Local and Nonlocal Symmetries and Inverse Problems for Ordinary Differential Equations

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***Abstract*—**

Some recent results and perspectives for development of contemporary group analysis for ordinary differential equations are considered. The article deals with regular algorithms of searching for the higher symmetries (tangential, Lie–Bäcklund and nonlocal – exponential and non-exponential). The solution for an inverse problem in class of equations admitting some non-exponential nonlocal operator is given.

***Index Terms*—**

Ordinary differential equations, tangential operators, Lie–Bäcklund and nonlocal operators, inverse problem.

I. Introduction

The present article generalizes and significantly develops the results of authors announced on international conferences MOGRAN (2000) [4], WSEAS (2002 [1], 2004 [2]) and Europment (2014 [3]).

It is well known that the group analysis originated in the end of XIX century (Sophus Lie) and initially it considered only point transformation leaving ordinary differential equation (ODE) invariant. It is possible to equally describe almost all classic methods of integration known at that period on this basis. However, there didn't exist any new methods of solving practically important model equations and further development of group analysis became possible only after Ovsiannikov L. V. (1950 – 1960), who cogently demonstrated that usage of the group methods could give the huge number of physically significant partial solutions of nonlinear model equations in mathematical physics.

Since the search of particular solutions of partial equations (for example, self–model solutions) in partial derivatives is often reduced to the solution of nonlinear ODE, there appeared an urgent need to elaborate the new methods for ODE solving in closed form. Leaving meanwhile the occurrence of discrete–group analysis, it's possible to note that classic group analysis was developed in two directions: nonpoint transformations (local and nonlocal) research and inverse problem solution. Wherein the first direction allows significantly expand the number of "solvable" ODE and the second – to describe the multiplicity of **all** ODE of the given class coinciding with several a priori conditions , for example, those having a first integral (conservation law) of given structure or having symmetry corresponding to the symmetry of certain application. This in its turn gives the possibility to purposefully build the model equations regarding the requirement of maximum adequacy to describing phenomenon.

II. Local transformations

Point transformations do not describe all possible symmetries of ODE, even local. Natural generalization of point transformations in this case are **tangential** or **contact** transformations. The well-known example of such transformations – is the Legendre transformation. Let's consider the G group of point transformations in the space of independent variables (x, y, y')

$$\tilde{x} = \varphi(x, y, y', a), \quad \tilde{y} = \psi(x, y, y', a), \quad \tilde{y}' = \chi(x, y, y', a), \tag{1}$$

where

$$\varphi\Big|_{a=0} = x, \qquad \psi\Big|_{a=0} = y, \qquad \chi\Big|_{a=0} = y'. \tag{2}$$

Transformations (1) are called **contact**, if the G group preserves the following equation $\omega = dy - p\,dx$, i. e. the equation $dy - p\,dx = \rho(d\tilde{y} - \tilde{p}\,d\tilde{x})$ completes. This equation expresses the tangency condition of the first order.

Similarly it's possible to try to enter the tangential transformation of the highest order, but, apparently S. Lie already knew that there could **not exist any tangential transformation of the highest order** because transformations corresponding to them in form and properties turn out in prolongation of point transformations or contact transformations. Moreover, for functions which depend on more than one variable such contact transformations are indeed prolongation of point transformation.

***Theorem* 1** [5]. Operator

$$X = \xi(x, y, y')\frac{\partial}{\partial x} + \eta(x, y, y')\frac{\partial}{\partial y} + \zeta(x, y, y')\frac{\partial}{\partial y'} \tag{3}$$

is an infinitesimal operator of the group of contact transformations if and only if

$$\xi = -\frac{\partial W}{\partial y'}, \qquad \eta = W - y'\frac{\partial W}{\partial y'}, \qquad \zeta = \frac{\partial W}{\partial x} + y'\frac{\partial W}{\partial y} \tag{4}$$

with several function $W = W(x, y, y')$.

From the form (3) it is obvious that for equations of the second order the regular algorithm of search for contact transformations there do not exist, as the defining equation (invariance condition) does not split on the system due to the

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lack of an independent variable – all the required functions depend on all variables included. But even in the case when it is possible to find an assumed operator which coincides in its shape with the from (3) it is not guaranteed that it could be contact. For example, in article [6] there is an example of equation

$$y'' + \frac{1}{3}xy^{-5/3} = 0, \tag{5}$$

which permits the existence of the second operator (along with point $X_1 = 8x\,\partial_x + 9y\,\partial_y$)

$$X_2 = \left[(y')^2 - ty^{-2/3}\right]\partial_x - \frac{3}{2}y^{1/3}\,\partial_y + \frac{1}{2}y^{-2/3}y'\,\partial_{y'} \tag{6}$$

(this operator was found using a discrete group of transformations for the equation of Emden–Fowler [7]). However, it's impossible to find the function W – the system of equations (4) for coordinates (6) is incompatible. Therefore the operator (6) is tangent **only on the manifold of solutions** for equation (5), i. e. essentially it's Lie–Bäcklund's operator.

For equations of the third and higher order contact transformations could be received using the algorithm of Lie. However, despite of the significant increase of possible dimension of permissible operators algebra (from 7 to 10 for equations of the third order), the class of equations which admit local transformations is indeed rather narrow. Let's illustrate this statement considering an inverse problem for class of contact transformations allowed by such equation:

$$y''' = F(x, y, y'). \tag{7}$$

This problem means the solving of system

$$\begin{cases} W_{y'y'y'} = 0, \\ W_{yy'y'}y' + W_{xy'y'} + W_{yy'} = 0, \\ W_{yyy'}(y')^2 + 2W_{xyy'}y' + W_{xxy'} + W_{yy}y' + \\ +W_{xy} + W_{y'y'}F = 0, \\ W_{y'}F_x + (y'W_{y'} - W)F_y - (y'W_y - W_x)F_{y'} + \\ +(3y'W_{yy'} + W_y + 3W_{xy'})F + \\ +W_{xxx} + 3W_{xxy}y' + 3W_{xyy}(y')^2 + W_{yyy}(y')^3 = 0. \end{cases} \tag{8}$$

The solution of first three equations (8) gives two solutions: 1) if $W_{y'y'} = 0$, so F is arbitrary and operator converts to point operator. This problem is already solved (see, f. ex. [4]) and therefore is not interesting for us; 2) $W_{y'y'} \neq 0$, and then

$$W = \frac{1}{2}f(x)(y')^2 - f'(x)yy' + g(x)y' + H(x, y),$$
$$F = \frac{1}{f(x)}\Big[\big(2f''(x) - H_{yy}\big)y' + f'''(x)y - g''(x) - H_{xy}\Big]. \tag{9}$$

The substitution of expressions (9) in the last equation of system (8) gives (after splitting on exponents y') a new system, from the first two equations of which follows

$$H(x, y) = \frac{[f'(x)]^2 + C}{2f(x)}y^2 + a_1(x)y + a_0(x),$$

i. e. the equation (7) is linear.

III. Nonlocal operators

Let's find out which structure should be for infinitesimal operator to describe all ODE of the second order, allowing reduction of the number of order by moving to new variables – invariants for admitted operator. Let the operator be written in canonical form, then a universal invariant $I_0 = x$, and let the first differential equations be $I_1 = H(x, y, y')$. Then the function H satisfies the equation

$$\Phi\frac{\partial H}{\partial y} + D_x\Phi\frac{\partial H}{\partial y'} = 0, \tag{10}$$

where Φ – is a coordinate of formal canonical operator. The equation (10) is an equation of the total derivatives of the first order regarding unknown coordinate for Φ, and it could be considered as an equation with ramifying variables. Its solution has the following form:

$$\Phi = \exp\left(-\int\frac{\partial H/\partial y}{\partial H/\partial y'}\,dx\right). \tag{11}$$

It's obvious that the formal operator defined by this coordinate is an **exponential nonlocal operator (ENO)**. Thus, to describe all equations of the second order, which allow a reduction to the equation of the first order by submitting invariants of possible operator, an exponential nonlocal operator is enough. For equations of higher order this is not true, but it's possible to prove more general statements playing an important role in the general theory of nonlocal operators.

Theorem 2 ***(first factorization theorem)*** [9, 10]. Any differential equation of the n–th order $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$, factorized up to a special system:

$$\begin{cases} z^{(n-1)} = G(x, z, z', \ldots, z^{(n-2)}), \\ z = H(x, y, y'), \end{cases} \tag{12}$$

If and only if it admits the ENO

$$X = \exp\left(-\int\frac{H_y}{H_{y'}}\,dx\right)\partial y. \tag{13}$$

Remark 1. Any differential equation admitting an operator can be written in invariants of this operator. A priori we suppose that admissible operator has differential invariant of the first order.

Remark 2. The Theorem 2 in a slightly different form is given in the book of P. Olver [11].

Theorem 3 [9]. The equation

$$y'' = f(x)y + g'(x)y^{-1} - [g(x)]^2y^{-3}, \tag{14}$$

where $f(x)$, and $g(x)$ are arbitrary functions is a unique (up to the equivalence transformations of Kummer–Liouville) equation of the form $y'' = F(x, y)$ admitting ENO

$$X = \exp\left(\int\zeta(x, y)dx\right)\eta(x, y)\partial_y.$$

In this case it is factorized up to the system

$$\begin{cases} z' + z^2 = f(x), \\ y' = z(x, C)y + g(x)y^{-1}. \end{cases}$$

Thus the first equation of the system is a Riccati equation, and it can be solved independently of the second equation. The last equation is a Bernoulli equation, which is integrated in quadratures for arbitrary coefficient $z(x, C)$, i. e. for a general solution of the first equation.

Corollary. The equation (14) is a direct generalization of the **Ermakov equation**

$$y'' = f(x)y + Ay^{-3}$$

transforms to it for $g = \mathrm{const}$ and has all its properties except for an admissible 3-dimensional Lie algebra. For arbitrary f and g the equation (14) does not admit a point group at all with the exception of the trivial one. Its general solution

$$y = u\left[C_1 + 2\int \frac{g(x)\,dx}{u^2}\right]^{1/2},$$

where u is a general solution of the "shortened", linear equation $u'' = f(x)u$. Naturally, one of three arbitrary constants contained in the solution is not independent. The equation (14) admits one-dimensional (!) point algebra only (!) if

$$f = \frac{1}{2}\frac{g'''}{g'} + \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - \frac{1}{2}\frac{g''}{g} + \frac{k}{4}\left(\frac{g'}{g}\right)^2,$$

k is an arbitrary constant; 3-dimensional algebra is admitted **only** by the classical Ermakov equation (i. e. for $g = \mathrm{const}$).

The mechanism indicated above allows to solve problem arisen more than 100 years ago nomely the problem of construction of Ermakov equation's analog of any order.

Theorem 4 ***(second factorization theorem)*** [9,10]. Random differential equation of n–th order $y^{(n)} = F\big(x, y, y', \dots, y^{(n-1)}\big)$, could be factorized till the system of special form:

$$\left\{\begin{aligned} & z^{(n-k)} = G\big(x, z, z', \dots, z^{(n-k-1)}\big), \\ & z = H\big(x, y, y', \dots, y^{(k)}\big), \qquad \frac{\partial z}{\partial y^{(k)}} \neq 0, \end{aligned}\right. \tag{15}$$

if it admits some formal operator for which $H\big(x, y, y', \dots, y^{(k)}\big)$ is the ***youngest*** differential invariant on the manifold given by the equation. If the equation is factorized into the system (15), then it admits some formal operator for which $H\big(x, y, y', \dots, y^{(k)}\big)$ is a differential invariant of k –order on the manifold.

Remark. If $k = n - 1$ and the first equation (15) has the form $z' = 0$, so the function H is the first integral of the original equation.

The search of the formal operator having a predetermined invariant in general consists in the solution of differential equation with full derivatives

$$\Phi\frac{\partial H}{\partial y} + D_x[\Phi]\frac{\partial H}{\partial y'} + \dots + D_x^k[\Phi]\frac{\partial H}{\partial y^{(k)}} = 0.$$

A perspective approach of this problem solution (alternative generalized symmetries) is proposed by one of the authors of the present article [8,9], but here we will consider only one special case leading to a closed form of nonlocal operator different from ENO.

IV. Nonexponential nonlocal operators

Let's consider the task of searching the classes of the third order equations admitting nonlocal nonexponential operator (NNO) of such form

$$X = \eta(x, y, y')\left(\int \zeta(x, y, y')\,dx\right)\partial_y. \tag{16}$$

Theorem 5. Any equation allowing the operator (16) admits also the local operator $\bar{X} = \eta(x, y, y')\,\partial_y$.

Consequence. To solve this problem we could consider the class of autonomous equations, i. e. put $\eta \equiv y'$ and look for the operator as

$$X = y'\left(\int \zeta(x, y, y')\,dx\right)\partial_y, \tag{17}$$

and then find all classes of equations using the principle of similarity of oneparametric point groups on the plane.

The following statement is fair.

Theorem 6 [14]. There is no nontrivial equation of the form $y''' = F(y)$, admitting the NNO of the form (16).

Therefore, let's consider an autonomous equation of the third order without an "elder" derivative admitting nonlocal nonexponential operator (17).

Theorem 7 [14]. The equation (7) with $F_x = 0$ admits NNO (17) if and only if the right part has the form

$$F(y, y') = y'\Big(C\big(y'\big)^2 + G(y)\Big)H(y) - \frac{1}{2C}G''(y)y', \tag{18}$$

wherein

$$\zeta(x, y, y') = C + \frac{G(y)}{\big(y'\big)^2}, \tag{19}$$

where $G(y)$ and $H(y)$ – are an arbitrary functions, $C \neq 0$ – is an arbitrary constant.

Remark. The value $C = 0$ is possible only if $G''(y) \equiv 0$. But in this case the original equation is trivial and can be easily integrated.

It's easy to prove [8] that the operator (17) does not have the first differential invariant (more precisely – an invariant, depending only on the first derivative). To calculate the second differential invariant of found operator it's useful to solve the equation

$$\tilde{\eta}\frac{\partial \Phi}{\partial y} + \tilde{\eta}_1\frac{\partial \Phi}{\partial y'} + \tilde{\eta}_2\frac{\partial \Phi}{\partial y''} = 0.$$

Substituting the received coordinates of the operator and splitting the equation by the nonlocal invariable I, we obtain a system of two equations

$$\begin{cases} y'\left[C+\dfrac{G(y)}{\left(y'\right)^2}\right]\dfrac{\partial\Phi}{\partial y'}+\left(2Cy''+G'(y)\right)\dfrac{\partial\Phi}{\partial y''}=0, \\ y'\dfrac{\partial\Phi}{\partial y}+y''\dfrac{\partial\Phi}{\partial y'}+ \\ +\left[y'\Big(C\left(y'\right)^2+G(y)\Big)H(y)-\frac{1}{2C}G''(y)y'\right]\frac{\partial\Phi}{\partial y''}=0. \end{cases} \tag{20}$$

It's necessary to note that in the second equation instead of y''' is used the right part of the equation (18), i. e. an invariant is placed **on the manifold** of solutions of the original equation. The solution of the first equation of the system (20) is a function

$$\Omega\left(y,\frac{2Cy''+G'(y)}{C\left(y'\right)^2+G(y)}\right), \tag{21}$$

the substitution of (21) in the second equation of the system leads to a linear equation of the first order with partial derivatives regarding the function Ω

$$\frac{\partial\Omega}{\partial y}+\left[H(y)-2C\omega^2\right]\frac{\partial\Omega}{\partial\omega}=0, \tag{22}$$

where ω – is the second argument of the function Ω. The equation in characteristics of (22) is a Ricatti's equation in its canonical form, consequences, it is always could be solved as an linear equation of the second order. In a big amount of cases the solution of equation (22) could be expressed in a closed form – through an elementary or special functions. The type of submission subsequently depends on the function $H(y)$. For example, if $H(y)=y^k$ or $H(y)=e^y$, the second differential invariant is expressed through the Bessel functions, while in the case of degree function we obtain a special Ricatti's equation – if the expression $\frac{k+3}{k+2}$ is a half–integer, then the second differential invariant is an elementary function. For example, when $k=0$

$$\Omega=\sqrt{2C}y-\operatorname{arth}\left(\frac{2Cy''+G'(y)}{\sqrt{2C}\left(C\left(y'\right)^2+G(y)\right)}\right).$$

Direct verification shows that because of the original equation is $\Omega'=0$, i. e. there exists a factorization

$$\begin{cases} \Omega'=0, \\ \Omega=\sqrt{2C}y-\operatorname{arth}\left(\dfrac{2Cy''+G'(y)}{\sqrt{2C}\left(C\left(y'\right)^2+G(y)\right)}\right). \end{cases}$$

Thus, the function Ω is an **autonomous first integral** of the original equation and the found symmetry is an analogue of the variation symmetry.

V. An analogue of variation symmetry

It is known that among symmetries of ordinary differential equations (ODE) the special place is taken by **variation** or **noether** symmetries in the case when the Lagrangian is invariant towards the group of symmetries of the corresponding Euler equations. Terms that define this type of symmetries are inherent to mechanics and variation calculus but not to the theory of ODE. But if we are interested in the method of integration of equations and not the search of extremes there is no need to use the Hamiltonian formalism. Moreover, its necessary to abstract from traditional ideas about the properties of variation symmetries could be inherent only to symmetries of equations of even order. Obviously it is not so. For example, the simple equation of the 3rd order

$$y'''=2yy'.$$

Is autonomous and has an autonomous first integral

$$y''=y^2+C,$$

i. e. the symmetry of this equation is absolutely similar to variation symmetry in sense that its first integral "inherits" allowing to reduce the order of original equation immediately in two units. Nontrivial examples of such equations were found by P. P. Avrashkov [8].

Recently developed algorithms allow to find classes of such equations and prove their maximum with an additional conditions for the form of original equation and its first integral. By the virtue of the principal of similarity of one-parameter groups on the plane its enough to consider the **autonomous** equations and to look for an **autonomous** first integrals and than to multiply an obtained results by reversible point transformations which are the elements of equivalency group. Let's consider the following equation:

$$y'''=F(y). \tag{23}$$

***Theorem* 8** [12, 13]. There is no nontrivial equation (23) (i. e. $F(y)\neq 0$), having linear by y'' an autonomous first integral and the equation (23) having quadratic autonomous first integral and it could **unique** be:

$$y'''=\left(ay^2+by+c\right)^{-5/4}, \tag{24}$$

where a, b, c – are random constants. The first integral has the form of

$$P=R\left(y''\right)^2-\frac{1}{2}R'\left(y'\right)^2y''+\frac{1}{8}R''\left(y'\right)^4-2R^{-1/4}y'.$$

The requirement of the existence of a cubic autonomous first integral for equation (23) leads to condition for coefficients a, b and c in formula 24, should be the ratio $b^2=4ac$, and the formula in brackets should be the full quadrate, i. e. it could be written as

$$y'''=ay^{-5/2}.$$

In this case the equation has two functionally independent cubic first integrals which allows to integrate it fully, the elimination of second derivative leads to an autonomous equation of the first order. Its important to mention that to reach this form of equation its possible using a model equations of a border layer in sedative liquids. For class

$$y'''=F(y,y'), \tag{25}$$

a linear autonomous first integral exists for equations with

$$F = \frac{R''\bigl(y'\bigr)^3 - 2S'y'}{2R},$$

where R and S – are arbitrary functions of variable y, and quadratic – for a whole range of equations

$$F = R^{-3/2}y'\Phi(u) + \frac{2RR'' - \bigl(R'\bigr)^2}{8R^2}\bigl(y'\bigr)^3 - \frac{2RT' - R'T}{4R^2}y',$$

where

$$u = R^{-1/2}\bigl(y'\bigr)^2 + \int TR^{-3/2}\,dy,$$

R and T – are arbitrary functions of the variable y, Φ – is a random function of a variable u.

Finally for the class

$$y''' = F(y)\bigl(y''\bigr)^2 + G(y)y'' + H(y) \tag{26}$$

the following assertion is certain.

Theorem 9. There exists a unique equation of the class (26) with $F \equiv 0$, $H \not\equiv 0$, namely

$$y''' = \frac{cy''}{ay+b} + \frac{k}{(ay+b)^{5/2}}.$$

and the unique equation of the class (26) $F, H \not\equiv 0$, namely

$$y''' = \alpha\bigl(y''\bigr)^2 - \frac{ay''}{\alpha(ay+b)} + \frac{c}{(ay+b)^4},$$

having quadratic autonomous first integral , respectively

$$P = \left[(ay+b)y'' + \frac{1}{2}a\bigl(y'\bigr)^2 - cy'\right]^2 - \frac{2k}{a}\frac{ay' + 2c}{(ay+b)^{1/2}}$$

and

$$P = \left\{\bigl[\alpha(ay+b)y'' + ay'\bigr]^2 + \frac{\alpha c}{(ay+b)^2}\right\}e^{-2\alpha y'}.$$

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