Approximation to Hypergeometric Distribution with Modified Binomial Distributions

Juthaphorn Sinsomboonthong

Abstract—Two modified binomial approximations to a hypergeometric distribution—modified binomial distributions 2 and 3—are proposed and their accuracy is investigated in terms of the total variation distance. In addition, an efficiency comparison with a binomial approximation was conducted using a simulation study for 162 situations. It is found that the total variation distances of the two modified binomial approximations are less than that of a binomial approximation for almost all situations and tend to zero for a small sampling fraction whatever the levels of population size. Even for the large population size of 20,000, there seems to be no difference in the efficiencies of the two modified binomial approximations and the binomial approximation at all levels of the sampling fraction and the proportion of the population that has the specified attribute.

Index Terms—Hypergeometric distribution, sampling fraction, binomial distribution, accuracy

I. Introduction

A RANDOM sample from a finite population is one in which each member of the population can be classified as either having or not having a specified attribute. If sampling is with replacement, the process of observing whether or not each member selected has the specified attribute that constitutes a Bernoulli trial. Hence, for random sampling with replacement, the number of members contained that have a specified attribute is a binomial random variable. Specifically, the number of members obtained that have the specified attribute has a binomial distribution with parameters n and p where n is the sample size and p is the proportion of the population that has the specified attribute. However, if sampling is without replacement, the trials are not independent. Hence, the process of observing whether or not each member selected has the specified attribute does not constitute a Bernoulli trial. In particular, the number of members obtained that have the specified attribute has a hypergeometric distribution with parameters N, n and D where N is the population size, n is the sample size and D is the number of items that fall into a class of interest when $D \le N$ [1].

A family of hypergeometric random variables is closely related to a family of binomial random variables. Weiss [2] mentioned that if a sample size n is small relative to a population size N, the hypergeometric distribution with parameters N, n and D can be approximated by the binomial distribution. As a rule of thumb, the hypergeometric distribution can be adequately approximated by the binomial distribution, provided that the sample size does not exceed 5% of the population size [2]. Additionally, Montgomery [3] and Evans *et al*. [4] mentioned that if the sampling fraction $\left(f = \frac{n}{N}\right)$ is small—it is not greater than 0.1—then the binomial distribution with parameters $p = \frac{D}{N}$ and n is a good approximation to a hypergeometric distribution.

The hypergeometric distribution is particularly important in statistical quality control and the statistical estimation of population proportions for sampling survey theory [5], [6]. In certain quality control problems, it is sometimes useful to approximate a hypergeometric distribution with a binomial distribution or an asymptotic binomial distribution [7]. This is particularly helpful in situations where the original distribution is difficult to manipulate analytically. For example, this approximation is useful in the design of acceptance–sampling plans [3], [8], [9]. However, if the sampling fraction is greater than 0.1, then the approximation to a hypergeometric distribution with a binomial distribution is no better.

Therefore, this study proposes two modified binomial approximations to the hypergeometric distribution—modified binomial distributions 2 and 3—using an expansion of hypergeometric probabilities in term of Krawtchouk's polynomial [10]. These are good approximations to a hypergeometric distribution whatever the sampling fraction.

II. Materials and Methods

This study proposes two modified binomial approximations to the hypergeometric distribution—modified binomial distributions 2 and 3—using an expansion of hypergeometric probabilities in term of Krawtchouk's polynomial. An accuracy comparison of the two modified binomial approximations in terms of the total variation distance [11] with a binomial approximation was empirically performed using a simulation study.

A. *Distributions of Random Variable*

Consider a finite population of size N in which each member is classified as either having or not having a specified attribute. Let D be the number in the population having the specified attribute $(D \le N)$, then $N - D$ corresponds to the number in the population not having the

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specified attribute. These numbers are not known to us in advance. A random sample of size n, $n \le N$, is taken without replacement from the population. The sample contains X elements that have the specified attribute. Then X is called a hypergeometric random variable and is said to have a hypergeometric distribution with parameters N, n and D. Additionally, the probability mass function of X is

$$h(x;n,D,N) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

where $\mathrm{Max}\{0,\ n-(N-D)\} \le x \le \mathrm{Min}\{n, D\}$, $n \in \mathbb{Z}^+$, $N \in \mathbb{Z}^+$ and $D \le N$ (see [1], [12]).

However, if a random sample of size n is taken with replacement from a finite population of size N where each element in the population has an equal and independent probability p of having a specified attribute, then X is called a binomial random variable and is said to have a binomial distribution with parameters n and p. The probability mass function of X is

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

where $p+q=1$, $p>0$, $q>0$, $n \in \mathbb{Z}^+$, and $x = 0, 1, 2, \ldots, n$ (see [1], [4], [12]).

B. *Modified Binomial Approximation to Hypergeometric Distribution*

This section proposes two modified binomial approximations to the hypergeometric distribution which are called modified binomial distribution 2 and modified binomial distribution 3. A procedure to derive the proposed distributions is as follows:

Let $\alpha(x)$ be a step function with the jump, at the point x, of

$$j(x) = \binom{n}{x} p^x q^{n-x}$$

where $p+q=1$, $p>0$, $q>0$, $n \in \mathbb{Z}^+$, and $x = 0, 1, 2, \ldots, n$. Then Krawtchouk' s polynomial is defined by

$$k_m(x;n,p) = \sum_{j=0}^{m} \binom{x}{j}\binom{n-x}{m-j}(-p)^{m-j} q^j \quad \text{for } m = 0,1,\ldots,n$$

(see [13], [14]).

For any integers x, N, D, and n satisfying the four conditions as follows:

1) $\mathrm{Max}\{0,\ n-(N-D)\} \le x \le \mathrm{Min}\{n, D\}$
2) $1 \le n \le N$
3) $D \le N$
4) $0<p<1$, $0<q<1$ and $p+q=1$, an expansion of the hypergeometric probability in term of Krawtchouk's polynomial is given by

$$h(x;n,D,N) = b(x;n,p)\sum_{m=0}^{n} \binom{N}{m}^{-1} (pq)^{-m} k_m(x;n,p) k_m(D;N,p).$$

An approximation of the relationship between the hypergeometric and the binomial probabilities is given by

$$\begin{aligned} b_r(x;n,p) &= b(x;n,p)\sum_{m=0}^{r} \binom{N}{m}^{-1} (pq)^{-m} k_m(x;n,p) k_m(D;N,p) \\ &= b(x;n,p)\sum_{m=0}^{r} \binom{N}{m}^{-1} (pq)^{-m} k_m(D;N,p)\, k_m(x;n,p) \\ &= b(x;n,p)\sum_{m=0}^{r} \delta_m\, k_m(x;n,p) \qquad (1) \end{aligned}$$

where $r = 0,1,\ldots,n$

$$\delta_m = \binom{N}{m}^{-1} (pq)^{-m} k_m(D;N,p) \quad \text{for } m = 0,1,\ldots,n$$

$$\delta_{m+1} = -\frac{m}{(N-m)pq}\left[(q-p)\delta_m + \delta_{m-1}\right] \quad \text{for } m = 1,\ldots,n-1$$

$$p = \frac{D}{N}$$

$$q = 1-p$$

$k_m(x;n,p)$ and $k_m(D;N,p)$ are Krawtchouk's polynomials [10].

For r = 0, 1, 2, 3, the $k_r(x;n,p)$, $k_r(D;N,p)$ and δ_r are as follows:

$$k_0(x;n,p) = \sum_{j=0}^{0}\binom{x}{j}\binom{n-x}{0-j}(-p)^{0-j}q^j = 1$$

$$k_1(x;n,p) = \sum_{j=0}^{1}\binom{x}{j}\binom{n-x}{1-j}(-p)^{1-j}q^j = x - np$$

$$\begin{aligned} k_2(x;n,p) &= \sum_{j=0}^{2}\binom{x}{j}\binom{n-x}{2-j}(-p)^{2-j}q^j \\ &= \frac{1}{2}\left[\begin{matrix} x(x-1)q^2 - 2x(n-x)pq \\ +(n-x)(n-x-1)p^2 \end{matrix}\right] \end{aligned}$$

$$\begin{aligned} k_3(x;n,p) &= \sum_{j=0}^{3}\binom{x}{j}\binom{n-x}{3-j}(-p)^{3-j}q^j \\ &= \frac{1}{6}\left[\begin{matrix} x(x-1)(x-2)q^3 \\ -3x(x-1)(n-x)pq^2 \\ +3x(n-x)(n-x-1)p^2q \\ -(n-x)(n-x-1)(n-x-2)p^3 \end{matrix}\right] \end{aligned}$$

$$k_0(D;N,p) = \sum_{j=0}^{0}\binom{D}{j}\binom{N-D}{0-j}(-p)^{0-j}q^j = 1$$

$$k_1(D;N,p) = \sum_{j=0}^{1}\binom{D}{j}\binom{N-D}{1-j}(-p)^{1-j}q^j = Dq-(N-D)p$$

$$k_2(D;N,p) = \sum_{j=0}^{2}\binom{D}{j}\binom{N-D}{2-j}(-p)^{2-j}q^j = \frac{1}{2}\begin{bmatrix} D(D-1)q^2 - 2D(N-D)pq \\ + (N-D)(N-D-1)p^2 \end{bmatrix}$$

$$k_3(D;N,p) = \sum_{j=0}^{3}\binom{D}{j}\binom{N-D}{3-j}(-p)^{3-j}q^j = \frac{1}{6}\begin{bmatrix} D(D-1)(D-2)q^3 \\ - 3D(D-1)(N-D)pq^2 \\ + 3D(N-D)(N-D-1)p^2q \\ -(N-D)(N-D-1)(N-D-2)p^3 \end{bmatrix}$$

$$\delta_0 = \binom{N}{0}^{-1}(pq)^{-0}k_0(D;N,p) = 1$$

$$\delta_1 = \binom{N}{1}^{-1}(pq)^{-1}k_1(D;N,p) = 0$$

$$\delta_2 = \delta_{1+1} = -\frac{1}{(N-1)pq}\left[(q-p)\delta_1+\delta_{1-1}\right] = -\frac{1}{(N-1)pq}$$

$$\delta_3 = \delta_{2+1} = -\frac{2}{(N-2)pq}\left[(q-p)\delta_2+\delta_{2-1}\right] = \frac{2(q-p)}{(N-2)(N-1)(pq)^2}.$$

If $r = 0$, then $b_0(x;n,p) = b(x;n,p)\sum_{m=0}^{0}\delta_m\, k_m(x;n,p) = b(x;n,p)$ which is the binomial approximation to the hypergeometric distribution.

If $r = 1$, then $b_1(x;n,p) = b(x;n,p)\sum_{m=0}^{1}\delta_m\, k_m(x;n,p) = b(x;n,p)$ which is the binomial approximation to the hypergeometric distribution.

For $r = 2$, a modified binomial distribution 2 has a probability distribution in the form of

$$b_2(x;n,p) = \eta_2 \times b(x;n,p) \text{ for } x = 0, 1, 2, \ldots, n \quad (2)$$

where $b(x;n,p)$ is the probability mass function of a binomial random variable

$$\eta_2 = 1 - \frac{1}{2(N-1)pq}A$$

$$A = x(x-1)q^2 - 2xx'pq + x'(x'-1)p^2$$

$$x' = n-x$$

$p = \dfrac{D}{N}$ and $q = 1-p$.

For $r = 3$, a modified binomial distribution 3 has a probability distribution in the form of

$$b_3(x;n,p) = \eta_3 \times b(x;n,p) \text{ for } x = 0, 1, 2, \ldots, n \quad (3)$$

where $b(x;n,p)$ is the probability mass function of a binomial random variable

$$\eta_3 = 1 - \frac{1}{2(N-1)pq}A - \frac{(p-q)}{3(N-2)(N-1)p^2q^2}B$$

$$A = x(x-1)q^2 - 2xx'pq + x'(x'-1)p^2$$

$$B = x(x-1)(x-2)q^3 - 3x(x-1)x'pq^2 + 3xx'(x'-1)p^2q - x'(x'-1)(x'-2)p^3$$

$$x' = n-x$$

$p = \dfrac{D}{N}$ and $q = 1-p$.

The probability functions $b_2(x;n,p)$ and $b_3(x;n,p)$ as shown in equations (2) and (3) are called the modified binomial probability functions 2 and 3, respectively.

III. Simulation Results

A simulation study was conducted to empirically evaluate the validity and reliability of the two modified binomials and the binomial approximation to the hypergeometric distribution. In the study, finite populations of size N = 100, 500 and 20,000 were generated in the form of a hypergeometric distribution with the sampling fractions $\left(f = \frac{n}{N}\right)$ at 0.01, 0.02, 0.06, 0.1, 0.2 and 0.5. Moreover, proportions having a specified attribute $\left(p = \frac{D}{N}\right)$ were studied at 0.02, 0.04, 0.06, 0.08, 0.1, 0.2, 0.4, 0.6 and 0.8. Thus a total of 162 situations were created for the simulation study. Then, the total variation distances were compared empirically among the modified binomial 2, the modified binomial 3 and the binomial approximations to the hypergeometric distribution.

Let $d(b,h)$ be a total variation distance between the binomial and the hypergeometric distributions,

$d(b_2,h)$ be a total variation distance between the modified binomial distribution 2 and the hypergeometric distribution,

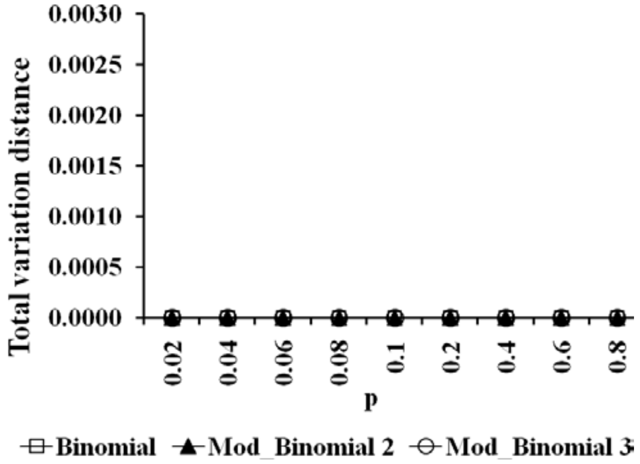
$d(b_3,h)$ be a total variation distance between the modified binomial distribution 3 and the hypergeometric distribution.

The simulation results in Fig. 1 to 6 reveal the total variation distances of the binomial, the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution. In Fig. 1(a), when the population size and the sampling fraction are 100 and 0.01 respectively, the total variation distances of the three estimations seem to be the same at all levels of p. When the sampling fraction is greater than 0.01, Fig 2(a) to 6(a) indicate that $d(b_2,h)$ and $d(b_3,h)$ are less than $d(b,h)$ for the population size of 100 and all levels of p. Additionally, the total variation distances of the three estimations seem to increase whenever the sampling fraction increases.

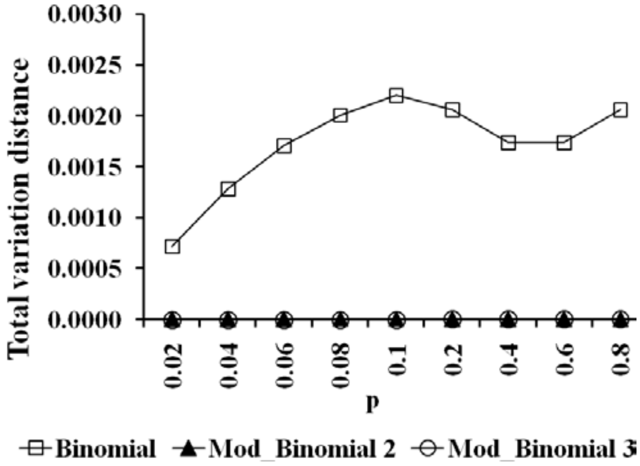
Fig. 1(b) to 6(b) indicate that $d(b_2,h)$ and $d(b_3,h)$ are less than $d(b,h)$ for population size of 500 at all levels of sampling fraction and p. The total variation distances of the three estimations tend to increase whenever the sampling fraction increases. For a large population size of N = 20,000, Fig. 1(c) to 6(c) indicate that the total variation distances of the three estimations seem to be slightly different at all levels of the sampling fraction and p.

Further, Fig. 1 to 4 indicate that the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution are not greater than 0.005 when the sampling fraction is not greater than 0.1 for population sizes of N = 100, 500 and 20,000 whatever the values of p. In addition, the total variation distances of the binomial approximation compared to the hypergeometric distribution are not greater than 0.025 when the sampling fraction is not greater than 0.1 for population sizes of N = 100 and 500 whatever the values of p. Even for the large population size of N = 20,000, the total variation distances of the binomial approximation compared to the hypergeometric distribution are small (less than 0.005) when the sampling fraction is not greater than 0.1 whatever the values of p. When the sampling fraction is greater than 0.1, Fig. 5 and 6 indicate that the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution are not greater than 0.05 for population sizes of N = 100, 500 and 20,000.

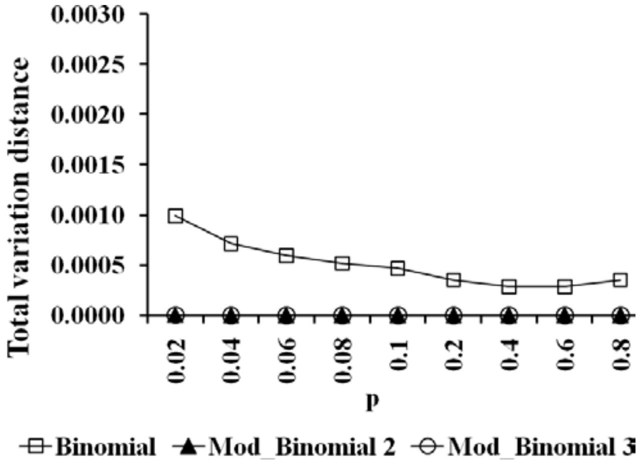
In addition, the total variation distances of the binomial approximation to the hypergeometric distribution seem to be large for population sizes of N = 100, 500 at a small value of p and the sampling fraction is greater than 0.1. Further, the $d(b_2,h)$ and $d(b_3,h)$ seem to be the same for almost all situations of the study, especially a large population size whatever the values of the sampling fraction and p.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.01 for N=100.

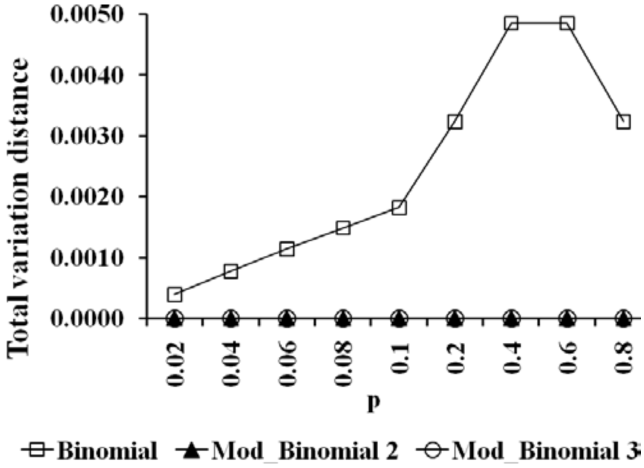


(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.01 for N=500.

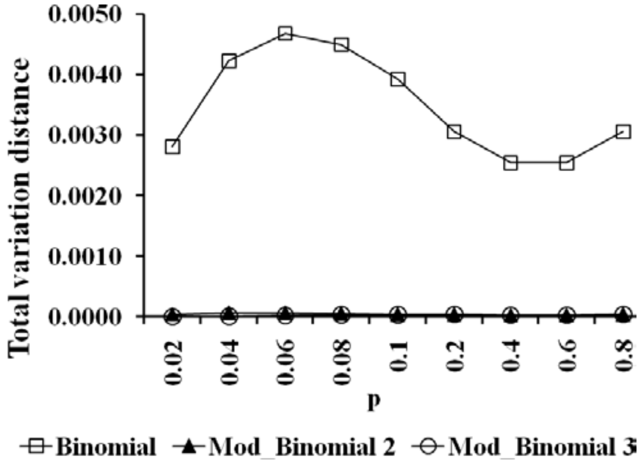


(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.01 for N=20,000.

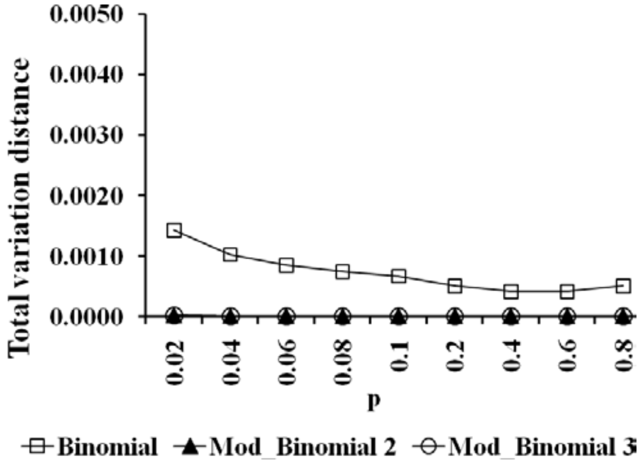
Fig. 1. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.01 for different values of the population size of N.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.02 for N=100.

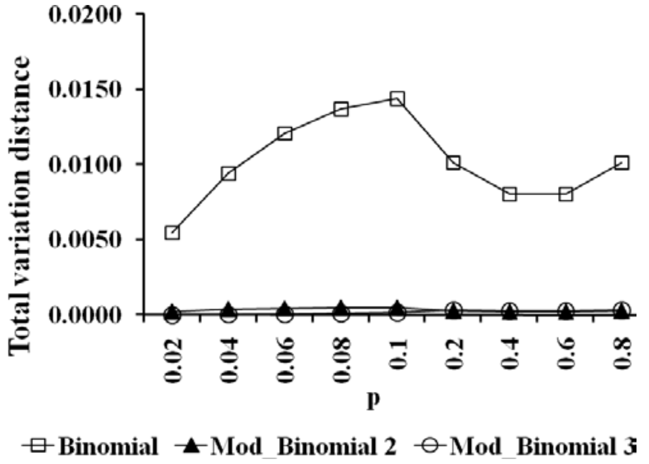


(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.02 for N=500.

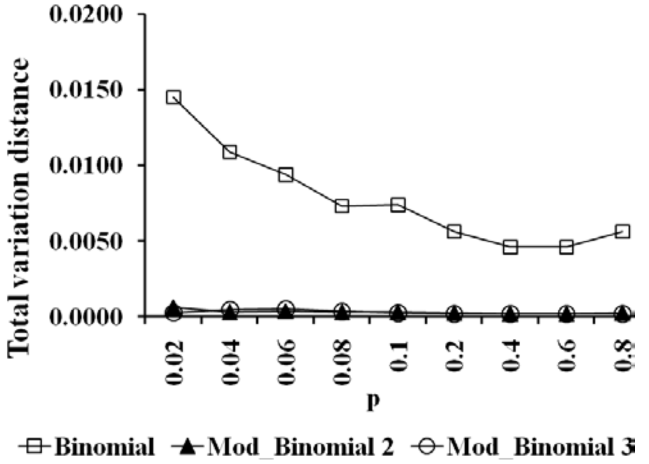


(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.02 for N=20,000.

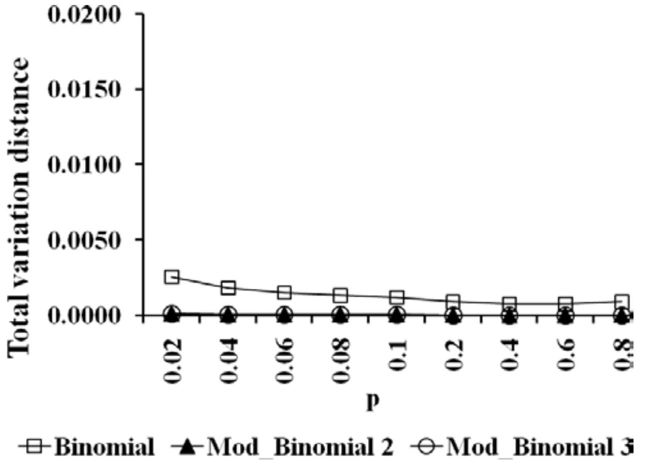
Fig. 2. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.02 for different values of the population size of N.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.06 for N=100.

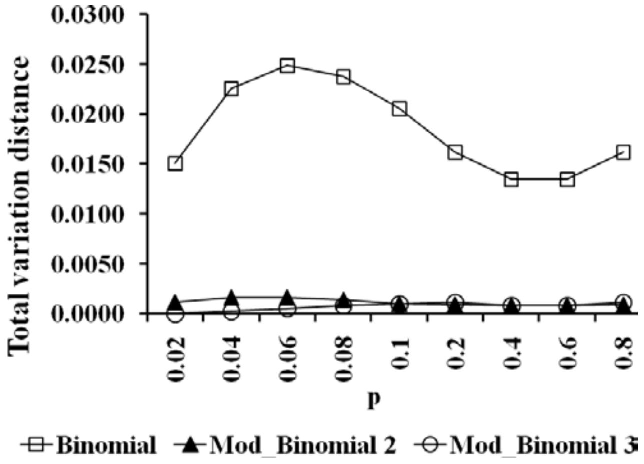


(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.06 for N=500.

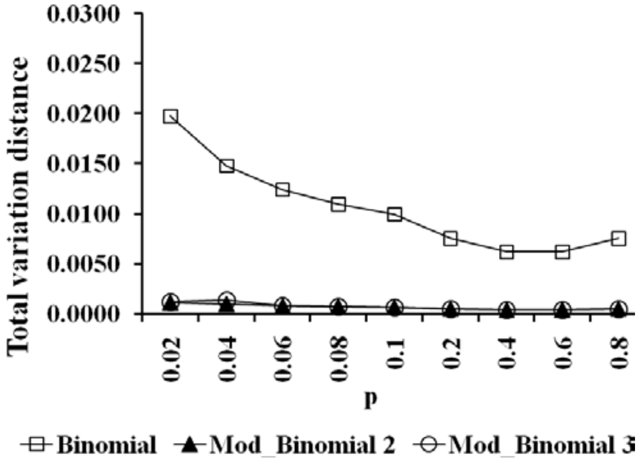


(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.06 for N=20,000.

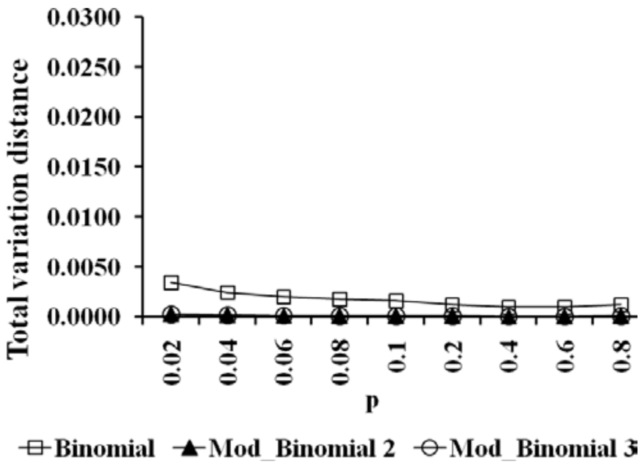
Fig. 3. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.06 for different values of the population size of N.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.1 for N=100.

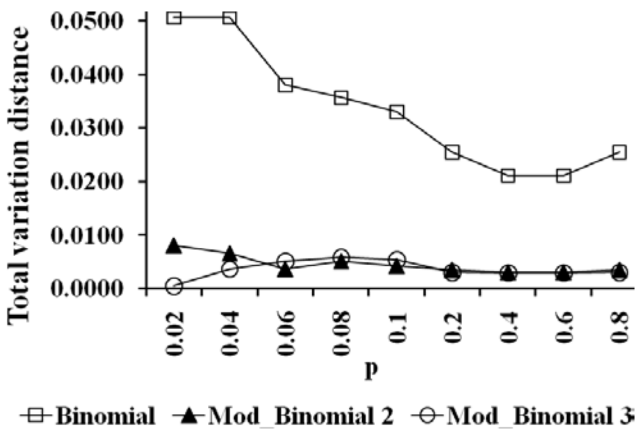


(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.1 for N=500.

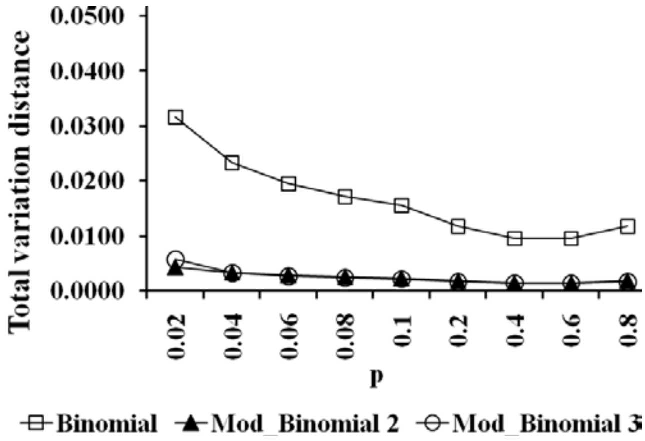


(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.1 for N=20,000.

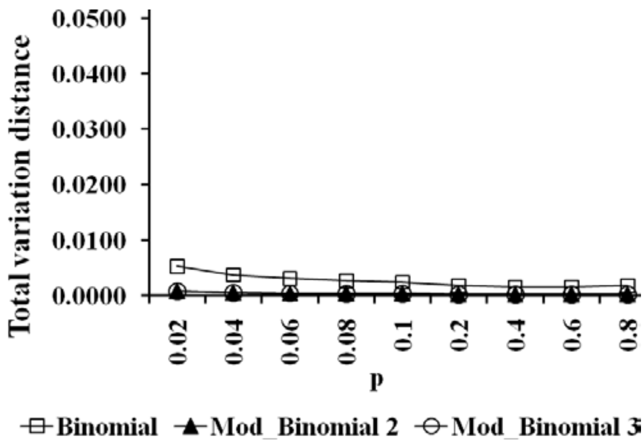
Fig. 4. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.1 for different values of the population size of N.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.2 for N=100.

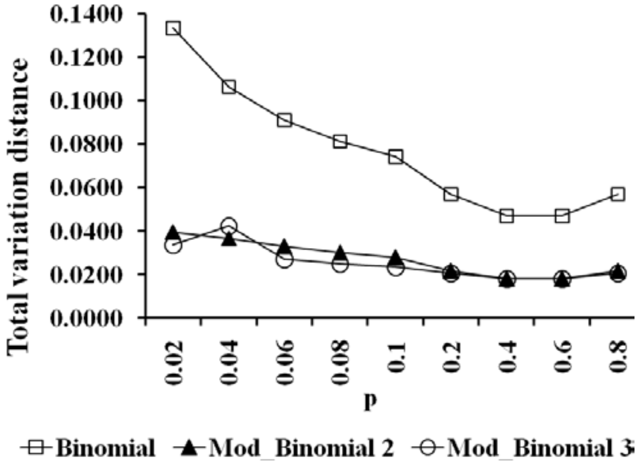


(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.2 for N=500.

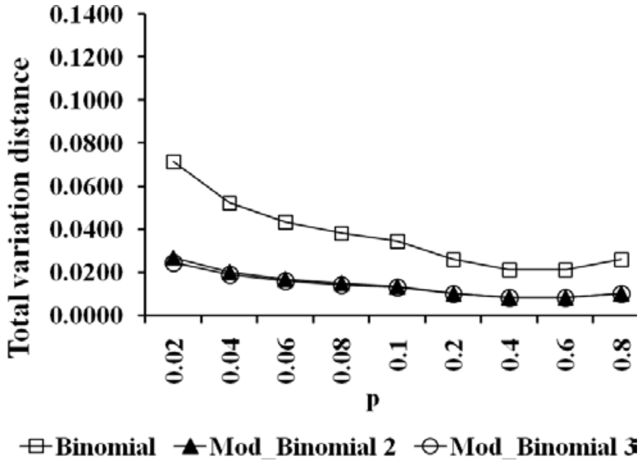


(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.2 for N=20,000.

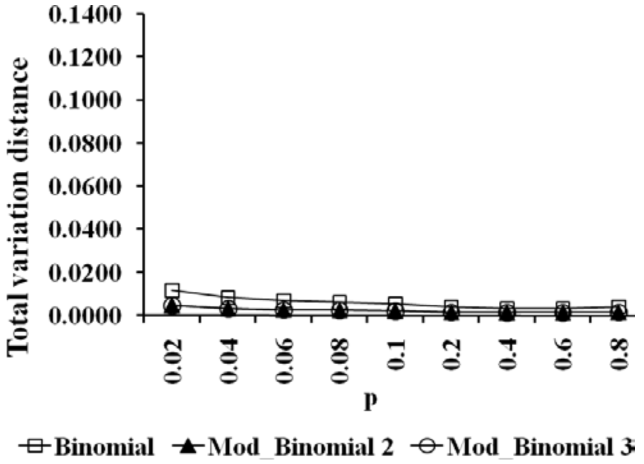
Fig. 5. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.2 for different values of the population size of N.



(a) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.5 for N=100.



(b) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.5 for N=500.



(c) Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when f = 0.5 for N=20,000.

Fig. 6. Total Variation Distances $d(b,h)$, $d(b_2,h)$ and $d(b_3,h)$ when the sampling fraction of f = 0.5 for different values of the population size of N.

Additionally, Fig. 1 to 6 indicate that the total variation distances of the three estimations tend to decrease whenever the population size increases whatever the levels of sampling fraction and p.

Further, Fig. 6(a) indicates that the total variation distance of the binomial approximation to the hypergeometric distribution approximates 0.14 when the sampling fraction of f = 0.5 for population sizes of N = 100 and a small value of p = 0.02. Even for the same situation, the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution approximate 0.04 which is smaller than that of a binomial approximation.

IV. Discussion

The total variation distances of the binomial approximations to the hypergeometric distribution are not greater than 0.025 when the sampling fraction is not greater than 0.1 for all levels of the population size in this study whatever the values of p. However, when the sampling fraction is greater than 0.1, the total variation distances of the binomial approximations to the hypergeometric distribution seem to be large for population sizes of N = 100, 500 and a small value of p.

Hence, the binomial distribution with parameters $p = \frac{D}{N}$ and n is a suitable approximation to the hypergeometric distribution for a small sampling fraction (the sampling fraction is not greater than 0.1) as mentioned by Weiss [2], Montgomery [3] and Evans *et al*. [4].

Even the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution are not greater than 0.005 when the sampling fraction is not greater than 0.1 for all levels of the population size in this study, whatever the values of p. When the sampling fraction is greater than 0.1, the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution are not greater than 0.05 for all levels of the population size in this study. Therefore, the efficiencies of the two modified binomial approximations to the hypergeometric distribution—the modified binomial distributions 2 and 3—using an expansion of hypergeometric probabilities in terms of Krawtchouk's polynomial are better than that of a binomial approximation to the hypergeometric distribution whatever the sampling fraction and the population size.

V. Conclusion

This study proposes two modified binomial approximations to the hypergeometric distribution—the modified binomial distributions 2 and 3—using an expansion of hypergeometric probabilities in term of Krawtchouk's polynomial. The results of the simulation study indicate that the total variation distances of the modified binomial 2 and the modified binomial 3 approximations to the hypergeometric distribution are less than that of a binomial approximation for almost all situations of the study and tend to zero for a small sampling fraction whatever the levels of population size. Even for a large population size of N = 20,000, the total variation distances of three estimations seem to be only slightly different at all levels of the sampling fraction and p. This work is particularly important in quality control problems. For example, the modified binomial distribution 2 and 3 are useful in the design of acceptance–sampling plans. In particular, if the sampling fraction is greater than 0.1, then the approximation to a hypergeometric distribution with a binomial distribution is no better.

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