Integral Fourier transforms with discontinuous coefficients

Oleg Yaremko, Natalia Yaremko

Abstract— This paper presents new methods for direct and inverse Fourier integral transform in piecewise- homogeneous axis. New formulas are obtained in the form of Hermite- type polynomial series. Method proposed by authors has features that distinguish it from well- known Fourier integral transform method. In particular, obtained formulas for direct and inverse Fourier integral transform in the form of Hermite- type polynomial series have symmetry and can be the basis for regularizing algorithms. In the article it is proved that the analogues of Hermite polynomials and Hermite functions form biorthogonal system.

Keywords— direct/inverse Fourier integral transform in piecewise-homogeneous axis, Hermite polynomials, Hermite functions, Dirichlet integral
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I. INTRODUCTION

New methods for direct and inverse Fourier integral transform for piecewise-homogeneous axis are developed in this article. Solutions of the problems are obtained in the form of Hermite- type polynomial series. A well-known classical Fourier integral transform in homogeneous axis are represented in the form of Dirichlet integral. In this case Dirichlet formula is proved on the basis of classical Fourier integral trasform method. For our main results, we need to develop a Fourier integral trasforms with discontinuous coefficients and based on them to prove the expansion theorems in piecewise-homogeneous axis. Integral transforms with discontinuous coefficients are appeared in the mathematical literature in the 70th of the last century in the works of Uflyand Y.S. [1], Lenuk M.P. [2]; Nayda L.S. [3] , Protsenko V.S. [4], etc.

In section -1 the author's result is given from work [7]. In this section the direct and adjoint Sturm-Liouville problems with inner contact conditions are considered, their solutions serve as a kernels of direct and inverse Fourier integral transforms with discontinuous coefficients. Expansion theorems are formulated.

In section -2 analogues of polynomials and Hermite functions are constructed.

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In sections - 3 the main results are proved, the formulas of direct and inverse Fourier integral transform in a piecewise-homogeneous axis are obtained.

II. ONE-DIMENSIONAL INTEGRAL FOURIER TRANSFORMS WITH DISCONTINUOUS COEFFICIENTS

We will use required information from the author's work [7]. First note that the structure of integral transforms with the relevant variables are determined by the type of differential equation and the kind of environment where the problem is considered. Therefore decision of integral transforms with discontinuous coefficients are the problem for mathematic modeling in piece-wise homogeneous axis. It is clear this method is an effective for obtaining the exact solution of boundary-value problems for piece-wise homogeneous structures mathematical physics.

Integral transforms with discontinuous coefficients are constructed in accordance with author's work [7].

Let $\varphi(x,\lambda)$ and $\varphi^*(x,\lambda)$ be eigenfunctions of primal and dual Sturm-Liouville problems for Fourier operator on piece-wise homogeneous axis I_n,

$$I_n = \bigcup_{j=1}^{n+1}(l_{j-1}, l_j), l_0 = -\infty, l_{n+1} = \infty.$$

Let us remark that eigenfunction $\varphi(x,\lambda)$,

$$\varphi(x,\lambda) = \sum_{k=2}^{n}\theta(x-l_{k-1})\theta(l_k - x)\varphi_k(x,\lambda) + \\ + \theta(l_1 - x)\varphi_1(x,\lambda) + \theta(x - l_n)\varphi_{n+1}(x,\lambda)$$

is the solution for separated system of differential equations

$$\left(B+\lambda^2\right)\varphi(x,\lambda) = 0, \ \ x \in I_n, \tag{1}$$

by the inner contact conditions

$$\left[\alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k\right]\varphi_k = \left[\alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k\right]\varphi_{k+1}, \tag{2}$$

$$x = l_k, \ k = 1,...,n; \ m = 1,2,$$

on the boundary conditions

$$\left|\varphi_1\right|_{x=-\infty} < \infty, \left|\varphi_{n+1}\right|\big|_{x=\infty} < \infty, \tag{3}$$

where

$$B=\sum_{k=2}^{n}\theta(x-l_{k-1})\theta(l_k-x)a_k^2\frac{d^2}{dx^2}+$$

$$+\theta(l_1-x)\varphi_1(x,\lambda)a_1^2\frac{d^2}{dx^2}+\theta(x-l_n)a_{n+1}^2\frac{d^2}{dx^2}$$

Similarly, the eigenfunction $\varphi^*(x,\lambda)$,

$$\varphi^*(\xi,\lambda)=\sum_{k=2}^{n}\theta(\xi-l_{k-1})\theta(l_k-\xi)\varphi_k^*(\xi,\lambda)+$$

$$+\theta(l_1-\xi)\varphi_1^*(\xi,\lambda)+\theta(\xi-l_n)\varphi_{n+1}^*(\xi,\lambda)$$

is the solution for separated system of differential equations

$$(B+\lambda^2)\varphi_m^*(x,\lambda)=0,\ \ x\in I_n \qquad (4)$$

by the inner contact conditions

$$\frac{1}{\Delta_{1,k}}\left[\alpha_{m1}^k\frac{d}{dx}+\beta_{m1}^k\right]\varphi_k^*=$$

$$=\frac{1}{\Delta_{2,k}}\left[\alpha_{m2}^k\frac{d}{dx}+\beta_{m2}^k\right]\varphi_{k+1}^*,\quad x=l_k, \qquad (5)$$

on the boundary conditions

$$\left|\varphi_1\right|_{x=-\infty}<\infty,\ \left|\varphi_{n+1}\right||_{x=\infty}<\infty, \qquad (6)$$

where

$$\Delta_{i,k}=\det\begin{pmatrix}\alpha_{1i}^k & \beta_{1i}^k\\ \alpha_{2i}^k & \beta_{2i}^k\end{pmatrix},$$

$$k=1,...,n;\qquad i,m=1,2,$$

Further normalization eigenfunctions are accepted by the following:

$$\varphi_{n+1}(x,\lambda)=e^{ia_{n+1}^{-1}x\lambda},\qquad \varphi_{n+1}^*(x,\lambda)=e^{-ia_{n+1}^{-1}x\lambda}.$$

Let direct F_n and inverse F_n^{-1} Fourier transforms on the Cartesian axis with n contact points be defined by the rules, [7]:

$$F_n[f](\lambda)=\int_{-\infty}^{\infty}\varphi^*(\xi,\lambda)f(\xi)d\xi\equiv\hat f(\lambda), \qquad (7)$$

$$F_n^{-1}\left[\tilde f(\lambda)\right](x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x,\lambda)\hat f(\lambda)\lambda d\lambda\equiv f(x). \qquad (8)$$

We refer to the expansion theorems from [7] for function $f(x)$ and spectral function $\hat f(\lambda)$.

Theorem 1.1 *Let function $f(x)$ be defined, piece-wise continuous, absolutely integrable and has bounded variation on I_n, then for each $x\in I_n$ the integral representation*

$$\frac{1}{2}\left[f(x-0)+f(x+0)\right]=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x,\lambda)\int_{-\infty}^{\infty}\varphi^*(\xi,\lambda)f(\xi)d\xi d\lambda,$$

is hold.

Theorem 2.2 *Let function $\hat f(\lambda)$ be defined, piece-wise continuous, absolutely integrable and has bounded variation on $(-\infty,\infty)$, then for each $\lambda\in(-\infty,\infty)$, the integral representation*

$$\frac{1}{2}\left[\hat f(\lambda-0)+\hat f(\lambda+0)\right]=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi^*(x,\lambda)\int_{-\infty}^{+\infty}\varphi(x,\beta)\hat f(\beta)d\beta dx$$

is hold.

III. Piece-wise homogeneous analogues of Hermite polynomials and Hermite functions

Definition 1. 3 *Right and left analogs of power function satisfying the conditions (2) or (5) are defined by formulas*

$$\xi_n^{*k}=i^kD_\lambda^k v^*(\xi,0),$$

$$x_n^k=(-i)^kD_\lambda^k v(x,0),$$

respectively.

The function

$$e^{\lambda^2\beta}v^*(\xi,\lambda)$$

is a generating function for Hermite polynomials [6] with piece-wise contstant coefficients, this means that

$$e^{\lambda^2\beta}v^*(\xi,\lambda)=\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}H_{j,n}^*(\xi,\beta), \qquad (9)$$

Definition 2. 4 *The Hermite polynomials with piece-wise constant coefficients are called the the sequence of functions $H_{j,n}^*(z,\beta)$ from (9).*

In the homogeneous case

$$v^*(\xi,\lambda)=e^{-i\lambda\xi},$$

$$H_{j,0}^*(\xi,\beta)=\beta^{\frac{j}{2}}H_j\left(\frac{\xi}{2\sqrt{\beta}}\right),$$

where $H_j(z)-$ classical Hermite polynomial, [6]. Expansion of piece-wise homogeneous analogues of Hermite polynomials on the right piece-wise homogeneous analogues of the power function is followed from Definitions 1,2.

Theorem 5 3. *If*

$$H^*_{j,0}(\xi,\beta)=\sum_{k=0}^{j} h_{k,j}\xi^k$$

is an expansion of Hermite polynomial with respect to ξ *, then for their piece-wise homogeneous analogues* $H^*_{j,n}(\xi,\beta)$ *the representation*

$$H^*_{j,n}(\xi,\beta)=\sum_{k=0}^{j} h_{k,j}\xi_n^{*k},$$

is hold.

Definition 2. 6 *For each fixed j = 0,1,2, ... we define a piece-wise homogeneous analogue of Hermite function* $H_{j,n}(x,\beta),\ j=0,1,2,\ldots$ *as follows:*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta} v(x,\lambda)d\lambda = H_{j,n}(x,\beta). \qquad (10)$$

In the homogeneous case we have

$$v(x,\lambda)=e^{i\lambda x},$$

$$H_{j,0}(x,\beta)=\frac{e^{-\frac{x^2}{4(\beta)}}}{2\sqrt{\pi\beta}}\frac{1}{(2\sqrt{\beta})^j}H_j\left(\frac{x}{2\sqrt{\beta}}\right),$$

where $H_j(z)-$ classical Hermite polynomial, [6]. Expansion of piece-wise homogeneous analogues of Hermite polynomials on the left piece-wise homogeneous analogues of the power function is followed From Definitions 1,3.

Theorem 7 4. *If*

$$H_{j,0}(x,\beta)=\sum_{k=0}^{\infty} h_{k,j}x^k$$

is the expansion of the Hermite function into Taylor series with respect to x *, then for its piece-wise homogeneous analogue* $H_{j,n}(x,\beta)$ *the representation*

$$H_{j,n}(x,\beta)=\sum_{k=0}^{\infty} h_{k,j}x_n^k,$$

holds true.

Theorem 5.8 *System of functions* $H_{j,n}(x,\beta)$, $H^*_{k,n}(x,\beta)$ *is biorthogonal, i.e.*

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)H^*_{k,n}(x,\beta)dx=\delta_{j,k}.$$

Proof. Consider the integral

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)e^{s^2\beta}v^*(\xi,s)dx =$$
$$=\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}\cdot$$
$$\cdot v(x,\lambda)d\lambda e^{s^2\beta}v^*(\xi,\beta)dx.$$

Changing the order of integration and applying the decomposition theorem, we obtain the equality

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)e^{s^2\beta}v^*(\xi,s)dx=(-is)^j.$$

To complete the proof we use equation (10) and the uniqueness of Taylor's expansion. The theorem is proved.

IV. New Expansion Theorems

We use the expansion theorem for function $f(x)$ in Fourier- type integral from, [7]:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}v(x,\lambda)\left(\int_{-\infty}^{\infty}v^*(\xi,\lambda)f(\xi)d\xi\right)d\lambda, \qquad (11)$$

Write the last equality in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}v^*(\xi,\lambda)f(\xi)d\xi\right)d\lambda, \qquad (12)$$

where $\beta>0$.

In accordance with (9) formula (12) takes the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)f(\xi)d\xi d\lambda.$$

Then we use definition -- 2 and finally get new analytical representation at the point x

$$f(x)=\sum_{j=0}^{\infty}H_{j,n}(x,\beta)\frac{f_j}{j!}, \qquad (13)$$

where

$$f_j=\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)f(\xi)d\xi. \qquad (14)$$

We get a new expansion theorem.

Classical expansion theorem takes the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty}v^*(\xi,\lambda)f(\xi)d\xi\right)d\lambda.$$

If $\beta>0$, then the last formula takes the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}v^*(\xi,\lambda)f(\xi)d\xi\right)d\lambda, \qquad (15)$$

Because of formula (9) we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)f(\xi)d\xi d\lambda.$$

We will change the order of integration and calculate the inner integral with respect to λ. On the basis of (10) we can write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}v(x,\lambda)d\lambda = H_{j,n}(x,\beta).$$

Finally, second new expansion theorem takes the form

$$f(x)=\sum_{j=0}^{\infty}H_{j,n}(x,\beta)\frac{f_j}{j!}, \qquad (16)$$

where

$$f_j=\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)f(\xi)d\xi.$$

Now we get third new formula. To do this, formula (11) can be written in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}v(x,\lambda)\cdot\right.$$
$$\left.\cdot v^*(\xi,\lambda)f(\xi)d\xi\right)d\lambda,$$

where $\beta>0$.

Use a Taylor series expansion with respect to λ

$$e^{\lambda^2\beta}v(x,\lambda)v^*(\xi,\lambda)=\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}H_{j,n}(x,\xi,\beta). \qquad (17)$$

In the homogeneous case we have

$$v(x,\lambda)v^*(\xi,\lambda)=e^{-i\lambda(\xi-x)},$$
$$H_{j,n}(x,\xi,\beta)=\beta^{\frac{j}{2}}H_j\left(\frac{\xi-x}{2\sqrt{\beta}}\right),$$

where $H_j(z)-$ classical Hermite polynomial.

Let

$$H_{j,0}(\xi-x,\beta)=$$
$$=\sum_{k=0}^{j}h_{k,j}\sum_{\alpha+\beta=k}(-1)^{\beta}\frac{(\alpha+\beta)!}{\alpha!\beta!}\xi^{\alpha}x^{\beta}-$$

is Hermite polynomial expansion on powers ξ, x, then for their piece-wise homogeneous analogues $H_{j,n}(\xi,x,\beta)$, so called Hermite-type polynomials, we have the representation

$$H_{j,n}(\xi,x,\beta)=\sum_{k=0}^{j}h_{k,j}\sum_{\alpha+\beta=k}(-1)^{\beta}\frac{(\alpha+\beta)!}{\alpha!\beta!}\xi_n^{*\alpha}x_n^{\beta}.$$

In view of (17) we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2\beta}\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\cdot$$
$$\cdot\int_{-\infty}^{\infty}H_{j,n}(x,\xi,\beta)f(\xi)d\xi d\lambda.$$

To simplify last formula, we change the order of integration and compute the inner integral with respect to λ, substitute $x=0$ in (10). Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}d\lambda=\frac{1}{(2\sqrt{\beta})^{j+1}}H_j(0).$$

Taking into account the well-known formula from [6]

$$H_{2j}(0)=\frac{(-1)^n(2n)!}{2^n n!}, H_{2j+1}(0)=0;\ j=0,1,2,...$$

we get finally new analytical representation for $f(x)$

$$f(x)=\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^{2j+1}}\frac{(-1)^j f_{2j}}{2^j j!}, \qquad (18)$$

where

$$f_{2j}=\int_{-\infty}^{\infty}H_{2j,n}(x,\xi,\beta)f(\xi)d\xi. \qquad (19)$$

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