Spectral Equivalence of $\mathcal{A}_S$-Scalar Operators. $\mathcal{A}_S$-Decomposable and $\mathcal{A}_S$-Spectral Operators

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Abstract: This paper is dedicated to the study of some properties of the operators which admit residually non-analytic functional calculus initiated in [16]. We shall also define and study the spectral *s*-capacities, and give several *s*-decomposability criteria. We shall further study the restrictions and the *S*-decomposable operators' quotients.

The concepts of $\mathcal{A}_S$-spectral function, respectively $\mathcal{A}_S$-decomposable and $\mathcal{A}_S$-spectral operators are introduced and characterized here and several elementary properties concerning them are studied. These operators are natural generalizations of the notions of $\mathcal{A}$-scalar, $\mathcal{A}$-decomposable and $\mathcal{A}$-spectral operators studied in [8] and appear, in generally, as restrictions or quotients of the last one.

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1. INTRODUCTION

Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathbb{C}$ be the complex plane. If $T \in \mathbf{B}(X)$ and $Y \subset X$ is a (closed) invariant subspace to T, let us denote by $T|Y$ the restriction of T to Y, respectively by $\dot{T}$ the operator induced by T in the quotient space $\dot{X} = X/Y$. In what follows, by subspace of X we understand a closed linear manifold of X. Recall that Y is a *spectral maximal space* of T if it is an invariant subspace such that for any other subspace $Z \subset X$ also invariant to T, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$ ([8]). A family of open sets $G_S \cup \{G_i\}_{i=1}^n$ is an *S-covering* of the closed set $\sigma \subset \mathbb{C}$ if $G_S \cup \left(\bigcup_{i=1}^n G_i\right) \supset \sigma \cup S$ and $\overline{G}_i \cap S = \varnothing$ $(i = 1, 2, ..., n)$ (where $S \subset \mathbb{C}$ is also closed) ([14]).

The operator $T \in \mathbf{B}(X)$ is *S-decomposable* (where $S \subset \sigma(T)$ is compact) if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $Y_S \cup \{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that $\sigma(T|Y_S) \subset G_S$, $\sigma(T|Y_i) \subset G_i$ $(i = 1, 2, ..., n)$ and $X = Y_S + \sum_{i=1}^n Y_i$ ([4]). If $\dim S = 0$, then $S = \varnothing$ and T is decomposable ([8]). An open set $\Omega \subset \mathbb{C}$ is said to be a *set of analytic uniqueness* for $T \in \mathbf{B}(X)$ if for any open set $\omega \subset \Omega$ and any analytic function $f_0 : \omega \to X$ satisfying the equation $(\lambda I - T) f_0(\lambda) \equiv 0$ it follows that $f_0(\lambda) \equiv 0$ in ω ([14]). For $T \in \mathbf{B}(X)$ there is a unique maximal open set Ω_T of analytic uniqueness ([14]). We shall denote by $S_T = \complement\Omega_T = \mathbb{C} \setminus \Omega_T$ and call it *the analytic spectral residuum of* T. For $x \in X$, a point λ is in $\delta_T(x)$ if in a neighborhood V_λ of λ, there is at least an analytic X-valued function f_x (called *T-associated to* x) such that $(\mu I - T) f_x(\mu) \equiv x$, for $\mu \in V_\lambda$. We shall put

$\gamma_T(x) = \complement\delta_T(x) = \mathbb{C} \setminus \delta_T(x)$, $\rho_T(x) = \delta_T(x) \cap \Omega_T$

$\sigma_T(x) = \complement\rho_T(x) = \mathbb{C} \setminus \rho_T(x) = \gamma_T(x) \cup S_T$ and

$X_T(F) = \{x \in X;\ \sigma_T(x) \subset F\}$

where $S_T \subset F \subset \mathbb{C}$ ([14], [15]).

An operator $T \in \mathbf{B}(X)$ is said to have *the single-valued extension property* if for any analytic function $f : \omega \to X$ (where $\omega \subset \mathbb{C}$ is an open set), with $(\lambda I - T) f(\lambda) = 0$, it follows that $f(\lambda) \equiv 0$ ([10]). T has the single-valued extension property if and only if $S_T = \emptyset$; then we have $\sigma_T(x) = \gamma_T(x)$ and there is in $\rho_T(x) = \delta_T(x)$ a unique analytic function $x(\lambda)$, T-associated to x, for any $x \in X$. We shall recall that if $T \in \mathbf{B}(X)$, $S_T \neq \emptyset$, $S_T \subset F$ and $X_T(F)$ is closed, for $F \subset \mathbb{C}$ closed, then $X_T(F)$ is a spectral maximal space of T ([14]).

We say that two operators $T_1, T_2 \in \mathbf{B}(X)$ are *quasinilpotent equivalent* if

$$\lim_{n\to\infty} \left\| (T_1 - T_2)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| (T_2 - T_1)^{[n]} \right\|^{\frac{1}{n}} = 0$$

where

$$(T_1 - T_2)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_1^k \, T_2^{n-k} \quad ([8]).$$

Definition 1.1. ([16]) Let Ω be a set of the complex plane $\mathbb{C}$ and let $S \subset \overline{\Omega}$ be a compact subset. An algebra $\mathcal{A}_S$ of $\mathbb{C}$-valued functions defined on Ω is called S*-normal* if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\overline{\Omega}$, there are the functions, f_S, $f_i \in \mathcal{A}_S$ $(1 \le i \le n)$ such that:

1) $f_S(\Omega) \subset [0, 1]$, $f_i(\Omega) \subset [0, 1]$ $(1 \le i \le n)$;

2) $\operatorname{supp}(f_S) \subset G_S$, $\operatorname{supp}(f_i) \subset G_i$ $(1 \le i \le n)$;

3) $f_S + \sum_{i=1}^{n} f_i = 1$ on Ω

where *the support* of $f \in \mathcal{A}_S$ is defined as: $\operatorname{supp}(f) = \overline{\{\mu \in \Omega;\ f(\mu) \neq 0\}}$.

Definition 1.2. ([16]) An algebra $\mathcal{A}_S$ of $\mathbb{C}$-valued functions defined on Ω is called S-*admissible* if:

1) $\lambda \in \mathcal{A}_S, 1 \in \mathcal{A}_S$ (where λ and 1 denote the functions $f(\lambda) = \lambda$ and $f(\lambda) = 1$);

2) $\mathcal{A}_S$ is S-normal;

3) for any $f \in \mathcal{A}_S$ and any $\xi \notin \operatorname{supp}(f)$, the function

$$f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda}, & \text{for } \lambda \in \Omega \setminus \{\xi\} \\ 0, & \text{for } \lambda \in \Omega \cap \{\xi\} \end{cases}$$

belongs to $\mathcal{A}_S$.

Definition 1.3. ([16]) An operator $T \in \mathbf{B}(X)$ is said to be $\mathcal{A}_S$*-scalar* if there are an S-admissible algebra $\mathcal{A}_S$ and an algebraic homomorphism $U : \mathcal{A}_S \to \mathbf{B}(X)$ such that $U_1 = I$ and $U_\lambda = T$ (where 1 is the function $f(\lambda) = 1$ and λ is the function $f(\lambda) = \lambda$). The mapping U is called $\mathcal{A}_S$*-spectral homomorphism* ($\mathcal{A}_S$*-spectral function* or $\mathcal{A}_S$-*functional calculus*) for T.
If $S = \emptyset$, then we put $\mathcal{A} = \mathcal{A}_\emptyset$ and we obtain an $\mathcal{A}$-spectral function and an $\mathcal{A}$-scalar operator ([8]).

The support of an $\mathcal{A}_S$-spectral function U is denoted by $\operatorname{supp}(U)$ and it is defined as the smallest closed set $F \subset \overline{\Omega}$ such that $U_f = 0$ for $f \in \mathcal{A}_S$ with $\operatorname{supp}(f) \cap F = \emptyset$.

A subspace Y of X is said to be *invariant with respect to* an $\mathcal{A}_S$-spectral function $U : \mathcal{A}_S \to \mathbf{B}(X)$ if $U_f Y \subseteq Y$, for any $f \in \mathcal{A}_S$.

We recall several important properties of an $\mathcal{A}$-spectral function U ([8]), because we want to obtain similar properties for an $\mathcal{A}_S$-spectral function:

1. U_λ has the single-valued extension property, where λ is the identical function $f(\lambda) \equiv \lambda$;

2. $\sigma_{U_\lambda}(U_f x) \subset \operatorname{supp}(f)$, for any $f \in \mathcal{A}$ and $x \in X$;

3. If $\sigma_{U_\lambda}(x) \cap \operatorname{supp}(f) = \emptyset$, then $U_f(x) = 0$, for any $f \in \mathcal{A}$ and $x \in X$;

4. $x \in X_{U_\lambda}(F) = \{x \in X;\ \sigma_{U_\lambda}(x) \subset F\}$

$\Leftrightarrow U_f(x)=0$, for any $f\in\mathcal{A}$ with property $\operatorname{supp}(f)\cap F=\emptyset$, $F\subset\Omega$ closed;

5. $\operatorname{supp}(U)=\sigma(U_\lambda)$;

6. U_λ is decomposable.

Theorem 1.4. *Let $T\in\mathbf{B}(X)$ be an $\mathcal{A}_S$-scalar operator and let U be an $\mathcal{A}_S$-spectral function for T. Then we have*:
$\operatorname{supp}(U)\subset\sigma(T)\cup S$ *and* $\sigma(T)\subset\operatorname{supp}(U)\cup S$.

Proof. Let us consider $f\in\mathcal{A}_S$ such that $\operatorname{supp}(f)\cap(\sigma(T)\cup S)=\emptyset$. If $\xi\notin\operatorname{supp}(f)$ and λ
is the identical function $f(\lambda)=\lambda$, then we have

$$(\xi I-U_\lambda)U_{f_\xi}=U_{(\xi-\lambda)f_\xi}=U_f$$

hence

$$U_{f_\xi}=\Re(\xi,U_\lambda)\,U_f,\ \text{for}$$
$$\xi\in\rho(U_\lambda)\cap\complement\operatorname{supp}(f).$$

The function

$$F(\xi)=\begin{cases}\Re(\xi,T)U_f, & \text{for } \xi\in\rho(U_\lambda)\\ U_{f_\xi}, & \text{for } \xi\in\complement\operatorname{supp}(f)\end{cases}$$

is entire and $\lim\limits_{|\xi|\to\infty}\|F(\xi)\|=0$, therefore $F\equiv 0$. It follows that $U_{f_\xi}=0$ on $\complement\operatorname{supp}(f)$ and $U_f=0$, hence

$$\operatorname{supp}(U)\subset\sigma(T)\cup S.$$

Let now $\xi_0\notin\operatorname{supp}(U)\cup S$, let V_{ξ_0} be an open neighborhood of ξ_0 and let W be an open neighborhood of $\operatorname{supp}(U)\cup S$ such that $V_{\xi_0}\cap W=\emptyset$. Because the algebra $\mathcal{A}_S$ is S-normal, then there is a function $f\in\mathcal{A}_S$ with $f(\mu)=1$ on W and $f(\mu)=0$ for $\mu\in V_{\xi_0}$. Consequently

$$\operatorname{supp}(1-f)\cap(\operatorname{supp}(U)\cup S)=\emptyset$$

hence

$$U_{1-f}=0,\ \text{i.e. } U_f=I.$$

Whence

$$U_{f_{\xi_0}}(\xi_0 I-U_\lambda)=(\xi_0 I-U_\lambda)U_{f_{\xi_0}}=U_f=I$$

therefore we finally have $\xi_0\notin\sigma(U_\lambda)=\sigma(T)$ and hence $\sigma(T)\subset\operatorname{supp}(U)\cup S$.

Theorem 1.5. (Properties of $\mathcal{A}_S$-spectral functions)
Let U be an $\mathcal{A}_S$-spectral function (particularly, U is an $\mathcal{A}_S$-spectral function for an $\mathcal{A}_S$-scalar operator $T\in\mathbf{B}(X)$, $T=U_\lambda$). Then we have the following properties:

(1) *The spectral analytic residuum S_T has the property: $S_T\subset S$; when $S_T=\emptyset$ (particularly, $S=\emptyset$), then T has the single-valued extension property*;

(2) *If $(\lambda_0 I-U_\lambda)x_0=0$, with $x_0\neq 0$ and $f\in\mathcal{A}_S$ with $f(\lambda)=c$, for $\lambda\in G\cap\Omega$, where G is a neighborhood of λ_0, then $U_f x_0=c x_0$*;

(3) *If $f\in\mathcal{A}_S$ and $x\in X$, then $\gamma_T(U_f x)\subset\operatorname{supp}(f)$; moreover, if $\operatorname{supp}(f)\supset S$, then $\sigma_T(U_f x)\subset\operatorname{supp}(f)$*;

(4) *If $f\in\mathcal{A}_S$ such that $\sigma_{U_\lambda}(x)\cap\operatorname{supp}(f)=\emptyset$ and $S_T=\emptyset$, then $U_f x=0$*;

(5) *If $F\subset\Omega$ closed, with $F\supset S$, $x\in X$ and $S_T=\emptyset$, then $x\in X_{U_\lambda}(F)$ if and only if $U_f x=0$, for any $f\in\mathcal{A}_S$ with the property $\operatorname{supp}(f)\cap F=\emptyset$*;

(6) *U_λ is S-decomposable.*

Proof. The assertions (1) and (2) are proved in [16], Theorem 3.2, respectively Lemma 3.1.

(3) We observe that for any $\xi\notin\operatorname{supp}(f)$ we have $f_\xi\in\mathcal{A}_S$ and the X-valued function $\xi\to U_{f_\xi}x$ is analytic. Consequently,

$$(\xi I-T)U_{f_\xi}x=(\xi I-U_\lambda)U_{f_\xi}x=U_f x,$$

therefore $\xi\in\delta_T(U_f x)$, hence

$$\gamma_T(U_f x)\subset\operatorname{supp}(f).$$

Furthermore, for $f\in\mathcal{A}_S$ with $\operatorname{supp}(f)\supset S$, we deduce

$$\sigma_T(U_f x) = S_T \cup \gamma_T(U_f x) \subset S \cup \gamma_T(U_f x) \subset \mathrm{supp}(f).$$

(4) Let $x(\xi)$ be the unique analytic X-valued function defined on $\rho_{U_\lambda}(x)$ which satisfies the equality

$$(\xi I - U_\lambda)x(\xi) = x \text{ on } \rho_{U_\lambda}(x).$$

It results that

$$(\xi I - U_\lambda)U_f x(\xi) = U_f(\xi I - U_\lambda)x(\xi) = U_f x \text{ on } \rho_{U_\lambda}(x)$$

hence the following inclusions are obtained

$$\rho_{U_\lambda}(x) \subset \rho_{U_\lambda}(U_f x) \text{ and } \sigma_{U_\lambda}(U_f x) \subset \sigma_{U_\lambda}(x).$$

From assertion (3),

$$\sigma_{U_\lambda}(U_f x) = \sigma_T(U_f x) = S_T \cup \gamma_T(U_f x) = \gamma_T(U_f x) \subset \mathrm{supp}(f),$$

hence

$$\sigma_{U_\lambda}(U_f x) \subset \mathrm{supp}(f) \cap \sigma_{U_\lambda}(x) = \varnothing,$$

therefore according to Proposition 1.1.2, [8], it follows that $U_f x = 0$.

The property (5) can be obtained by using (4), as in the proof of Proposition 3.1.17, [8] and will be omitted.

The proof of (6) is presented in [14], Theorem 3.3.

Lemma 1.6. *Let* U *be an* $\mathcal{A}_S$*-spectral function. If* G_1 *is an open neighborhood of* $\mathrm{supp}(U)$, $G_1 \supset \mathrm{supp}(U)$ *and* G_2 *is an open set such that* $G_1 \cup G_2 \supset \overline{\Omega}$, $G_2 \cap \mathrm{supp}(U) = \varnothing$ (i.e. $\{G_1, G_2\}$ *is an open covering of* $\overline{\Omega}$)*, then by* S*-normality of the algebra* $\mathcal{A}_S$ *it results that there are tow functions* $f_1, f_2 \in \mathcal{A}_S$ *such that*:

$$0 \le f_1(\lambda) \le 1,\ 0 \le f_2(\lambda) \le 1,\ \lambda \in \Omega,$$
$$\mathrm{supp}(f_1) \subset G_1,\ \mathrm{supp}(f_2) \subset G_2 \text{ and}$$
$$f_1 + f_2 = 1 \text{ on } \Omega.$$

With these conditions we have:

a) $U_{f_1} = I$, $U_{f_2} = 0$

b) *For* $f \in \mathcal{A}_S$ *having the property that* $f = 1$ *on a neighborhood of* $\mathrm{supp}(U)$*, it results that* $U_f = I$.

Proof. We have

$$\mathrm{supp}(1 - f_1) = \mathrm{supp}(f_2) \subset G_2$$
$$\mathrm{supp}(1 - f_1) \cap \mathrm{supp}(U) = \varnothing$$

hence

$$0 = U_{1-f_1} = U_1 - U_{f_1}$$

therefore

$$U_{f_1} = U_1 = I \text{ and } U_{f_2} = 0.$$

Moreover, for $f \in \mathcal{A}_S$ with the property that $f = 1$ on a neighborhood of $\mathrm{supp}(U)$ we have

$$U_f = I$$

because it can be chosen in this case: $f_1 = f$, $f_2 = g$, with $\mathrm{supp}(g) \cap \mathrm{supp}(U) = \varnothing$, hence $U_g = 0$ and accordingly

$$U_{f+g} = U_1 = I = U_f + U_g, \text{ whence } U_f = I.$$

Remark 1.7. From Lemma 1.6, it results that if $f \in \mathcal{A}_S$ and $f = 1$ in a neighborhood of $\mathrm{supp}(U)$, then $U_f = I$. If we denote by $\bigvee_{f \in \mathcal{A}_0} U_f Y$ the linear subspace of X generated by $U_f Y$, where $Y \subset X$ and $\mathcal{A}_0$ is the set of all functions in $\mathcal{A}_S$ with compact support, then we have:

$$\bigvee_{f \in \mathcal{A}_0} U_f X = X.$$

Definition 1.8. Let U be an $\mathcal{A}_S$-spectral function. For any open set $G \in \mathcal{G}_S$ we denote

$$X_{[U]}(G) = \bigvee_{\mathrm{supp}(f) \subset G} U_f X$$

and for any closed set $F \in \mathcal{F}_S$ we put

$$X_{[U]}(F) = \bigcap_{G \supset F} X_{[U]}(G).$$

where $\mathcal{F}_S$ (respectively, $\mathcal{G}_S$) is the family of all closed (respectively, open) subsets $F \subset \mathbb{C}$ (respectively, $G \subset \mathbb{C}$) having the property: either $F \cap S = \varnothing$ or $F \supset S$ (respectively, $G \cap S = \varnothing$ or $G \supset S$).

Theorem 1.9. *Let* U *be an* $\mathcal{A}_S$*-spectral function. Then*

$$X_{[U]}(F) = X_{U_\lambda}(F) = \{x \in X;\ \sigma_{U_\lambda}(x) \subset F\},\ \textit{for}\ F \in \mathcal{F}_S,\ F \supset S.$$

Proof. If $\sigma_{U_\lambda}(x) \subset F$, for $F \in \mathcal{F}_S$, with $F \supset S$, let us consider $G \in \mathcal{G}_S$ an open set with $G \supset F \supset S$. Then by S-normality of $\mathcal{A}_S$ there is a function $f \in \mathcal{A}_S$ such that

$$f(\xi) = \begin{cases} 1, \text{ for } \xi \text{ in a neighborhood of } \Omega \cap F \\ 0, \text{ for } \xi \text{ in a neighborhood of } \Omega \setminus (G \cap \Omega) \end{cases}$$

and therefore $\text{supp}(f) \subset G$, whence

$$\text{supp}(1-f) \cap \sigma_{U_\lambda}(x) \subset \text{supp}(1-f) \cap F = \varnothing.$$

According to Theorem 1.5, $U_{1-f}\, x = 0$, hence

$$x = U_f\, x \in X_{[U]}(G).$$

$G \in \mathcal{G}_S$ being an arbitrary open set with $G \supset F$, $F \in \mathcal{F}_S$, we have

$$x \in X_{[U]}(F), \text{ i.e. } X_{U_\lambda}(F) \subseteq X_{[U]}(F).$$

Conversely, let us show that $X_{[U]}(F) \subset X_{U_\lambda}(F)$, for any $F \in \mathcal{F}_S$, $F \supset S$.

Let $x \in X_{[U]}(F) \subset X_{[U]}(G)$, for any open set $G \in \mathcal{G}_S$, $G \supset F \supset S$, and let $G_1 \in \mathcal{G}_S$ be an arbitrary open set containing $\overline{G}$. By S-normality of $\mathcal{A}_S$, there is a function $f_1 \in \mathcal{A}_S$ such that

$$f_1(\xi) = \begin{cases} 1, \text{ for } \xi \in \overline{G} \cap \Omega \\ 0, \text{ for } \xi \in \Omega \setminus (G_1 \cap \Omega) \end{cases}$$

hence $\text{supp}(f_1) \subset \overline{G}_1$. Therefore for any $f \in \mathcal{A}_S$ with $\text{supp}(f) \subset G$ we have $f_1 f = f$, so that

$$U_{f_1} U_f = U_f, \text{ i.e.}$$

$$U_{f_1} \mid X_{[U]}(G) = I \mid X_{[U]}(G)$$

whence

$$U_{f_1} x = x.$$

According to Theorem 1.5 it follows that

$$\sigma_{U_\lambda}(x) = \sigma_{U_\lambda}(U_{f_1} x) = \gamma_{U_\lambda}(U_{f_1} x) \cup S_{U_\lambda} = \gamma_{U_\lambda}(U_{f_1} x) \subset \text{supp}(f_1) \subset \overline{G}_1$$

and hence

$$\sigma_{U_\lambda}(x) = \bigcap_{\substack{G_1 \in \mathcal{G}_S \\ G_1 \supset \overline{G}}} \overline{G}_1 = \overline{G}.$$

$G \in \mathcal{G}_S$ being an arbitrary open set, $G \supset F \supset S$, we obtain

$$\sigma_{U_\lambda}(x) \subset \bigcap_{\substack{G \in \mathcal{G}_S \\ G \supset F \supset S}} \overline{G} = F, \text{ hence } x \in X_{U_\lambda}(F).$$

Corollary 1.10. *If U is an $\mathcal{A}_S$-spectral function, then for any $F \in \mathcal{F}_S$ with $F \supset S$, $X_{[U]}(F)$ is a maximal spectral space for U_λ.*

Proof. It results easily from the previous theorem.

Theorem 1.11. *Let $T_1, T_2 \in \mathbf{B}(X)$. If T_1 is S-decomposable (in particular, decomposable) and T_1, T_2 are spectral equivalent, then T_2 is also S-decomposable (in particular, decomposable) and*

$$X_{T_1}(F) = X_{T_2}(F),$$

for any $F \subset \mathbb{C}$ closed, $F \supset S$ (when $S = \varnothing$, for any $F \subset \mathbb{C}$ closed).

If T_1 and *T_2 are decomposable, then T_1 is spectral equivalent to T_2 if and only if their spectral spaces $X_{T_1}(F)$ and $X_{T_2}(F)$ are equal,* i.e. *$X_{T_1}(F) = X_{T_2}(F)$, for any $F \subset \mathbb{C}$ closed* ([8], 2.2.1, 2.2.2).

If T_1 and T_2 are S-decomposable and spectral equivalent, then their spectral spaces are equal, i.e. *$X_{T_1}(F) = X_{T_2}(F)$, for any $F \subset \mathbb{C}$ closed, $F \supset S$, but conversely is not true.*

2. THE STRUCTURE OF SPECTRAL MAXIMAL SPACES OF *S*-DECOMPOSABLE OPERATORS

This paragraph is devoted to the study of the *S*-decomposable operators defined in the introduction (see [6], [7]). First, we reveal some structural properties of spectral maximal spaces of the *S*-decomposable operators. Then, we shall present the behavior of these operators at direct sums, at projections, at separate parts of the spectrum, at the Riesz-Dunfort functional calculus and at the quasinilpotent equivalence. We will also give proof of an important structural theorem of spectral maximal spaces, generalising the following from [11] and [12].

Lemma 2.1. *Let $T \in B(X)$ be a S-decomposable operator, and let G be an open set such that:*

$$G \cap (\sigma(T) \setminus S) = \varnothing$$

then there exists a maximal spectral space $Y \neq \{0\}$ of T such that $\sigma(T / Y) \subset G$. If $\dim S \leq 1$ and $G \cap \operatorname{Int} \sigma(T) \neq \varnothing$ (G being an open set), then there exists a maximal spectral space $Y \neq \{0\}$ of T such that $\sigma(T / Y) \subset G$.

Proof. Let G_S be an open set such that:

$$S \subset G_S \not\supset \sigma(T)$$

and

$$G_S \cup G \supset \sigma(T).$$

T being S-decomposable, there exists a system of spectral maximal spaces Y_S, Y from T such that:

$$\sigma(T \mid Y_S) \subset G_S,\ \sigma(T \mid Y) \subset G$$

and

$$X = X_S + Y.$$

If $Y = \{0\}$, we have $Y_S = X$ and $\sigma(T \mid Y_S) = \sigma(T) \subset G_S$, contradiction, hence $Y \neq \{0\}$. When $\dim S \leq 1$ and $G \cap \operatorname{Int} \sigma(T) \neq \varnothing$ it follows that $G \cap (\sigma(T \mid Y) \setminus S) \neq \varnothing$, consequently $Y \neq \{0\}$.

Theorem 2.2.. *If $T \in B(X)$ is S-decomposable where $\dim S \leq 1$, then*

$\sigma_p^0(T) = \sigma_r^0(T) = \varnothing$ (see [8],Theorem 1.3.6),

T has the single-valued extension property ($S_T = \varnothing$) and $\sigma(T) = \sigma_l(T)$. If $S_T \neq \varnothing$, then $S_T \subset S$ and $\dim S = 2$.

Proof. If $\sigma_p^0(T) = \varnothing$, let G be a component of $\sigma_p^0(T)$. Then, by [37] Proposition 1.3.7, there doesn't exist any spectral maximal space $Y \neq \{0\}$ of T such that

$$\sigma(T \mid Y) \subset G;$$

by the preceding lemma, $G \cap \sigma(T) = \varnothing$, therefore $G \cap \sigma_p^0(T) = \varnothing$ which is impossible (since $G \subset \sigma_p^0(T) \subset \operatorname{Int} \sigma(T)$). Same for $\sigma_r(T)$. Consequently

$$\sigma_p^0(T) = \sigma_r^0(T) = \varnothing$$

since $S_T = \overline{\sigma_p^0(T)}$, and $\sigma_r^0(T) = \sigma(T) \setminus \sigma_l(T)$, we have $S_T = \varnothing$ (meaning that T has the single-valued extension property) and

$$\sigma(T) = \sigma_l(T).$$

Now let $S_T \neq \varnothing$. In order to verify the inclusion $S_T \subset S$ it will suffice to verify that $\sigma_p^0(T) \subset S$. Suppose that $\sigma_p^0(T) \not\subset S$; then there exists a component G_0 of $\sigma_p^0(T)$ such that:

$$G_0 \not\subset S \text{ and } G_0 \cap (\sigma(T) \setminus S) \neq \varnothing.$$

By the preceding lemma there follows that there exists a spectral maximal space Y_0 of *T*, $Y_0 \neq \{0\}$ such that:

$$\sigma(T \mid Y_0) \subset G_0;$$

contradicts [8] Proposition 1.3.7, consequently $S_T \subset S$. But $S_T \neq \varnothing$ implies $\dim S = 2$ (we have $\operatorname{Int} S_T \neq \varnothing$) hence $\operatorname{Int} S \neq 0$.

Theorem 2.3. *Let $T \in B(X)$ be a S-decomposable operator and let $F \subset \mathbb{C}$ be a closed set such that*

$$S \subset F \subset \sigma(T).$$

Then $X_T(F)$ is a spectral maximal space of T and

$$\sigma(T \mid X_T(F)) \subset F.$$

Conversely, for any spectral maximal space Y of T such that $\sigma(T \mid Y) \supset S$ we have

$$Y = X_T(\sigma(T \mid Y)).$$

Proof. Let $F \subset \sigma(T)$ be closed such that $S \subset F$ ($S_T \subset S \subset F$) and let G_S, H be two open sets satisfying conditions $G_S \supset F$, $H \cap F = \varnothing$ and $G_S \cup H \supset \sigma(T)$. We shall consider

$$G_1 = G_S,\ G_2 = H.$$

Let $\{Y_i\}_1^2$ be a corresponding system of spectral maximal spaces of T such that:

$$\sigma(T \mid Y_i) \subset G_i \ (i = 1,2)$$

and

$$X = Y_1 + Y_2.$$

If $x \in X_T(F)$, then $x = y_1 + y_2$, $y_i \in Y_i$ ($i = 1,2$) and $\sigma_T(x) \subset F$; for $\lambda \in \rho_T(x)$ $x(\lambda)$ has meaning and

$$(\lambda I - T)x(\lambda) = x$$

hence for $\lambda \in \complement F \cap \rho(T \mid Y_2)$ we have

$$(\lambda I - T)(R(\lambda, T \mid Y_2)y_2 - x(\lambda)) = y_2 - x = -y_1,$$

from which it follows that $\lambda \in \rho_T(y_1)$. But $\lambda \notin S \supset S_T$, consequently $\lambda \in \delta_T(y_1) \cap \cap \Omega_T = \rho_T(y_1)$ and from this it derives that

$$\sigma_T(y_1) \subset F \cup \sigma(T \mid Y_2) \subset F \cup \overline{G}_2$$

therefore

$$\complement F \cap \complement \overline{G}_2 \subset \rho_T(y_1).$$

Let now Γ be a bounded system of simple closed curves surrounding F and included in $\complement F \cap \complement \overline{G}_2$. For $\lambda \in \Gamma$ we have

$$y_1(\lambda) = -R(\lambda, T \mid Y_2)y_2 + x(\lambda), \text{ Hence}$$

$$\frac{1}{2\pi i}\int_\Gamma y_1(\lambda)\ d\lambda = -\frac{1}{2\pi i}\int_\Gamma R(\lambda, T \mid Y_2)y_2\ d\lambda + \frac{1}{2\pi i}\int_\Gamma x(\lambda)\ d\lambda.$$

The spectral maximal space Y_1 of T being T-absorbing ([14], Proposition 3.1), if $y_1 \in Y_1$, then $y_1(\lambda) \in Y_1$ for $\lambda \in \rho_T(y_1)$ and since $\sigma(T \mid Y_2)$ is "outside" Γ we obtain

$$\frac{1}{2\pi i}\int_\Gamma y_1(\lambda)\ d\lambda \in Y_1,$$

$$\frac{1}{2\pi i}\int_\Gamma R(\lambda, T \mid Y_2)y_2\ d\lambda = 0.$$

Consequently

$$x = \frac{1}{2\pi i}\int_{|\lambda| = \|T\|+1} R(\lambda, T)x\ d\lambda = \frac{1}{2\pi i}\int_\Gamma x(\lambda)\ d\lambda,$$

$$= \frac{1}{2\pi i}\int_\Gamma y_1(\lambda)\ d\lambda \in Y_1$$

thus

$$X_T(F) \subset \bigcap_{G_1 \supset F} Y = Z.$$

By other means, if $z \in \mathbf{Z}$ then from the inclusions

$$\gamma_T(z) \subset \gamma_{T|Y_1}(z) \subset \sigma(T \mid Y_1) \subset G_1$$

it follows that

$$\sigma_T(z) = \gamma_T(z) \cup S_T \subset \bigcap_{G_1 \supset F} G_1 = F_1$$

hence $z \in X_T(F)$ and $Z \subset X_T(F)$; so we conclude that

$$X_T(F) = \bigcap_{G_1 \supset F} Y_1,$$

from where it follows that $X_T(F)$ is closed. By [14] Proposition 3.4, $X_T(F)$ is a spectral maximal space of T and $\sigma(T \mid X_T(F)) \subset F$. Conversely, if Y is a spectral maximal space of T such that $\sigma(T \mid Y) \supset S$, then according to those proved before we obtain that

$$\sigma(T \mid X_T(\sigma(T \mid Y))) \subset \sigma(T \mid Y)$$

hence

$$X_T(\sigma(T \mid Y)) \subset Y.$$

But from the evident inclusion $Y \subset X_T(\sigma(T \mid Y))$ one finally obtains

$$Y = X_T(\sigma(T \mid Y)).$$

At this moment the theorem is completely proved. When T has the single-valued extension property ($S_T = \varnothing$) we have the following

COROLLARY 2.4. *Let $T \in B(X)$ a s-decomposable operator with $S_T = \varnothing$ and let $F \in \mathbb{C}$ be such that either $S \cap F = \varnothing$ or $F \supset S_1$ and $F \cap (S \setminus S_1) = \varnothing$, where S_1 is a separated part of S. Then $X_T(F)$ is a spectral maximal space of T and $\sigma(T \mid X_T(F)) \subset F$. Conversely, if Y is a spectral maximal space of T such that $\sigma(T \mid Y) = F$ and F has one of the two properties above, then $Y = X_T(\sigma(T \mid Y))$.*

Proof. If $F \cap S = \varnothing$ ($F \subset \sigma(T)$ closed), by the preceding theorem $X_T(S)$ and $X_T(F \cup S)$ are spectral maximal spaces of T and

$$X_T(F \cup S) = X_T(F) + X_T(S),$$

whence it follows that $X_T(F)$ is also a spectral maximal space for T (see [4], Proposition 4.9) and $\sigma(T(X_T(F))) \subset F$.

If

$$S = S_1 \cup (S \setminus S_1),$$

where S_1 is a separated part of S and $F \supset S_1$, $F \cap (S \setminus S_1) = \varnothing$, then

$$X_T(F \cup (S \setminus S_1)) = X_T(F) + X_T(S \setminus S_1);$$

therefore $X_T(F)$ is again a spectral maximal space of T. The final part of the corollary results identically as in the preceding theorem namely from the evident inclusions $Y \subset X_T(\sigma(T \mid Y))$ and $\sigma(T \mid X_T(\sigma(T \mid Y))) \subset \sigma(T \mid Y)$.

PROPOSITION 2.5. *Let $T \in B(X)$ a S-decomposable operator and S_1 a separated part of S with $\dim S_1 = 0$. Then T is S'-decomposable where $S' = S \setminus S_1$.*

Proof. The case $S_T = \varnothing$ has been proved in Proposition 1.2.9. Keeping the notations from the Proposition 1.2.9 proof, we will obtain the spectral maximal spaces $\{Y_S\} \cup \{Y'\}_1^n$ of T such that $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i') \subset G_i'$ $(i = 1,2,...,n)$ and

$$X = Y_S + Y_1' + Y_2' + ... + Y_n'.$$

But $Y_S = Y_{\sigma'} + Y_{\sigma_1} + Y_{\sigma_2} + ... + Y_{\sigma_n}$, where $\sigma(T \mid Y_S) = \sigma' \cup \sigma_1 \cup \sigma_2 \cup ... \cup \sigma_n$, $\sigma(T \mid Y_{\sigma'}) = \sigma'$, $\sigma(T \mid Y_{\sigma_i}) = \sigma_i$ $(i = 1,2,...,n)$. $Y_{\sigma'}, Y_{\sigma_i}$ being spectral maximal spaces of *T*, and $\sigma' \subset G_{S'}$, $\sigma_i \subset G_i' \subset G_i$.

Let $\hat{\sigma}_i = \sigma_i \cup \sigma(T \mid Y_i')$. Since $\hat{\sigma}_i \cap S' = \varnothing$, we have $X_T(S' \cup \hat{\sigma}_i) = X_T(S') + Y_{\hat{\sigma}_i}$, where $Y_{\hat{\sigma}_i}$ are spectral maximal spaces of *T*, $\sigma(T \mid Y_{\hat{\sigma}_i}) \subset \hat{\sigma}_i \subset G_i$ $(i = 1,2,...,n)$. We have $Y_1' + Y_{\sigma_i} \subset Y_{\hat{\sigma}_i}$ and $X_T(S') + Y_{\sigma'} \subset X_T \quad (\sigma' \cup S') = Y_{S'}$, therefore $X = Y_{S'} + Y_{\hat{\sigma}_1} + ... + Y_{\hat{\sigma}_n}$, and *T* is S'-decomposable.

Remark 2.6. Let $T \in B(X)$ be a *S*-decomposable operator and $S_1 \subset S$ the closing of the set of *S*'s points in which *S* has the dimension 0, $\dim S_1 = 0$ and thus that $S' = S \setminus S_1$ be closed (and thus separated from S_1); then from the preceding proposition it follows that *T* is S'-decomposable.

Proposition 2.7 *Let* $T_\alpha \in (X_\alpha)$ $(\alpha = 1,2)$ *and let* $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$. *If* $Y \subset X_1 \oplus X_2$ *is a spectral maximal space of* $T_1 \oplus T_2$, *then* $Y = Y_1 \oplus Y_2$, *where* Y_1, Y_2 *are spectral maximal spaces of* T_1 *respectively* T_2.

Proof. Let P_1 and P_2 be the corresponding projections: $X_1 = P_1(X_1 \oplus X_2)$, $X_2 = P_2(X_1 \oplus X_2)$. It is easy to verify that P_1 and P_2 switch with $T_1 \oplus T_2$ and since *Y* is ultrainvariant at $T_1 \oplus T_2$, it follows that *Y* is invariant to P_1 and P_2. By putting $Y_1 = P_1 Y$ and $Y_2 = P_2 Y$, we have $Y_1 \subset Y$, $Y_2 \subset Y$, $Y_1 \oplus Y_2 \subset Y$, P_1 and P_2 also being projections in the Banach space *Y*, Y_1, Y_2 closed. If $y \in Y$, then $y = P_1 y \oplus P_2 Y \in Y_1 \oplus Y_2$, so $Y = Y_1 \oplus Y_2$. Let Z_α $(\alpha = 1,2)$ two invariant at *T* subspace such that

$$\sigma(T_\alpha \mid Z_\alpha) \subset \sigma(T_\alpha \mid Y_\alpha) \ (\alpha = 1,2).$$

Then $Z = Z_1 \oplus Z_2$ is an (closed) invariant subspace at $T_1 \oplus T_2$ and

$$\sigma(T_1 \oplus T_2 \mid Z_1 \oplus Z_2) \subset \sigma(T_1 \oplus T_2 \mid Y_1 \oplus Y_2),$$

hence $Z_1 \oplus Z_2 \subset Y_1 \oplus Y_2$. From this inclusion it obviously follows that

$$Z_1 \subset Y_1,\ Z_2 \subset Y_2$$

consequently Y_1 and Y_2 are spectral maximal spaces of T_1, respectively T_2.

3. SPECTRAL EQUIVALENCE OF $\mathcal{A}_S$-SCALAR OPERATORS. $\mathcal{A}_S$-DECOMPOSABLE AND $\mathcal{A}_S$-SPECTRAL OPERATORS

For decomposable (respectively, spectral, S-decomposable, S-spectral) operators, we have several important results with respect to spectral equivalence property. Thus if $T_1, T_2 \in \mathbf{B}(X)$, T_1 is decomposable (respectively, spectral, S-decomposable, S-spectral) and T_1, T_2 are spectral equivalent, then T_2 is also decomposable (respectively, spectral, S-decomposable, S-spectral). Furthermore, if T_1 and T_2 are decomposable (respectively, spectral), then T_1, T_2 are spectral equivalent if and only if the spectral maximal spaces $X_{T_1}(F)$, $X_{T_2}(F)$ of T_1 and T_2, corresponding to any closed set $F \subset \mathbb{C}$, are equal (respectively, the spectral measures E_1, E_2 of T_1 and T_2 are equal) ([8], 2.2.1, 2.2.2, 2.2.4). For S-decomposable (respectively, S-spectral) operators, the equality of the spectral spaces (respectively, the equality of S-spectral measures) does not induce the spectral equivalence of the operators, but only their S-spectral equivalence.

The behaviour of $\mathcal{A}$-scalar and $\mathcal{A}_S$-scalar operators with respect to spectral equivalence is completely different. If $T_1 \in \mathbf{B}(X)$ is $\mathcal{A}$-scalar (respectively, $\mathcal{A}_S$-scalar) and $T_2 \in \mathbf{B}(X)$ is spectral equivalent to T_1, then T_2 is not $\mathcal{A}$-scalar (respectively, $\mathcal{A}_S$-scalar), in general; in this situation, we still know that T_2 is decomposable (respectively, S-decomposable) and then T_2 is said to be $\mathcal{A}$-decomposable (respectively, $\mathcal{A}_S$-decomposable). If in addition T commutes with one of its $\mathcal{A}$-spectral (respectively, $\mathcal{A}_S$-spectral) functions U, i.e. $TU_f = U_f T$, for any $f \in \mathcal{A}$ (respectively, for any $f \in \mathcal{A}_S$), then T is said to be $\mathcal{A}$-spectral (respectively, $\mathcal{A}_S$-spectral).

Definition 3.1. An operator $T \in \mathbf{B}(X)$ is called $\mathcal{A}_S$*-decomposable* if there is an $\mathcal{A}_S$-spectral function U such that T is spectral equivalent to U_λ.
In case that $S = \varnothing$, we have $\mathcal{A}_\varnothing = \mathcal{A}$, $\mathcal{A}_\varnothing$-spectral function is $\mathcal{A}$-spectral function, $\mathcal{A}_\varnothing$-decomposable operator is $\mathcal{A}$-decomposable operator ([8]).

Theorem 3.2. *Let* $T \in \mathbf{B}(X)$ *such that we consider the following two assertions*:
(I) *There is an* $\mathcal{A}_S$*-spectral function* U *such that* T *is spectral equivalent to* U_λ (i.e. T *is* $\mathcal{A}_S$*-decomposable*);
(II) *There is an* $\mathcal{A}_S$*-spectral function* U *such that for any closed set* $F \subset \mathbb{C}$, $F \supset S$, *we have*:
(a) $TX_{U_\lambda}(F) \subset X_{U_\lambda}(F)$
(b) $\sigma\left(T \middle| X_{U_\lambda}(F)\right) \subset F$.
Then the assertion (I) *implies the assertion* (II), *and for case* $S = \varnothing$, *the assertions* (I) *and* (II) *are equivalent*.
Proof. Let us suppose that there is an $\mathcal{A}_S$-spectral function U such that T and U_λ are spectral equivalent. Since U_λ is S-decomposable (Theorem 1.5), then, according to Theorem 1.11, it results that T is S-decomposable and we have

$X_T(F) = X_{U_\lambda}(F)$ for any $F \subset \mathbb{C}$ closed, $F \supset S$. But $X_T(F)$ is invariant to T and $\sigma\left(T \middle| X_T(F)\right) \subset F$ (Theorem 2.1.3, [6]), whence it follows (by (1)) that

$$TX_{U_\lambda}(F) \subset X_{U_\lambda}(F)$$

and

$$\sigma\left(T \middle| X_{U_\lambda}(F)\right) \subset F.$$

In case $S = \varnothing$, if the assertion (II) is fulfilled, according to Theorem 2.2.6, [8], we deduce that T is decomposable and that the equality (1) holds for any closed set $F \subset \mathbb{C}$. Then T is spectral equivalent to U_λ (Theorem 2.2.2, [8]) and therefore (I) is verified.

Remark 3.3. If $T \in \mathbf{B}(X)$ is $\mathcal{A}_S$-decomposable and U is one of its $\mathcal{A}_S$-spectral functions, then:

1) T is S-decomposable;
2) $X_T(F) = X_{U_\lambda}(F)$, for any $F \subset \mathbb{C}$ closed, $F \supset S$;
3) If V is another $\mathcal{A}_S$-spectral function of T, then U_λ and V_λ are spectral equivalent (in particular, V_λ is spectral equivalent to T);
4) For $S = \varnothing$, if $\mathcal{A}$ is an inverse closed algebra of continuous functions defined on a closed subset of $\mathbb{C}$ and V is another $\mathcal{A}$-spectral function of T, then U_f and V_f are spectral equivalent, for any $f \in \mathcal{A}$ (see [8]).

Definition 3.4. An operator $T \in \mathbf{B}(X)$ is called $\mathcal{A}_S$*-spectral* if it is $\mathcal{A}_S$-decomposable and commutes with one of its $\mathcal{A}_S$-spectral functions, hence T is $\mathcal{A}_S$-spectral if there is an $\mathcal{A}_S$-spectral function U commuting with T such that T is spectral equivalent to U_λ.
For $S = \varnothing$, we have that an $\mathcal{A}_\varnothing$-spectral operator is an $\mathcal{A}$-spectral operator ([8]).

Theorem 3.5. *For an operator* $T \in \mathbf{B}(X)$ *we consider the following four assertions*:
(I) T *is* $\mathcal{A}_S$*-decomposable and commutes with one of its* $\mathcal{A}_S$*-spectral functions* (i.e. T *is* $\mathcal{A}_S$*-spectral*);
(II) (II1) T *is* S*-decomposable*;
(II2) *There is an* $\mathcal{A}_S$*-spectral function* U *commuting with* T, i.e. $U_f T = TU_f$, *for any* $f \in \mathcal{A}_S$;
(II3) $X_T(F) = X_{U_\lambda}(F)$, *for any* $F \subset \mathbb{C}$ *closed*, $F \supset S$;
(III) (III1) *There is an* $\mathcal{A}_S$*-spectral function* U *commuting with* T;

$(\mathrm{III2})$ $\sigma\left(T \mid X_{U_\lambda}(F)\right) \subset F$, *for any* $F \subset \mathbb{C}$ *closed,* $F \supset S$;

(IV) $T = S + Q$, *where* S *is an* $\mathcal{A}_S$*-scalar operator and* Q *is a quasinilpotent operator commuting with an* $\mathcal{A}_S$*-spectral function of* S (*not to be confused the compact subset* S *with the operator* S *from the equality* $T = S + Q$, S *being the scalar part of T and* Q *the radical part of* T).

Then the assertions (I) *and* (IV), *respectively* (II) *and* (III) *are equivalent*, (I) *implies* (II), *respectively* (III), *and finally* (IV) *implies* (II).

Proof. $(\mathrm{I}) \Rightarrow (\mathrm{II}),(\mathrm{III})$. Assuming (I) fulfilled, we prove that the assertions (II) and (III) are verified. If T is $\mathcal{A}_S$-decomposable and commutes with one of its $\mathcal{A}_S$-spectral functions U, then U_λ is spectral-equivalent to T. Furthermore, U_λ being S-decomposable (Theorem 1.5), then T is S-decomposable (Theorem 1.12) and we have the equality:

$$X_T(F) = X_{U_\lambda}(F)$$

for any $F \subset \mathbb{C}$ closed, $F \supset S$, hence (II) is fulfilled. From Theorem 2.2, it follows that

$$\sigma\left(T \middle| X_{U_\lambda}(F)\right) = \sigma\left(T \middle| X_T(F)\right) \subset F$$

for any $F \subset \mathbb{C}$ closed, $F \supset S$, hence (III) is also verified.

$(\mathrm{I}) \Rightarrow (\mathrm{IV})$ T being $\mathcal{A}_S$-spectral, there is an $\mathcal{A}_S$-spectral function U commuting with T, i.e. $TU_f = U_f T$, for any $f \in \mathcal{A}_S$ (in particular, $TU_\lambda = U_\lambda T$) such that T is spectral equivalent to U_λ. But the operator U_λ is S-decomposable (Theorem 1.5), hence by Theorem 1.12, T is also S-decomposable and the following equality is verified

$$X_T(F) = X_{U_\lambda}(F), \text{ for any } F \subset \mathbb{C} \text{ closed, } F \supset S.$$

Using the fact that T and U_λ commute, it follows that $T - U_\lambda$ is a quasinilpotent operator commuting with U, because

$$(T - U_\lambda)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} T^k U_\lambda^{n-k} = (T - U_\lambda)^n$$

and the quasinilpotent equivalence of T and U_λ is given by

$$\lim_{n\to\infty} \left\| (T - U_\lambda)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| (U_\lambda - T)^{[n]} \right\|^{\frac{1}{n}} = 0$$

(we remember that an operator T is quasinilpotent if $\lim_{n\to\infty} \left\| T^n \right\|^{\frac{1}{n}} = 0$ or, equivalently, $\sigma(T) = 0$). We remark that if U is an $\mathcal{A}_S$-spectral function, then U_λ is an $\mathcal{A}_S$-scalar operator. Putting $S = U_\lambda$ and $Q = T - U_\lambda$, we have

$$T = S + Q$$

where S is $\mathcal{A}_S$-scalar and Q is quasinilpotent (S is the scalar part of T and Q is the radical part of T).

$(\mathrm{IV}) \Rightarrow (\mathrm{I})$ By the hypothesis of assertion (IV), since S is an $\mathcal{A}_S$-scalar operator, we deduce that there is at least one $\mathcal{A}_S$-spectral function U of S such that: $S = U_\lambda$, the quasinilpotent operator Q commutes with U and S is S-decomposable (Theorem 1.5). It also results that $T = S + Q$ commutes with U (since we obviously have $U_\lambda U_f = U_f U_\lambda = \ = U_{\lambda f}$) and since $Q = T - S$ is quasinilpotent, then T is spectral equivalent to S, consequently T is $\mathcal{A}_S$-spectral.

$(\mathrm{III}) \Rightarrow (\mathrm{II})$ Assume that there is an $\mathcal{A}_S$-spectral function U commuting with T such that $\sigma\left(T \mid X_{U_\lambda}(F)\right) \subset F$, for $F \subset \mathbb{C}$ closed, $F \supset S$. On account of the definition and the properties of an $\mathcal{A}_S$-spectral function and of an $\mathcal{A}_S$-scalar operator, we remark that U_λ is an $\mathcal{A}_S$-scalar operator, hence U_λ is S-decomposable (Theorem 1.5) and we have $X_{U_\lambda}(F) = X_{[U]}(F)$, $F \subset \mathbb{C}$ closed, $F \supset S$ (Theorem 1.9). But $X_{U_\lambda}(F)$ is a spectral maximal space of U_λ (Theorem 2.1.3, [6]), hence it is ultrainvariant to U_λ (Proposition 1.3.2, [8]); therefore $X_{U_\lambda}(F)$ is invariant to T and then the restriction $T \mid X_{U_\lambda}(F)$ makes sense and $\sigma\left(T \mid X_{U_\lambda}(F)\right) \subset F$.

$(\mathrm{II}) \Rightarrow (\mathrm{III})$ The operator T being S-decomposable, according to Theorem 2.1.3, [6], we have that $X_T(F)$ is a spectral maximal space of T, for any $F \subset \mathbb{C}$ closed, $F \supset S$ and

$$\sigma(T \mid X_T(F)) \subset F \cap \sigma(T)$$

hence (by ((II3))

$$\sigma\left(T \middle| X_{U_\lambda}(F)\right) = \sigma\left(T \middle| X_T(F)\right) \subset F.$$

$(\mathrm{IV}) \Rightarrow (\mathrm{II})$ S being $\mathcal{A}_S$-scalar, there is an $\mathcal{A}_S$-spectral function U such that $S = U_\lambda$. But from Theorem 1.5, S is S-decomposable and applying Theorem 1.11 to T and S, we get that T is S-decomposable and

$$X_T(F) = X_S(F) = X_{U_\lambda}(F)$$

for any $F \subset \mathbb{C}$ closed, $F \supset S$.

The function U commutes with the quasinilpotent operator Q, i.e. $Q U_f = U_f Q$, for $f \in \mathcal{A}_S$, hence $T = S + Q$ commutes with U.

Remark 3.6. With the same conditions as in Theorem 2.4, if $S = \varnothing$, then the four assertions above are equivalent (see [8]).

Remark 3.7. Let $T_1, T_2 \in \mathbf{B}(X)$ be two spectral equivalent operators. Then we have:

1) If $T_1 \in \mathbf{B}(X)$ is $\mathcal{A}_S$-scalar (respectively, $\mathcal{A}$-scalar), then T_2 is not $\mathcal{A}_S$-scalar (respectively, $\mathcal{A}$-scalar).

2) If $T_1 \in \mathbf{B}(X)$ is $\mathcal{A}_S$-decomposable(respectively, $\mathcal{A}$-decomposable), then T_2 is $\mathcal{A}_S$-decomposable (respectively, $\mathcal{A}$-decomposable).

3) If $T_1 \in \mathbf{B}(X)$ is $\mathcal{A}_S$-spectral (respectively, $\mathcal{A}$-spectral), then T_2 is not $\mathcal{A}_S$-spectral (respectively, $\mathcal{A}$-spectral).

4. CONCLUSIONS

We will underline the relevance, importance and necessity of studying the $\mathcal{A}_S$-scalar (respectively, $\mathcal{A}_S$-decomposable or $\mathcal{A}_S$-spectral) operators, showing the consistence of this class, in the sense of how many and how substantial its subfamilies are. These operators are natural generalizations of the notions of $\mathcal{A}$-scalar, $\mathcal{A}$-decomposable and $\mathcal{A}$-spectral operators studied in [8] and appear, in general, as restrictions or quotients of the last one.

We demonstrated some of their properties, leaving the challenge to proof and generalize many others.

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