^*_{i,i})_m\}$ converges uniformly to the function $F^*=\{f^*_{i,i}\}$ in ℓ^{∞}. Then , the sequence of approximate solutions $\{U_m\}=\{(u_{i,i})_m\}$ converges uniformly to the exact solution $U=\{u_{i,i}\}$ of the nonlinear algebraic system (31) in ℓ^{∞}.

Proof :

In view of Eq.(31) , we get

$$\left|u_{i,i}-(u_{i,i})_m\right|\le \sup_i\left|f_{i,i}-(f_{i,i})_m\right|+$$

$$\alpha\sup_i\sum_{j=0}^{N}\left|w_{i,j}\,Q^*_{i,j}\right|\sup_j\left|F_{j,j}(B_ju_{j,j})-F_{j,j}((B_ju_{j,j})_m)\right|,\ \forall i.$$

The above inequality with the aid of conditions (38) , (39) and (41), takes the form

$$\|U-U_m\|_{\ell^\infty}\le\frac{1}{1-\lambda_1}\left\|F^*-F^*_m\right\|_{\ell^\infty}\quad;\quad(\lambda_1<1)\ .$$

Since $\left\|F^*-F^*_m\right\|_{\ell^\infty}\to 0$ *as* $m\to\infty$, so that $\|U-U_m\|_{\ell^\infty}\to 0$.

This completes the proof of the theorem .

When $N\to\infty$, the sum

$\sum_{j=0}^{N}w_{i,j}\,Q^*_{i,j}\,F_{j,j}(B_ju_{j,j})$; $0\le i,j\le N$ becomes

$\int_0^t p(t,\eta)\,Q^*(t,\eta)\,F(x,\eta,B(\eta)u(x,\eta))\,d\eta$,consequently the solution of the algebraic system (31) is the same solution of the integral equation (26) . The next theorem shows the convergence of the sequence of approximate solutions to the exact solution of Eq.(26) in the Banach space $C_{\Re}(\Re\times[0,T])$.

Theorem 7 :

If the sequence of continuous functions $\left\{f^*_n(x,t)\right\}$ converges uniformly to the function $f^*(x,t)$, and the functions $Q^*(t,\eta)$, $p(t,\eta)$ and $F(x,t,B(t)u(x,t))$ satisfy, respectively , the conditions (2') , (3'- b) and (30). Then, the sequence of approximate solutions $\{u_n(x,t)\}$ converges uniformly to the exact solution of Eq.(4.1) in the Banach space $C_{\Re}(\Re\times[0,T])$.

Proof :

The formula (26) with its approximate solution give

$$\max_{x,t}\left|u(x,t)-u_n(x,t)\right|\le\max_{x,t}\left|f^*(x,t)-f^*_n(x,t)\right|$$

$$+\alpha\int_0^t\left|p(t,\eta)\right|\left|Q^*(t,\eta)\right|F^*\,d\eta\ ,$$

$$\forall\,0\le\eta\le t\le T\ ,\ -\infty<x<\infty. \qquad (45)$$

Where

$$F^*=\max_{x,\eta}\left|F(t,\eta,B(\eta)u(x,\eta))-F(t,\eta,B(\eta)u_n(x,\eta))\right|$$

In view of the conditions (2') and (3'-b) , there exist two constants c_1 and c_2 , such that

$$\left|Q^*(t,\eta)\right|\le c_1\ ,and\quad\int_0^t\left|p(t,\eta)\right|d\eta\le c_2\ . \qquad (46)$$

Hence, the inequality (45) with the aid of (15), (46) and (30) takes the form

$$\left\|u(x,t)-u_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\le\frac{1}{(1-D^*)}\times$$

$$\left\|f^*(x,t)-f^*_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])};\quad(D^*=\alpha\,c_1c_2\,M\,L^*)\,.$$

Since $\left\|f^*(x,t)-f^*_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\to 0$ as $n\to\infty$,

so that $\left\|u(x,t)-u_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\to 0$.

Definition 1 :

The consistency error R_N of product Nyström method is determined by the following equation

$$R_N=\left|\int_0^{t_N}p(t_N,\eta)\,Q^*(t_N,\eta)\,F(x,\eta,B(\eta)u(x,\eta))\,d\eta\right.$$

$$\left.-\sum_{j=0}^{N}w_{i,j}\,Q^*_{i,j}\,F_{j,j}(B_j\,u_{j,j})\right|, \qquad (47)$$

also, Eq.(47) gives

$$u(x,t)-u_N(x,t)=\sum_{j=0}^{N}w_{i,j}\,Q^*(t_i,\eta_j)$$

$$\times\left[F(x_j,\eta_j,B(\eta_j)u(x_j,\eta_j))-F(x_j,\eta_j,B(\eta_j)u_N(x_i,\eta_j))\right]+R_N\,. \qquad (48)$$

Where $u_N(x,t)$ is the approximate solution of Eq.(26) .

Theorem 8 :

Assume that , the hypothesis of Theorem (7) are satisfied , then

$$\lim_{N\to\infty}R_N=0\ . \qquad (49)$$

Proof :

The formula (48) leads to

$$\left|R_N\right|\le\sup_i\left|u_{i,i}-(u_{i,i})_N\right|+$$

$$\sup_i\sum_{j=0}^{N}\left|w_{i,j}\,Q^*_{i,j}\right|\sup_j\left|F_{j,j}(B_j\,u_{j,j})-F_{j,j}(B_j(u_{j,j})_N)\right|.$$

Using the conditions (38) , (39) and (41) , we get

$$\left\|R_N\right\|_{\ell^\infty}\le\left\|U-U_N\right\|_{\ell^\infty}+q^*DM\left\|U-U_N\right\|_{\ell^\infty},\ \forall\ N=1,2,3,...\ .$$

Since $\left\|U-U_N\right\|_{\ell^\infty}\to 0$ *as* $N\to\infty$ (see Theorem (7)) , it follows that $\left\|R_N\right\|_{\ell^\infty}\to 0$.

This completes the proof of theorem .

Application I :

In Eq.(26) , let :

$$0<\alpha<1\ ,\ Q^*(t,\eta)=1\ ,\ F(x,t,B(t)u(x,t))=(x+t)^2\,.$$

Thus , we get a nonlinear Volterra integral equation of the second kind with Abel kernel

$$u(x,t)=(x+t)-\frac{t^{\alpha}}{\alpha}\left(x^2+\frac{2xt}{(\alpha+1)}+\frac{2t^2}{(\alpha+1)(\alpha+2)}\right)$$
$$+\alpha\int_0^t (t-\eta)^{\alpha-1}(x+\eta)^2\, d\eta \quad , \qquad (50)$$

where the exact solution $u(x,t)=x+t$.

The results are obtained numerically in Table 1. which lists various values of $x,t\in[0,0.8]$ together with the values of the exact and approximate solutions and the error of Eq.(50) . It can be observed from this table that :

1. The error is 0 for $x=t=0$.

2. As x and t are increasing through [0,0.8] , the error is also increasing for $\alpha=0.98$, $\alpha=0.8$ and $\alpha=0.4$.

3. As the values of α are increasing , the values of error are also increasing for $x,t\in[0,0.8]$.

x=t	*Exact*	$\alpha=0.4$		$\alpha=0.8$	
		Approx. Solution	Error	Approx. Solution	Error
0	0	0	0	0	0
0.08	0.16	0.079507	0.000493	0.079929	7.0828e-05
0.16	0.32	0.158409	0.001591	0.159404	0.000596
0.24	0.48	0.234257	0.005743	0.238186	0.001814
0.32	0.64	0.312337	0.007663	0.315773	0.004227
0.4	0.8	0.376563	0.023437	0.39183	0.008170
0.48	0.96	0.452405	0.027595	0.466214	0.013786
0.56	1.12	0.513083	0.046917	0.538719	0.021281
0.64	1.28	0.589233	0.050767	0.608666	0.031334
0.72	1.44	0.621711	0.098289	0.675075	0.044925
0.8	1.6	0.693685	0.106315	0.739096	0.060904

x=t	*Exact*	$\alpha=0.98$	
		Approx. Solution	Error
0	0	0	0
0.08	0.16	0.079973	2.6855e-05
0.16	0.32	0.159638	0.000362
0.24	0.48	0.238938	0.001062
0.32	0.64	0.31712	0.002880
0.4	0.8	0.394762	0.005238
0.48	0.96	0.470214	0.009786
0.56	1.12	0.545146	0.014854
0.64	1.28	0.616527	0.023473
0.72	1.44	0.687461	0.032539
0.8	1.6	0.753268	0.046732

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