A Method of Estimating the p-Adic Sizes Polynomials

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***Abstract*— The exponential sum associated with f is defined as $S(f; q) = \sum_{\underline{x} \bmod q} e^{\frac{2\pi i f(\underline{x})}{q}}$, where the sum is taken over a complete set of residues modulo q. The value of $S(f; q)$ depends on the estimate of cardinality in the set $V = \{\underline{x} \bmod q \mid \underline{f}_{\underline{x}} \equiv \underline{0} \bmod q\}$ where $\underline{f}_{\underline{x}}$ is the partial derivatives of $\underline{f}$ with respect to $\underline{x}$. In order to determine the cardinality, the p-adic sizes of common zeros of the partial derivative polynomials need to be obtained. This paper will give an estimation of the p-adic sizes of common zeros of partial derivative polynomials of degree eight in $\mathbb{Z}_p[x, y]$ by using Newton polyhedron technique.

***Keywords*—**Exponential sums, Cardinality, p-adic sizes, Newton polyhedron.

I. Introduction

In our discussion, we use notations the ring of p-adic integers ($\mathbb{Z}_p$), the completion of algebraic closure of $\mathbb{Q}_p$ the field of rational p-adic numbers (Ω_p) and the p-adic size of x which means the highest power of p dividing x ($ord_p\, x$). It follows that for rational number x and y, $ord_p\, x = \infty$ if and only if $x = 0$; $ord_p\,(xy) = ord_p\, x + ord_p\, y$ and $ord_p\,(x + y) \geq min\ \{ord_p\, x, ord_p\, y\}$ with equality if $ord_p\, x \neq ord_p\, y$.

The researchers in [5] who investigate the exponential sums $S(f\,;\,q) = \sum_{\underline{x} \bmod q} \exp(2\pi i f / q)$ where f is a nonlinear polynomial in $Z[\underline{x}]$ showed that the number of common zeros of the partial derivative polynomials of f with respect to $\underline{x}$ modulo q gives the estimation of $S(f\,;\,q)$.

Then from the works of [4], they found that the p-adic sizes of common zeros to partial derivative polynomials associated with f in the neighborhood of points in the product space Ω_p^n, $n > 0$, can estimate the cardinality of V.

The estimations for lower degree two-variable polynomials by using Newton polyhedron technique are found by many researchers such as [6] who defines the p-adic sizes of common zeros, [1] who estimates the cardinality $N(f; p^\alpha)$ of the set of solutions to congruence equations modulo a prime power and also [2] who founds a better estimate for the exponential sums under various conditions on the coefficients of $f(x, y)$ and obtained the estimate of $S(f\,;\,p^\alpha)$.

Our approach entails the work developed by [7] who presented the p-adic Newton polyhedral method of finding the p-adic order of polynomials in $\Omega_p[x, y]$ which is an analogue of Newton polygon defined by [3].

The researchers in [8] improved the result from [7]. Then, [9] discussed a method of determining the p-adic sizes of partial derivative polynomial of degree n where n is odd based on the p-adic Newton polyhedron technique at simple points of intersections in the combination of indicator diagrams associated with a pair of polynomials in $Z_p[x,y]$ where $Z_p[x,y]$ denote the set of polynomials in x and y with coefficients in Z_p. From the works of [10], they estimates a method of the p-adic sizes of common zeros of partial derivative polynomials associated with a quintic form for $p > 5$ by using the similar technique as paper [9].

Recently, [11] showed that the p-adic sizes of common zeros of partial derivative polynomials associated with a cubic form can be found explicitly on the overlapping segment of the indicator diagrams associated with the polynomials by using Newton polyhedron technique.

Our work involves application of the Newton polyhedron technique at the point of intersection in the combination of indicator diagrams to determine explicitly the p-adic sizes of the component (ξ, η) a common root of partial derivative polynomials of $f(x, y)$ in $\mathbb{Z}_p[x, y]$ of an eighth degree form.

II. p-Adic Sizes of Zeros of a Polynomial

In this work, we discuss about the p-adic sizes of common zeros of partial derivative polynomials associated with a polynomial $f(x, y)$ of degree eight in $\mathbb{Z}_p[x, y]$. Researchers [8] proved that every point of intersection of the indicator diagrams, there exist common zeros of both polynomials in $\mathbb{Z}_p[x, y]$ whose p-adic orders correspond to point (μ_1, μ_2) as mention in Theorem 1 as follows:

Theorem 1
Let p be a prime. Suppose f and g are polynomials in $\mathbb{Z}_p[x, y]$. Let (μ_1, μ_2) be a point of intersection of the indicator diagrams associated with f and g at the vertices or simple points of intersections. Then there are ξ and η in Ω_p^2 satisfying $f(\xi, \eta) = g(\xi, \eta) = 0$ and $ord_p\, \xi = \mu_1$, $ord_p\, \eta = \mu_2$.

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Our work concentrates on the p-adic sizes of common zeros of partial derivative associated with a polynomial $f(x,y) = ax^8 + bx^7y + cx^6y^2 + sx + ty + k$. From our investigation, we found that the p-adic sizes of $f(x,y)$ is as the following theorem.

Theorem 2
Let $f(x,y) = ax^8 + bx^7y + cx^6y^2 + sx + ty + k$ be a polynomial in $\mathbb{Z}_p[x,y]$ with $p > 7$ be a prime. Let $\alpha > 0$, $\delta = max\{ord_p a, ord_p b, ord_p c\}$. If $ord_p\, f_x(0,0)$, $ord_p\, f_y(0,0) \geq \alpha > 8\delta$ then there exists (ξ,η) in Ω_p^2 such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ and

$ord_p\xi \geq \frac{1}{7}(\alpha - \delta)$ and	$ord_p\xi \geq \frac{1}{7}(\alpha - \delta - \varepsilon_1)$ and
$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - 5\varepsilon_0)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - 5\varepsilon_0)$	$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1 - 5\varepsilon_0)$.

for some $\varepsilon_0, \varepsilon_1 \geq 0$.

In order to prove Theorem 2, we need the result of the following lemmas.

Lemma 1
Let $p > 7$ be a prime. Let a, b and c in $\mathbb{Z}_p$ and λ_1, λ_2 zeros of $k(\lambda) = 4c^2\lambda^2 + 4bc\lambda + (49b^2 - 192ac)$ and suppose

$$\alpha_1 = \frac{7b + 2\lambda_1 c}{2(8a + \lambda_1 b)} \quad \text{and} \quad \alpha_2 = \frac{7b + 2\lambda_2 c}{2(8a + \lambda_2 b)}.$$

Therefore, $ord_p\,(\alpha_1 - \alpha_2) = \frac{1}{2}ord_p(192ac - 48b^2) + ord_p 2c - ord_p\Delta$ and $ord_p\,(\alpha_1 + \alpha_2) = ord_p 12bc - ord_p\Delta$, where $\Delta = 16ac - 7b^2$.

Proof.

Let $\lambda_i = \frac{-b \pm \sqrt{192ac - 48b^2}}{2c}$, $i = 1$ or 2. Given that $\alpha_i = \left(\frac{7b + 2\lambda_i c}{2(8a + \lambda_i b)}\right)$, $i = 1, 2$. Then we have

$$\alpha_1 - \alpha_2 = \frac{(192ac - 48b^2)^{1/2}(16ac - 7b^2)}{2c\,(8a + \lambda_1 b)(8a + \lambda_2 b)} \tag{1}$$

and

$$\alpha_1 + \alpha_2 = \frac{3b\,(7b^2 - 16ac)}{c(6a + \lambda_1 b)(6a + \lambda_2 b)}. \tag{2}$$

By substituting the values of λ_i, $i = 1, 2$, we have

$$(8a + \lambda_1 b)(8a + \lambda_2 b) = \frac{(16ac - 7b^2)^2}{4c^2}. \tag{3}$$

By substituting (3) into (1), we obtain

$$\alpha_1 - \alpha_2 = \frac{2c[192ac - 48b^2]^{1/2}}{\Delta}, \text{ where } \Delta = 16ac - 7b^2.$$

Therefore,

$ord_p\,(\alpha_1 - \alpha_2)$

$$= \frac{1}{2}ord_p(192ac - 48b^2) + ord_p 2c - ord_p\Delta.$$

By substituting (3) into (2), we obtain

$$\alpha_1 + \alpha_2 = -\frac{12bc}{\Delta}, \text{ where } \Delta = 16ac - 7b^2.$$

Thus,

$$ord_p\,(\alpha_1 + \alpha_2) = ord_p 12bc - ord_p\Delta.$$

as asserted. □

Throughout the following discussion,

$$\alpha_1 = \frac{7b + 2\lambda_1 c}{2(8a + \lambda_1 b)} \quad \text{and} \quad \alpha_2 = \frac{7b + 2\lambda_2 c}{2(8a + \lambda_2 b)} \tag{4}$$

with λ_1, λ_2 are the zeros of $k(\lambda) = 4c^2\lambda^2 + 4bc\lambda + (n-1)^2b^2 - 4n(n-2)ac$. $\alpha_1 \neq \alpha_2$ since $\lambda_1 \neq \lambda_2$.

Lemma 2
Suppose (U,V) in Ω_p^2. Let $p > 7$ be a prime, a, b and c coefficients of α_1 and α_2 as in (4) in $\mathbb{Z}_p$. Then

$ord_p\,(\alpha_1 V - \alpha_2 U) =$
$ord_p\left[6b(U - V) + \sqrt{192ac - 48b^2}(U + V)\right] + ord_p\, c - ord_p\,\Delta$

where $\Delta = 16ac - 7b^2$.

Proof.

We have

$$ord_p\,(\alpha_1 V - \alpha_2 U) = ord_p\left[\left(\frac{7b + 2\lambda_1 c}{2(8a + \lambda_1 b)}\right)V - \left(\frac{7b + 2\lambda_2 c}{2(8a + \lambda_2 b)}\right)U\right]$$

$$= ord_p\left[\frac{(7b + 2\lambda_1 c)(8a + \lambda_2 b)V - (7b + 2\lambda_2 c)(8a + \lambda_1 b)U}{2(8a + \lambda_1 b)(8a + \lambda_2 b)}\right]. \tag{5}$$

Let

$$\lambda_i = \frac{-b \pm \sqrt{192ac - 48b^2}}{2c}, \quad i = 1 \text{ or } 2.$$

From the numerator of the equation (5), it can be proved that $(7b + 2\lambda_1 c)(8a + \lambda_2 b)V - (7b + 2\lambda_2 c)(8a + \lambda_1 b)U = \left(\frac{16ac - 7b^2}{2c}\right)\left[6b(U - V) + \sqrt{192ac - 48b^2}\,(U + V)\right]$

Therefore, by (3) and (5), we obtain

$ord_p\,(\alpha_1 V - \alpha_2 U) =$
$ord_p\left[6b(U - V) + \sqrt{192ac - 48b^2}\,(U + V)\right] + ord_p\, c - ord_p\,\Delta$

where $\Delta = 16ac - 7b^2$ as asserted. □

Lemma 3
Suppose (x,y) in Ω_p^2 with $U = x^{\frac{7}{2}} + \alpha_1 x^{\frac{5}{2}}y$ and $V = x^{\frac{7}{2}} + \alpha_2 x^{\frac{5}{2}}y$ where α_1 and α_2 as in (4). Let $p > 7$ be a prime, a, b and c the coefficients of α_1 and α_2, in $\mathbb{Z}_p$. If $ord_p b^2 \neq ord_p ac$, then

$ord_p b^2 > ord_p ac$	$ord_p b^2 < ord_p ac$
$ord_p\, x \geq \frac{2}{7}W$	$ord_p\, x \geq \frac{2}{7}W$
$ord_p\, y \geq \frac{2}{7}\left[W - \frac{1}{2}ord_p\frac{cb^5}{a^6}\right]$	$ord_p\, y \geq \frac{2}{7}\left[W - \frac{1}{2}ord_p\frac{c^7}{b^7}\right]$
$ord_p\, y \geq \frac{2}{7}\left[W - \frac{1}{2}ord_p\frac{cb^5}{a^6} - \frac{5}{2}\varepsilon_0\right]$	$ord_p\, y \geq \frac{2}{7}\left[W - \frac{1}{2}ord_p\frac{c^7}{b^7} - \frac{5}{2}\varepsilon_0\right]$

where $W = min\{ord_p V, ord_p U\}$ for some $\varepsilon_0 \geq 0$ which can be specified explicitly.

Proof.

Solve $U = x^{\frac{7}{2}} + \alpha_1 x^{\frac{5}{2}} y$ and $V = x^{\frac{7}{2}} + \alpha_2 x^{\frac{5}{2}} y$ simultaneously, we have

$$x = \left(\frac{\alpha_1 V - \alpha_2 U}{\alpha_1 - \alpha_2}\right)^{\frac{2}{7}}, \quad y = \frac{V - U}{(\alpha_1 - \alpha_2) x^{\frac{5}{2}}}. \qquad (6)$$

From the result of Lemmas 1 and 2, equation (6) become

$$ord_p\, x = \tfrac{2}{7} ord_p \left[6b(U - V) + \sqrt{192ac - 48b^2}(U + V)\right] - \tfrac{1}{7} ord_p[192ac - 48b^2] \qquad (7)$$

and

$$ord_p\, y = \tfrac{2}{7} ord_p (U - V) - ord_p c - \tfrac{1}{7} ord_p (192ac - 48b^2) + ord_p (16ac - 7b^2) - \tfrac{5}{7} ord_p b. \qquad (8)$$

Now, let consider (7) and (8). From these two equations, we will consider two conditions with two cases for each condition as follows:

CONDITION 1: $ord_p b^2 > ord_p ac$

In this condition, we have to consider two cases. That is,

CASE 1: $ord_p\, 6b(U - V) \neq ord_p \sqrt{192ac - 48b^2}(U + V)$

CASE 2: $ord_p\, 6b(U - V) = ord_p \sqrt{192ac - 48b^2}(U + V)$.

Now, we consider CASE 1. In this case, we have to consider another two cases. That is,

(i) Suppose *min* $\{ord_p\, 6b(U - V),\ ord_p \sqrt{192ac - 48b^2}(U + V)\} = ord_p \sqrt{192ac - 48b^2}(U + V)$.

By applying the above condition into (7), we have

$$ord_p\, x = \tfrac{2}{7} ord_p \sqrt{192ac - 48b^2}(U + V) - \tfrac{1}{7} ord_p 192ac.$$

Since $p > 7$ a prime and $ord_p b^2 > ord_p ac$, then

$$ord_p\, x = \tfrac{1}{7} ord_p\, ac + \tfrac{2}{7} ord_p (U + V) - \tfrac{1}{7} ord_p ac.$$

Therefore,

$$ord_p\, x = \tfrac{2}{7} ord_p (U + V). \qquad (9)$$

It follows that,

$$ord_p\, x \geq \tfrac{2}{7} W$$

where $W = min\{ord_p V, ord_p U\}$.

By adding x and y in equation (6), we obtain

$$ord_p (U + V) = ord_p \left(2x^{\frac{7}{2}} + (\alpha_1 + \alpha_2) x^{\frac{5}{2}} y\right). \qquad (10)$$

Thus, from (9), we have

$$ord_p\, x^{\frac{7}{2}} = ord_p (U + V)$$

Therefore,

$$ord_p\, x \leq ord_p (\alpha_1 + \alpha_2) y.$$

From (8), we have

$$ord_p \geq \tfrac{2}{7} ord_p (U - V) - ord_p c - \tfrac{1}{7} ord_p 192ac + ord_p 12ac - \tfrac{5}{7} ord_p b. \qquad (11)$$

By solving equation (11), we obtain

$$ord_p\, y \geq \tfrac{2}{7}\left[W - \tfrac{1}{2} ord_p \frac{cb^5}{a^6}\right]$$

where $W = min\{ord_p V, ord_p U\}$.

(ii) Suppose *min* $\{ord_p\, 6b(U - V),\ ord_p \sqrt{192ac - 48b^2}(U + V)\} = ord_p\, 6b(U - V)$.

By applying the above condition into (7), we have

$$ord_p\, x = \tfrac{2}{7} ord_p (U - V) + \tfrac{1}{7}\left(ord_p\, 6b^2 - ord_p 192ac\right). \qquad (12)$$

Since $p > 7$ a prime and $ord_p b^2 > ord_p ac$, then

$$ord_p\, x = \tfrac{2}{7} ord_p (U - V). \qquad (13)$$

It follows that,

$$ord_p\, x \geq \tfrac{2}{7} W$$

where $W = min\{ord_p V, ord_p U\}$.

From (13), we have

$$ord_p\, x^{\frac{7}{2}} = ord_p (U + V).$$

By substituting equation (13) into (12), we obtain

$$ord_p\, x \leq \tfrac{2}{7} ord_p (U + V) + \tfrac{1}{7}\left(ord_p ac - ord_p ac\right), \qquad (14)$$

since $ord_p b^2 > ord_p ac$.

So that, it can be shown that

$$ord_p\, x \leq ord_p (\alpha_1 + \alpha_2) y. \qquad (15)$$

From (15), by using the similar method as equation (8), we have

$$ord_p\, y \geq \tfrac{2}{7}\left[W - \tfrac{1}{2} ord_p \frac{cb^5}{a^6}\right]$$

where $W = min\{ord_p V, ord_p U\}$.

Now, we consider CASE 2. That is,

$ord_p\, 6b(U - V) = ord_p \sqrt{192ac - 48b^2}(U + V)$.

From (7), we have

$$ord_p\, x \geq \tfrac{2}{7} min \{ord_p\, 6b(U - V), ord_p \sqrt{192ac - 48b^2}(U + V)\} - \tfrac{1}{7} ord_p 192ac.$$

Since $p > 7$ a prime and $ord_p b^2 > ord_p ac$, then

$$ord_p\, x \geq \tfrac{2}{7} ord_p (U + V).$$

Thus,

$$ord_p\, x \geq \tfrac{2}{7} W$$

where $W = min\{ord_p V, ord_p U\}$.

From (8), since $p > 7$ a prime and $ord_p b^2 > ord_p ac$, then

$$ord_p\, y = ord_p (U - V) - \tfrac{1}{7} ord_p \frac{c}{a^6} - \tfrac{5}{7} ord_p \left[6b(U - V) + \sqrt{192ac - 48b^2}(U + V)\right]. \qquad (16)$$

Let

$$\beta = ord_p\, 6b(U - V) = ord_p \sqrt{192ac - 48b^2}(U + V). \qquad (17)$$

There exists m and n such that

$$6b(U - V) = p^{\beta} m \text{ with } ord_p\, m = 0 \qquad (18)$$

$$\sqrt{192ac - 48b^2}(U + V) = p^{\beta} n \text{ with } ord_p\, n = 0. \qquad (19)$$

Hence, from (17), we have

$$ord_p (U - V) = \beta - ord_p\, b. \qquad (20)$$

Therefore, by substituting (18) and (19) into (16), we obtain

$$ord_p\, y = \tfrac{2}{7}\beta - ord_p\, b - \tfrac{1}{7} ord_p \frac{c}{a^6} - \tfrac{5}{7} ord_p (m + n).$$

Suppose $\varepsilon_0 = ord_p (m + n)$, then

$$ord_p\, y = \tfrac{2}{7}\beta - ord_p\, b - \tfrac{1}{7} ord_p \frac{c}{a^6} - \tfrac{5}{7}\varepsilon_0.$$

That is, from (20), we obtain

$$ord_p\, y = \tfrac{2}{7} ord_p (U - V) - \tfrac{1}{7} ord_p \frac{cb^5}{a^6} - \tfrac{5}{7}\varepsilon_0.$$

It follows that,

$$ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p\ \frac{cb^5}{a^6} - \frac{5}{2}\varepsilon_0\right]$$

where $W = min\{ord_p V, ord_p U\}$ and $\varepsilon_0 = ord_p\,(m+n)$.
Hence, for the condition $ord_p b^2 > ord_p ac$, we have

$$ord_p\ x \geq \frac{2}{7} W \text{ and } ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{cb^5}{a^6}\right] \text{ or}$$

$$ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{cb^5}{a^6} - \frac{5}{2}\varepsilon_0\right]$$

where $W = min\{ord_p V, ord_p U\}$ for $\varepsilon_0 \geq 0$ as asserted.

<u>CONDITION 2</u>: $ord_p b^2 < ord_p ac$
By using the similar process as CONDITION 1, we will obtain

$$ord_p\ x \geq \frac{2}{7} W \text{ and } ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{c^7}{b^7}\right] \text{ or}$$

$$ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{c^7}{b^7} - \frac{5}{2}\varepsilon_0\right]$$

where $W = min\{ord_p V, ord_p U\}$ for $\varepsilon_0 \geq 0$ as asserted. □

Lemma 4

Suppose (U,V) in Ω_p^2 and $U = x^{\frac{7}{2}} + \alpha_1 x^{\frac{5}{2}} y$, $V = x^{\frac{7}{2}} + \alpha_2 x^{\frac{5}{2}} y$ where α_1 and α_2 as in (4). Let $p > 7$ be a prime, a, b, c, s and t in $\mathbb{Z}_p$, $\delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p s, ord_p t \geq \alpha > 8\delta$. If $ord_p U = \frac{1}{2} ord_p \frac{s+\lambda_1 t}{8a+\lambda_1 b}$ and $ord_p V = \frac{1}{2} ord_p \frac{s+\lambda_2 t}{8a+\lambda_2 b}$ then

$ord_p x \geq \frac{1}{7}(\alpha - \delta)$ and	$ord_p x \geq \frac{1}{7}(\alpha - \delta - \varepsilon_1)$ and
$ord_p y \geq \frac{1}{7}(\alpha - 8\delta)$ or	$ord_p y \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1)$ or
$ord_p y \geq \frac{1}{7}(\alpha - 7\delta)$ or	$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1)$ or
$ord_p y \geq \frac{1}{7}(\alpha - 6\delta)$ or	$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1)$ or
$ord_p y \geq \frac{1}{7}(\alpha - 8\delta - 5\varepsilon_0)$ or	$ord_p y \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0)$ or	$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - 5\varepsilon_0)$	$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1 - 5\varepsilon_0)$

where $\varepsilon_0 = ord_p(m+n)$ and $\varepsilon_1 = ord_p(d_1 + d_2)$ for some $\varepsilon_0, \varepsilon_1 \geq 0$.

Proof.
From Lemma 3 for both conditions which are $ord_p b^2 > ord_p ac$ and $ord_p b^2 < ord_p ac$, we have

$$ord_p\ x \geq \frac{2}{7} W \qquad (21)$$

and $ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{cb^5}{a^6}\right]$ (22)

or $ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{cb^5}{a^6} - \frac{5}{2}\varepsilon\right]$ (23)

or $ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{c^7}{b^7}\right]$ (24)

or $ord_p\ y \geq \frac{2}{7}\left[W - \frac{1}{2} ord_p \frac{c^7}{b^7} - \frac{5}{2}\varepsilon_0\right]$ (25)

where $W = min\ \{ord_p U, ord_p V\}$.
Now,

$$ord_p U = \frac{1}{2} ord_p \frac{s+\lambda_1 t}{8a+\lambda_1 b} \text{ and } ord_p V = \frac{1}{2} ord_p \frac{s+\lambda_2 t}{8a+\lambda_2 b}. \qquad (26)$$

After substituting (26) into equations (21), (22), (23), (24) and (25), we obtain

$$ord_p\ x \geq \frac{1}{7}\left[ord_p(s+\lambda_i t) - ord_p(8a+\lambda_i b)\right] \qquad (27)$$

and

$$ord_p\ y \geq \frac{1}{7}\left[ord_p(s+\lambda_i t) - ord_p(8a+\lambda_i b) - ord_p cb^5 + ord_p a^6\right] \qquad (28)$$

or

$$ord_p\ y \geq \frac{1}{7}\left[ord_p(s+\lambda_i t) - ord_p(8a+\lambda_i b) - ord_p cb^5 + ord_p a^6 - 5\varepsilon_0\right] \qquad (29)$$

or

$$ord_p\ y \geq \frac{1}{7}\left[ord_p(s+\lambda_i t) - ord_p(8a+\lambda_i b) - ord_p c^7 + ord_p b^7\right] \qquad (30)$$

or

$$ord_p\ y \geq \frac{1}{7}\left[ord_p(s+\lambda_i t) - ord_p(8a+\lambda_i b) - ord_p c^7 + ord_p b^7 - 5\varepsilon_0\right] \qquad (31)$$

where $i = 1, 2$.
In order to solve the above equations, we have to consider two cases. That is,

CASE 1: $min\ \{ord_p s, ord_p\ \lambda_i t\} = ord_p s$, $i = 1, 2$.

CASE 2: $min\ \{ord_p s, ord_p\ \lambda_i t\} = ord_p \lambda_i t$, $i = 1, 2$.

Now, we consider the first case. In this case, we will consider another two cases. Let $p > 7$ be a prime.
From equation (27), we have
<u>Case (i)</u>: $\{ord_p 8a \neq ord_p \lambda_i b\}$

(i) $min\ \{ord_p 8a, ord_p \lambda_i b\} = ord_p a$.

$$ord_p\ x \geq \frac{1}{7}\left(ord_p s - ord_p a\right).$$

By applying the hypothesis, we obtain

$$ord_p\ x \geq \frac{1}{7}(\alpha - \delta).$$

(ii) $min\ \{ord_p 8a, ord_p \lambda_i b\} = ord_p \lambda_i b$.

$$ord_p\ x = \frac{1}{7}\left(ord_p s - ord_p \lambda_i b\right).$$

Since $ord_p\ \lambda_i b > ord_p a$, then

$$ord_p\ x \geq \frac{1}{7}\left(ord_p s - ord_p a\right).$$

Therefore,

$$ord_p\ x \geq \frac{1}{7}(\alpha - \delta).$$

<u>Case (ii)</u>: $\{ord_p 8a = ord_p \lambda_i b\}$
Let $\theta = ord_p 8a = ord_p \lambda_i b$. Then, there exist d_1 and d_2 such that

$$8a = p^\theta d_1 \text{ with } ord_p d_1 = 0 \qquad (32)$$

$$\lambda_i b = p^\theta d_2 \text{ with } ord_p d_2 = 0. \qquad (33)$$

Suppose $ord_p a = \theta$ for $p > 7$ a prime.
By substituting (32) and (33) into (27), we obtain

$$ord_p\ x \geq \frac{1}{7}\left(ord_p s - ord_p\left(p^\theta d_1 + p^\theta d_2\right)\right)$$

$$= \frac{1}{7}\left(ord_p s - ord_p a - ord_p(d_1 + d_2)\right),$$

where $ord_p a = \theta$.
Let $\varepsilon_1 = ord_p(d_1 + d_2)$. Therefore,

$$ord_p\ x \geq \frac{1}{7}(\alpha - \delta - \varepsilon_1).$$

From equation (28), we have

Case (i): $\{ord_p 8a \neq ord_p \lambda_i b\}$

(i) $min\{ord_p 8a, ord_p \lambda_i b\} = ord_p a$

$ord_p\ y \geq \frac{1}{7}[ord_p s - ord_p a - ord_p c - 5ord_p b + 6ord_p a]$

$= \frac{1}{7}\left[ord_p s - \left(ord_p c + ord_p \frac{b^5}{a^5}\right)\right].$ (34)

If $ord_p b^2 > ord_p ac$, then

$ord_p c + ord_p \frac{b^5}{a^5} < ord_p c + ord_p \frac{b^7}{a^6 c} = 7\ ord_p b - 6\ ord_p a < 7\ ord_p b.$ (35)

Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,

$ord_p s, ord_p t \geq \alpha > 8\delta$ and by substituting (35) into (34), we have

$$ord_p\ y \geq \frac{1}{7}[ord_p s - 7ord_p b].$$

By applying the hypothesis, we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 7\delta).$$

If $ord_p b^2 < ord_p ac$, then

$ord_p c + ord_p \frac{b^5}{a^5} < ord_p c + ord_p \frac{c^5}{b^5} = 6ord_p c - 5ord_p b < 6ord_p c.$ (36)

where $ord_p \frac{b^2}{a} < ord_p c$ implies $ord_p \frac{b}{a} < ord_p \frac{c}{b}$.

Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,

$ord_p s, ord_p t \geq \alpha > 8\delta$ and by substituting (36) into (34), we have

$$ord_p\ y \geq \frac{1}{7}[ord_p s - 6ord_p c].$$

By applying the hypothesis, we obtain

$$ord_p y \geq \frac{1}{7}(\alpha - 6\delta).$$

(ii) $min\{ord_p 8a, ord_p \lambda_i b\} = ord_p \lambda_i b$

$ord_p\ y = \frac{1}{7}[ord_p s - ord_p \lambda_i b - ord_p c - 5ord_p b + 6ord_p a]$

Since $ord_p\ \lambda_i b > ord_p a$, then

$ord_p\ y \geq \frac{1}{7}[ord_p s - ord_p a - ord_p c - 5ord_p b + 6ord_p a]$

Thus, by the similar method as (35) and (36), we have

$$ord_p\ y \geq \frac{1}{7}(\alpha - 7\delta) \quad \text{and} \quad ord_p y \geq \frac{1}{7}(\alpha - 6\delta).$$

Case (ii): $\{ord_p 8a = ord_p \lambda_i b\}$

Let $\theta = ord_p 8a = ord_p \lambda_i b$.

Therefore, by substituting (32) and (33) into (28), we obtain $ord_p\ y \geq \frac{1}{7}\left(ord_p s - ord_p\left(p^\theta d_1 + p^\theta d_2\right) - ord_p cb^5 + ord_p a^6\right)$

$= \frac{1}{7}\left(ord_p s - ord_p a - ord_p(d_1 + d_2) - ord_p c - 5ord_p b + 6ord_p a\right),$

where $ord_p a = \theta$.

Let $\varepsilon_1 = ord_p(d_1 + d_2)$. Therefore,

$ord_p\ y \geq \frac{1}{7}\left(ord_p s - ord_p a - \varepsilon_1 - ord_p c - 5ord_p b + 6ord_p a\right)$

We solve the above equation by applying the similar method as (34), thus we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1) \quad \text{and} \quad ord_p y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1).$$

From equation (29), by using the same process as equation (28) which is not involving ε_0, we obtain

$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0)$ or

$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$ or

$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - 5\varepsilon_0)$ or

$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1 - 5\varepsilon_0)$

where $\varepsilon_0 = ord_p(m + n)$ and $\varepsilon_1 = ord_p(d_1 + d_2)$ for some $\varepsilon_0, \varepsilon_1 \geq 0$.

From equation (30), we have two cases as follow.

Case (i): $\{ord_p 8a \neq ord_p \lambda_i b\}$

(i) Suppose $min\{ord_p 8a, ord_p \lambda_i b\} = ord_p a$.

$ord_p\ y \geq \frac{1}{7}[ord_p s - ord_p a - 7ord_p c + 7ord_p b]$

$= \frac{1}{7}\left[ord_p s - \left(ord_p a + ord_p \frac{c^7}{b^7}\right)\right].$ (37)

If $ord_p b^2 > ord_p ac$, then

$ord_p a + ord_p \frac{c^7}{b^7} < ord_p a + ord_p \frac{c^6}{ab^5} = 6ord_p c - 5ord_p b < 6ord_p c$ (38)

Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,

$ord_p s, ord_p t \geq \alpha > 8\delta$ and by substituting (38) into (37), we have

$$ord_p\ y \geq \frac{1}{7}[ord_p s - 6ord_p c].$$

By applying the hypothesis, we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 6\delta).$$

We have

$$\lambda_i = \frac{-b \pm \sqrt{192ac - 48b^2}}{2c},\ i = 1, 2.$$

Since $p > 7$ a prime, then we have

$$ord_p \lambda_i = ord_p\left[-b \pm \sqrt{192ac - 48b^2}\right] - ord_p c.$$

Since $ord_p b^2 < ord_p ac$, we have

$$ord_p \lambda_i = ord_p b - ord_p c = ord_p \frac{b}{c}.$$

Thus,

$$ord_p\ y \geq \frac{1}{7}[ord_p s - 6ord_p c].$$

By applying the hypothesis, we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 6\delta). \quad (39)$$

(ii) Suppose $min\{ord_p 8a, ord_p \lambda_i b\} = ord_p \lambda_i b$.

$ord_p\ y = \frac{1}{7}[ord_p s - ord_p \lambda_i b - 7ord_p c + 7ord_p b]$

Since $ord_p\ \lambda_i b > ord_p a$, then

$ord_p\ y \geq \frac{1}{7}[ord_p s - ord_p a - 7ord_p c + 7ord_p b]$

Thus, by the similar method as (38) and (39), we have

$$ord_p\ y \geq \frac{1}{7}(\alpha - 6\delta).$$

Case (ii): $\{ord_p 8a = ord_p \lambda_i b\}$
Let $\theta = ord_p 8a = ord_p \lambda_i b$.
Therefore, by substituting (32) and (33) into (30), we obtain
$ord_p\ y \geq \frac{1}{7}\left(ord_p s - ord_p\left(p^{\theta}d_1 + p^{\theta}d_2\right) - ord_p c^7 + ord_p b^7\right)$
$= \frac{1}{7}\left(ord_p s - ord_p a - ord_p(d_1 + d_2) - 7ord_p c + 7ord_p b\right),$
where $ord_p a = \theta$.
Let $\varepsilon_1 = ord_p(d_1 + d_2)$. Therefore,

$$ord_p\ y \geq \frac{1}{7}\left(ord_p s - ord_p a - \varepsilon_1 - 7ord_p c + 7ord_p b\right).$$

We solve the above equation by applying the similar method as (37), thus we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1).$$

From equation (31), by using the same process as equation (30) which not involving ε_0, we obtain

$$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - 5\varepsilon_0) \quad \text{or}$$
$$ord_p y \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1 - 5\varepsilon_0)$$

where $\varepsilon_0 = ord_p(m+n)$ and $\varepsilon_1 = ord_p(d_1 + d_2)$ for some $\varepsilon_0, \varepsilon_1 \geq 0$.
Now, we consider the second case. That is, *min* $\{ord_p s, ord_p\ \lambda_i t\} = ord_p \lambda_i t$, *i* = 1, 2. In this case, we also consider another two cases.
For equation (27), we use the similar method as the first case, and then we will obtain the same result as follows.
Case (i): $\{ord_p 8a \neq ord_p \lambda_i b\}$
By applying the hypothesis, we obtain

$$ord_p\ x \geq \frac{1}{7}(\alpha - \delta).$$

Case (ii): $\{ord_p 8a = ord_p \lambda_i b\}$
By applying the hypothesis, we obtain

$$ord_p\ x \geq \frac{1}{7}(\alpha - \delta - \varepsilon_1).$$

Next, from equation (28), we have
Case (i): $\{ord_p 8a \neq ord_p \lambda_i b\}$

(i) *min* $\{ord_p 8a, ord_p \lambda_i b\} = ord_p\ \lambda_i b$.

$$ord_p\ y \geq \frac{1}{7}\left[ord_p\ \lambda_i t - ord_p\ \lambda_i b - ord_p c - 5ord_p b + 6ord_p a\right]$$
$$= \frac{1}{7}\left[ord_p t - \left(ord_p c + ord_p \frac{b^6}{a^6}\right)\right]. \qquad (40)$$

If $ord_p b^2 > ord_p ac$, then
$ord_p c + ord_p \frac{b^6}{a^6} < ord_p c + ord_p \frac{b^8}{a^7 c} = 8\ ord_p b - 7\ ord_p a < 8\ ord_p b. \qquad (41)$
Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,
$ord_p s, ord_p t \geq \alpha > 8\delta$ and by substituting (41) into (40), we have

$$ord_p\ y \geq \frac{1}{7}\left[ord_p t - 8ord_p b\right].$$

By applying the hypothesis, we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 8\delta).$$

If $ord_p b^2 < ord_p ac$, then
$ord_p c + ord_p \frac{b^6}{a^6} < ord_p c + ord_p \frac{c^6}{b^6} = 7ord_p c - 6ord_p b < 7ord_p c. \qquad (42)$
where $ord_p \frac{b^2}{a} < ord_p c$ implies $ord_p \frac{b}{a} < ord_p \frac{c}{b}$.
Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,
$ord_p s, ord_p t \geq \alpha > 8\delta$ and by substituting (42) into (40), we have

$$ord_p\ y \geq \frac{1}{7}\left[ord_p t - 7ord_p c\right].$$

By applying the hypothesis, we obtain

$$ord_p y \geq \frac{1}{7}(\alpha - 7\delta).$$

(ii) Suppose *min* $\{ord_p 8a, ord_p \lambda_i b\} = ord_p a$.
$ord_p\ y = \frac{1}{7}\left[ord_p\ \lambda_i t - ord_p a - ord_p c - 5ord_p b + 6ord_p a\right]$
Since $ord_p a > ord_p\ \lambda_i b$, then
$ord_p\ y \geq \frac{1}{7}\left[ord_p\ \lambda_i t - ord_p\ \lambda_i b - ord_p c - 5ord_p b + 6ord_p a\right]$
Thus, by using the similar method as (41) and (42), we have

$$ord_p\ y \geq \frac{1}{7}(\alpha - 8\delta) \quad \text{and} \quad ord_p y \geq \frac{1}{7}(\alpha - 7\delta).$$

Case (ii): $\{ord_p 8a = ord_p \lambda_i b\}$
Let $\theta = ord_p 8a = ord_p \lambda_i b$.
Therefore, by substituting (32) and (33) into (28), we obtain
$ord_p\ y \geq \frac{1}{7}\left(ord_p \lambda_i t - ord_p\left(p^{\theta}d_1 + p^{\theta}d_2\right) - ord_p cb^5 + ord_p a^6\right)$
$= \frac{1}{7}\left(ord_p \lambda_i t - ord_p \lambda_i b - ord_p(d_1 + d_2) - ord_p c - 5ord_p b + 6ord_p a\right),$
where $ord_p \lambda_i b = \theta$.
Let $\varepsilon_1 = ord_p(d_1 + d_2)$. Therefore,
$ord_p\ y \geq \frac{1}{7}\left(ord_p \lambda_i t - ord_p \lambda_i b - \varepsilon_1 - ord_p c - 5ord_p b + 6ord_p a\right)$
We solve the above equation by applying the similar method as (40), thus we obtain

$$ord_p\ y \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1) \quad \text{and} \quad ord_p y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1).$$

From equation (29), by using the same process as equation (28) which is not involving ε_0, we obtain

$$ord_p y \geq \frac{1}{7}(\alpha - 8\delta - 5\varepsilon_0) \quad \text{or}$$
$$ord_p y \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1 - 5\varepsilon_0) \quad \text{or}$$
$$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0) \quad \text{or}$$
$$ord_p y \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$$

where $\varepsilon_0 = ord_p(m+n)$ and $\varepsilon_1 = ord_p(d_1 + d_2)$ for some $\varepsilon_0, \varepsilon_1 \geq 0$.
From equation (30), we have
Case (i): $\{ord_p 8a \neq ord_p \lambda_i b\}$

(i) Suppose *min* $\{ord_p 8a, ord_p \lambda_i b\} = ord_p\ \lambda_i b$.

$$ord_p\ y \geq \frac{1}{7}\left[ord_p\ \lambda_i t - ord_p\ \lambda_i b - 7ord_p c + 7ord_p b\right]$$
$$= \frac{1}{7}\left[ord_p t + 6ord_p b - 7ord_p c\right].$$

Therefore,

$$ord_p\ y \geq \frac{1}{7}\left[ord_p t - 7ord_p c\right]. \qquad (43)$$

Since $\delta = max\{ord_p a, ord_p b, ord_p c\}$,
$ord_p s, ord_p t \geq \alpha > 8\delta$, thus

$$ord_p\, y \ge \frac{1}{7}(\alpha - 7\delta).$$

(ii) Suppose $min\{ord_p 8a, ord_p \lambda_i b\} = ord_p a$.

$$ord_p\, y = \frac{1}{7}[ord_p\, \lambda_i t - ord_p a - 7ord_p c + 7ord_p b]$$

Since $ord_p a > ord_p\, \lambda_i b$, then

$$ord_p\, y \ge \frac{1}{7}[ord_p\, \lambda_i t - ord_p\, \lambda_i b - ord_p c^7 + ord_p b^7]$$

Thus, by using the similar method as (43), we have

$$ord_p\, y \ge \frac{1}{7}(\alpha - 7\delta).$$

Case (ii): $\{ord_p 8a = ord_p \lambda_i b\}$

Let $\theta = ord_p 8a = ord_p \lambda_i b$.

By substituting (32) and (33) into (30), we obtain

$$ord_p\, y \ge \frac{1}{7}\left(ord_p \lambda_i t - ord_p\left(p^\theta d_1 + p^\theta d_2\right) - ord_p c^7 + ord_p b^7\right)$$

$$= \frac{1}{7}\left(ord_p \lambda_i t - ord_p \lambda_i b - ord_p(d_1 + d_2) - 7\,ord_p c + 7ord_p b\right)$$

where $ord_p \lambda_i b = \theta$.

Let $\varepsilon_1 = ord_p(d_1 + d_2)$. Therefore,

$$ord_p\, y \ge \frac{1}{7}\left(ord_p \lambda_i t - ord_p \lambda_i b - \varepsilon_1 - 7ord_p c + 7ord_p b\right).$$

We solve the above equation by applying the similar method as (43), thus we obtain

$$ord_p\, y \ge \frac{1}{7}(\alpha - 7\delta - \varepsilon_1).$$

From equation (31), by using the same process as equation (30) which not involving ε_0, we obtain

$$ord_p y \ge \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0) \quad \text{or}$$

$$ord_p y \ge \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$$

where $\varepsilon_0 = ord_p(m + n)$ and $\varepsilon_1 = ord_p(d_1 + d_2)$ for some $\varepsilon_0, \varepsilon_1 \ge 0$ as asserted. □

Proof of Theorem 2

Let $g = f_x$ and $h = f_y$ and λ be a constant. Then

$$(g + \lambda h)(x, y) =$$

$$(8a + \lambda b)x^7 + (7b + 2\lambda c)x^6 y + 6cx^5y^2 + (s + \lambda t).$$

That is,

$$\frac{(g+\lambda h)(x,y)}{8a+\lambda b} = x^7 + \left(\frac{7b+2\lambda c}{8a+\lambda b}\right) x^4 y + \left(\frac{6c}{8a+\lambda b}\right) x^3 y^2 + \frac{s+\lambda t}{8a+\lambda b}.$$

By completing the square the above equation, we obtain

$$\frac{(g+\lambda h)(x,y)}{8a+\lambda b} = \left(x^{\frac{7}{2}} + \frac{7b+2\lambda c}{2(8a+\lambda b)} x^{\frac{5}{2}} y\right)^2 + \left(\frac{s+\lambda t}{8a+\lambda b}\right) \qquad (44)$$

if

$$\frac{6c}{8a+\lambda b} - \left(\frac{7b+2\lambda c}{2(8a+\lambda b)}\right)^2 = 0. \qquad (45)$$

By solving equation (45), we obtain

$$4c^2\lambda^2 + 4bc\lambda + 49b^2 - 192ac = 0.$$

Thus,

$$\lambda_1 = \frac{-b+\sqrt{192ac-48b^2}}{2c} \quad \text{and} \quad \lambda_2 = \frac{-b-\sqrt{192ac-48b^2}}{2c}$$

where λ_1, λ_2 be the zeros of the equation (45) whose expressions are given in Lemma 1. $\lambda_1 \neq \lambda_2$, since $ord_p b^2 > ord_p ac$ and $ord_p b^2 < ord_p ac$ implies $b^2 \neq ac$.

Now, let

$$U = x^{\frac{7}{2}} + \frac{7b + 2\lambda_1 c}{2(8a+\lambda_1 b)} x^{\frac{5}{2}} y \qquad (46)$$

$$V = x^{\frac{7}{2}} + \frac{7b + 2\lambda_2 c}{2(8a+\lambda_2 b)} x^{\frac{5}{2}} y \qquad (47)$$

$$F(U,V) = (g + \lambda_1 h)(x, y) \qquad (48)$$

$$G(U,V) = (g + \lambda_2 h)(x, y). \qquad (49)$$

Substitution of U and V in (44), for i = 1, 2, we have

$$(g + \lambda h)(x, y) =$$

$$\left(x^{\frac{7}{2}} + \frac{7b+2\lambda_i c}{2(8a+\lambda_i b)} x^{\frac{5}{2}} y\right)^2 (8a + \lambda_i b) + \frac{s+\lambda_i t}{8a+\lambda_i b}(8a + \lambda_i b)$$

gives the following polynomials in (U, V),

$$F(U,V) = (8a + \lambda_1 b)U^2 + (s + \lambda_1 t) \qquad (50)$$

$$G(U,V) = (8a + \lambda_2 b)V^2 + (s + \lambda_2 t). \qquad (51)$$

The combination of the indicator diagrams associated with the Newton polyhedron of (50) and (51) takes the form shown in Figure 1.

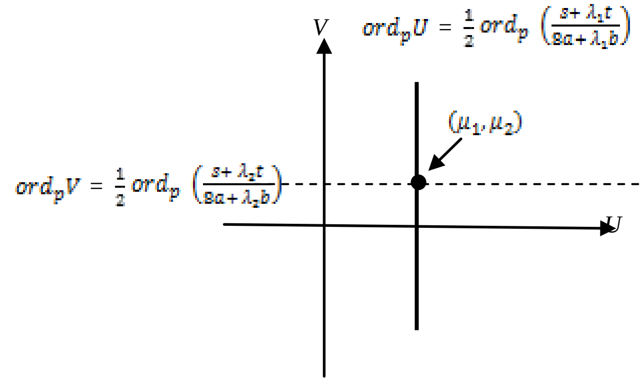

Fig 1. The indicator diagrams of
$\boldsymbol{F(U,V) = (8a + \lambda_1 b)U^2 + s + \lambda_1 t}$ (bold line) and
$\boldsymbol{G(U,V) = (8a + \lambda_2 b)V^2 + s + \lambda_2 t}$ (broken line) with $p > 7$.

From Fig. 1 and by Theorem 1, there exists $(\hat{U}, \hat{V})$ in Ω_p^2 such that $F(\hat{U}, \hat{V}) = 0$, $G(\hat{U}, \hat{V}) = 0$ and $ord_p \hat{U} = \mu_1$, $ord_p \hat{V} = \mu_2$ with $\mu_1 = \frac{1}{2} ord_p \left(\frac{s+\lambda_1 t}{8a+\lambda_1 b}\right)$ and also $\mu_2 = \frac{1}{2} ord_p \left(\frac{s+\lambda_2 t}{8a+\lambda_2 b}\right)$. Let $U = \hat{U}$ and $V = \hat{V}$ in (46) and (47). Thus, there exists (x_0, y_0) in Ω_p^2 such that

$$\hat{U} = x_0^{\frac{7}{2}} + \alpha_1 x_0^{\frac{5}{2}} y_0 \qquad (52)$$

$$\hat{V} = x_0^{\frac{7}{2}} + \alpha_2 x_0^{\frac{5}{2}} y_0. \qquad (53)$$

By multiplying α_2 to (52) and α_1 to (53), we have

$$\alpha_2 \hat{U} = \alpha_2 x_0^{\frac{7}{2}} + \alpha_1 \alpha_2 x_0^{\frac{5}{2}} y_0 \qquad (54)$$

$$\alpha_1 \hat{V} = \alpha_1 x_0^{\frac{7}{2}} + \alpha_1 \alpha_2 x_0^{\frac{5}{2}} y_0. \qquad (55)$$

Thus, by subtracting (55) and (54), we have

$$x_0 = \left(\frac{\alpha_1 \hat{V} - \alpha_2 \hat{U}}{\alpha_1 - \alpha_2}\right)^{\frac{2}{7}}.$$

Therefore,

$$ord_p x_0 = \frac{2}{7}\left[ord_p\left(\alpha_1 \hat{V} - \alpha_2 \hat{U}\right) - ord_p(\alpha_1 - \alpha_2)\right].$$

Then, by subtracting (52) and (53), we have

$$y_0 = \frac{\hat{U} - \hat{V}}{(\alpha_1 - \alpha_2) x_0^{\frac{5}{2}}}.$$

Therefore,

$$ord_p y_0 = ord_p\left(\hat{U} - \hat{V}\right) - ord_p(\alpha_1 - \alpha_2) - \frac{5}{2} ord_p x_0.$$

Since $ord_p U = \frac{1}{2} ord_p \left(\frac{s+\lambda_1 t}{8a+\lambda_1 b}\right)$ and $ord_p V = \frac{1}{2} ord_p \left(\frac{s+\lambda_2 t}{8a+\lambda_2 b}\right)$, then from Lemma 4, we let $x_0 = \xi$ and $y_0 = \eta$. Since $F(\hat{U},\hat{V}) = 0$ and $G(\hat{U},\hat{V}) = 0$, by back substitution in (50) and (51) we would have $g(\xi\ \eta) = f_x(\xi, \eta) = 0$ and $h(\xi, \eta) = f_y(\xi, \eta) = 0$. Thus,

$ord_p\xi \geq \frac{1}{7}(\alpha - \delta)$ and	$ord_p\xi \geq \frac{1}{7}(\alpha - \delta - \varepsilon_1)$ and
$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - 5\varepsilon_0)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 8\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - 5\varepsilon_0)$ or	$ord_p\eta \geq \frac{1}{7}(\alpha - 7\delta - \varepsilon_1 - 5\varepsilon_0)$ or
$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - 5\varepsilon_0)$	$ord_p\eta \geq \frac{1}{7}(\alpha - 6\delta - \varepsilon_1 - 5\varepsilon_0)$.

where (ξ, η) is a common zero of f_x and f_y and

$\delta = max\{ord_p a, ord_p b, ord_p c\}$ for some $\varepsilon_0, \varepsilon_1 \geq 0$. □

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