Infinite Series Forms of the Derivatives of Two Types of Functions

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***Abstract*—**This article takes the mathematical software Maple as the auxiliary tool to study the differential problem of two types of functions. We can obtain the infinite series forms of any order derivatives of these two types of functions by using binomial series and differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order derivative values. On the other hand, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying our answers by using Maple.

***Keywords*—**derivatives, infinite series forms, binomial series, differentiation term by term theorem, Maple.

I. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics courses, determining the n -th order derivative value $f^{(n)}(c)$ of a function $f(x)$ at $x=c$, in general, needs to go through two procedures: firstly finding the n -th order derivative $f^{(n)}(x)$ of $f(x)$, and then taking $x=c$ into $f^{(n)}(x)$. These two procedures will make us face with increasingly complex calculations when calculating higher order derivative values of this function (i.e. n is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we mainly study the differential problem of the following two types of trigonometric functions

$$f(x)=\cosh^r(ax+b)\exp[\beta\cosh^s(ax+b)] \quad (1)$$

$$g(x)=\sinh^r(ax+b)\exp[\beta\sinh^s(ax+b)] \quad (2)$$

where r,s,β,a,b are real numbers. We can obtain the infinite series forms of any order derivatives of these two types of functions by using binomial series and differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of calculating their higher order derivative values. As for the study of related differential problems can refer to [1-21]. On the other hand, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

II. Main Results

Firstly, we introduce a formula used in this study.

A. Formula

Taylor series expansion of the exponential function

$e^y=\sum_{m=0}^{\infty}\frac{1}{m!}y^m$, where y is any real number.

Next, we introduce two important theorems used in this paper.

B. Theorems

Binomial series :

If x,r are real numbers, $|x|<1$, then

$$(1+x)^r=\sum_{k=0}^{\infty}\frac{(r)_k}{k!}x^k,$$

where $(r)_k=r(r-1)\cdots(r-k+1)$ for any positive integer k, and $(r)_0=1$.

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Differentiation term by term theorem ([22])

For all non-negative integers k, if the functions $g_k:(a,b)\to R$ satisfy the following three conditions : (i) there exists a point $x_0\in(a,b)$ such that $\sum_{k=0}^{\infty}g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty}g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty}g_k(x)=\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

Before deriving our first major result, we need a lemma.

Lemma 1 Suppose a,b,p are real numbers.

Case (1) If $ax+b>0$, then

$$\cosh^{p}(ax+b)=\frac{1}{2^{p}}\cdot\sum_{k=0}^{\infty}\frac{(p)_k}{k!}e^{(p-2k)(ax+b)} \tag{3}$$

Case (2) If $ax+b<0$, then

$$\cosh^{p}(ax+b)=\frac{1}{2^{p}}\cdot\sum_{k=0}^{\infty}\frac{(p)_k}{k!}e^{-(p-2k)(ax+b)} \tag{4}$$

Proof *Case (1)* If $ax+b>0$,

$$\cosh^{p}(ax+b)$$

$$=\left[\frac{1}{2}[e^{(ax+b)}+e^{-(ax+b)}]\right]^{p}$$

$$=\frac{1}{2^{p}}\cdot e^{p(ax+b)}\left[1+e^{-2(ax+b)}\right]^{p}$$

$$=\frac{1}{2^{p}}\cdot e^{p(ax+b)}\sum_{k=0}^{\infty}\frac{(p)_k}{k!}e^{-2k(ax+b)}$$

(By binomial series)

$$=\frac{1}{2^{p}}\cdot\sum_{k=0}^{\infty}\frac{(p)_k}{k!}e^{(p-2k)(ax+b)}$$

Case (2) If $ax+b<0$,

$$\cosh^{p}(ax+b)$$

$$=\cosh^{p}[-(ax+b)]$$

$$=\frac{1}{2^{p}}\cdot\sum_{k=0}^{\infty}\frac{(p)_k}{k!}e^{-(p-2k)(ax+b)}$$ ■

The following is the first major result of this study, we obtain the infinite series forms of any order derivatives of the function (1).

Theorem 1 Suppose r,s,β,a,b are real numbers, and n is a positive integer. Let the domain of the function

$$f(x)=\cosh^{r}(ax+b)\exp[\beta\cosh^{s}(ax+b)]$$

be $\{x\in R|ax+b\neq 0\}$.

Case (1) If $ax+b>0$, then the n-th order derivative of $f(x)$,

$$f^{(n)}(x)$$

$$=a^{n}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^{m}(r+sm-2k)^{n}}{m!k!2^{r+sm}}e^{(r+sm-2k)(ax+b)} \tag{5}$$

Case (2) If $ax+b<0$, then

$$f^{(n)}(x)$$

$$=(-a)^{n}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^{m}(r+sm-2k)^{n}}{m!k!2^{r+sm}}e^{-(r+sm-2k)(ax+b)} \tag{6}$$

Proof *Case (1)* If $ax+b>0$, because

$$f(x)$$

$$=\cosh^{r}(ax+b)\exp[\beta\cosh^{s}(ax+b)]$$

$$=\cosh^{r}(ax+b)\cdot\sum_{m=0}^{\infty}\frac{1}{m!}[\beta\cosh^{s}(ax+b)]^{m}$$

(By Formula)

$$=\sum_{m=0}^{\infty}\frac{\beta^{m}}{m!}\cosh^{r+sm}(ax+b)$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^{m}}{m!k!2^{r+sm}}e^{(r+sm-2k)(ax+b)} \tag{7}$$

(Using (3))

By differentiation term by term theorem, differentiating n-times with respect to x on both sides of (7), we obtain

$$f^{(n)}(x)$$

$$=a^n\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^m(r+sm-2k)^n}{m!k!2^{r+sm}}e^{(r+sm-2k)(ax+b)}$$

Case (2) If $ax+b<0$, then

$f(x)$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^m}{m!k!2^{r+sm}}e^{-(r+sm-2k)(ax+b)} \quad (8)$$

(By (4))

Also, by differentiation term by term theorem, differentiating n-times with respect to x on both sides of (8), we have

$f^{(n)}(x)$

$$=(-a)^n\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(r+sm)_k\beta^m(r+sm-2k)^n}{m!k!2^{r+sm}}e^{-(r+sm-2k)(ax+b)}$$

■

To prove the second major result of this paper, we also need a lemma.

Lemma 2 Suppose a,b,p are real numbers.

Case (1) If $ax+b>0$, then

$$\sinh^p(ax+b)=\frac{1}{2^p}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!}e^{(p-2k)(ax+b)} \quad (9)$$

Case (2) If $ax+b<0$ and $(-1)^p$ exists,

$$\sinh^p(ax+b)=\frac{(-1)^p}{2^p}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!}e^{-(p-2k)(ax+b)} \quad (10)$$

Proof *Case (1)* If $ax+b>0$,

$\sinh^p(ax+b)$

$$=\left[\frac{1}{2}[e^{(ax+b)}-e^{-(ax+b)}]\right]^p$$

$$=\frac{1}{2^p}\cdot e^{p(ax+b)}\left[1-e^{-2(ax+b)}\right]^p$$

$$=\frac{1}{2^p}\cdot e^{p(ax+b)}\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!}e^{-2k(ax+b)}$$

$$=\frac{1}{2^p}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!}e^{(p-2k)(ax+b)}$$

Case (2) If $ax+b<0$ and $(-1)^p$ exists,

$\sinh^p(ax+b)$

$$=(-1)^p\sinh^p[-(ax+b)]$$

$$=\frac{(-1)^p}{2^p}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!}e^{-(p-2k)(ax+b)}$$ ■

Next, we determine the infinite series forms of any order derivatives of the function (2).

Theorem 2 If the assumptions are the same as Theorem 1. Let the domain of the function

$$g(x)=\sinh^r(ax+b)\exp[\beta\sinh^s(ax+b)]$$

be $\{x\in R|ax+b\neq 0\}$.

Case (1) If $ax+b>0$, then the n-th order derivative of $g(x)$,

$g^{(n)}(x)$

$$=a^n\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(r+sm)_k\beta^m(r+sm-2k)^n}{m!k!2^{r+sm}}e^{(r+sm-2k)(ax+b)} \quad (11)$$

Case (2) If $ax+b<0$ and $(-1)^r$, $(-1)^s$ exist, then

$g^{(n)}(x)$

$$=(-a)^n\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+sm+k}(r+sm)_k\beta^m(r+sm-2k)^n}{m!k!2^{r+sm}}e^{-(r+sm-2k)(ax+b)} \quad (12)$$

Proof *Case (1)* If $ax+b>0$, because

$g(x)$

$$=\sinh^r(ax+b)\cdot\sum_{m=0}^{\infty}\frac{1}{m!}[\beta\sinh^s(ax+b)]^m$$

$$=\sum_{m=0}^{\infty}\frac{\beta^m}{m!}\sinh^{r+sm}(ax+b)$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(r+sm)_k\beta^m}{m!k!2^{r+sm}}e^{(r+sm-2k)(ax+b)} \quad (13)$$

(Using (9))

By differentiation term by term theorem, differentiating

n -times with respect to x on both sides of (13), we obtain

$$g^{(n)}(x)$$

$$= a^n \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k (r+sm)_k \beta^m (r+sm-2k)^n}{m!k!2^{r+sm}} e^{(r+sm-2k)(ax+b)}$$

Case (2) If $ax+b<0$ and $(-1)^r$, $(-1)^s$ exist, because

$$g(x)$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+sm+k}(r+sm)_k \beta^m}{m!k!2^{r+sm}} e^{-(r+sm-2k)(ax+b)}$$

(By (10)) (14)

Using differentiation term by term theorem, differentiating n -times with respect to x on both sides of (14), we get

$$g^{(n)}(x)$$

$$= (-a)^n \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+sm+k}(r+sm)_k \beta^m (r+sm-2k)^n}{m!k!2^{r+sm}} e^{-(r+sm-2k)(ax+b)}$$

■

III. Examples

In the following, for the differential problem of the two types of functions in this study, we provide two examples and use Theorems 1, 2 to determine the infinite series forms of any order derivatives of these functions and evaluate some of their higher order derivative values. On the other hand, we employ Maple to calculate the approximations of these higher order derivative values and their solutions for verifying our answers.

Example 1 Suppose the domain of the function

$$f(x)=\cosh^{2/3}(2x-4)\exp[5\cosh^{4/7}(2x-4)] \quad (15)$$

is $\{x \in R | x \neq 2\}$.

Case (1) If $x>2$, then by (5), we obtain any n -th order derivative of $f(x)$,

$$f^{(n)}(x)$$

$$= 2^n \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2/3+4m/7)_k 5^m (2/3+4m/7-2k)^n}{m!k!2^{2/3+4m/7}} e^{(2/3+4m/7-2k)(2x-4)}$$

(16)

Thus, we can evaluate the 5-th order derivative value of $f(x)$ at $x=3$,

$$f^{(5)}(3)$$

$$= 32 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2/3+4m/7)_k 5^m (2/3+4m/7-2k)^5}{m!k!2^{2/3+4m/7}} e^{4/3+8m/7-4k}$$

(17)

Next, we use Maple to verify the correctness of (17).

```
>f:=x->(cosh(2*x-4))^(2/3)*exp(5*(cosh(2*x-4))^(4/7));
>evalf((D@@5)(f)(3),18);
```

$$7.97013490429966681 \cdot 10^{10}$$

```
>evalf(32*sum(sum(product(2/3+4*m/7-j,j=0..(k-1))*5^m*(2/3+4*m/7-2*k)^5/(m!*k!*2^(2/3+4*m/7))*exp(4/3+8*m/7-4*k),k=0..infinity),m=0..infinity),18);
```

$$7.97013490429966656 \cdot 10^{10}$$

Case (2) If $x<2$, using (6), we have

$$f^{(n)}(x)$$

$$= (-2)^n \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2/3+4m/7)_k 5^m (2/3+4m/7-2k)^n}{m!k!2^{2/3+4m/7}} e^{-(2/3+4m/7-2k)(2x-4)}$$

(18)

Therefore,

$$f^{(6)}(-1)$$

$$= 64 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2/3+4m/7)_k 5^m (2/3+4m/7-2k)^6}{m!k!2^{2/3+4m/7}} e^{(4+24m/7-12k)}$$

(19)

```
>evalf((D@@6)(f)(-1),18);
```

$$1.33216189686737323 \cdot 10^{59}$$

```
>evalf(64*sum(sum(product(2/3+4*m/7-j,j=0..(k-1))*5^m*(2/3+4*m/7-2*k)^6/(m!*k!*2^(2/3+4*m/7))*exp(4+24*m/7-12*k),k=0..infinity),m=0..infinity),18);
```

$$1.33216189686737370 \cdot 10^{59}$$

Example 2 Assume the domain of the function

$$g(x)=\sinh^{2/5}(3x+2)\exp[4\sinh^{2/7}(3x+2)] \quad (20)$$

is $\{x \in R | x \neq -2/3\}$.

Case (1) If $x>-2/3$, by (11), we obtain any n -th order derivative of $g(x)$,

$$g^{(n)}(x)$$

$$= 3^n \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k (2/5+2m/7)_k 4^m (2/5+2m/7-2k)^n}{m!k!2^{2/5+2m/7}} e^{(2/5+2m/7-2k)(3x+2)}$$

(21)

Hence,

$$g^{(4)}(2)$$

$$=81\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2/5+2m/7)_k 4^m(2/5+2m/7-2k)^4}{m!k!2^{2/5+2m/7}}e^{16/5+16m/7-16k} \quad (22)$$

>g:=x->((sinh(3*x+2))^2)^(1/5)*exp(4*((sinh(3*x+2))^2)^(1/7));

>evalf((D@@4)(g)(2),18);

$$1.56012980875727726\cdot 10^{21}$$

>evalf(81*sum(sum((-1)^k*product(2/5+2*m/7-j,j=0..(k-1))*4^m*(2/5+2*m/7-2*k)^4/(m!*k!*2^(2/5+2*m/7))*exp(16/5+16*m/7-16*k),k=0..infinity),m=0..infinity),18);

$$1.56012980875727719\cdot 10^{21}$$

Case (2) If $x<-2/3$, using (12), we obtain

$$g^{(n)}(x)$$

$$=(-3)^n\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2/5+2m/7)_k 4^m(2/5+2m/7-2k)^n}{m!k!2^{2/5+2m/7}}e^{-(2/5+2m/7-2k)(3x+2)} \quad (23)$$

Thus,

$$g^{(5)}(-3)$$

$$=-243\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2/5+2m/7)_k 4^m(2/5+2m/7-2k)^5}{m!k!2^{2/5+2m/7}}e^{14/5+14m/7-14k} \quad (24)$$

>evalf((D@@5)(g)(-3),18);

$$-3.04312591337212672\cdot 10^{18}$$

>evalf(-243*sum(sum((-1)^k*product(2/5+2*m/7-j,j=0..(k-1))*4^m*(2/5+2*m/7-2*k)^5/(m!*k!*2^(2/5+2*m/7))*exp(14/5+14*m/7-14*k),k=0..infinity),m=0..infinity),18);

$$-3.04312591337212665\cdot 10^{18}$$

IV. Conclusion

In this study, we provide a new technique to evaluate any order derivatives of some functions. We hope this technique can be applied to solve another differential problems. On the other hand, the binomial series and the differentiation term by term theorem play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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