Operational techniques for Laguerre and Legendre polynomials

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Abstract— By starting from the concepts and the related formalism of the Monomiality Principle, we exploit methods of operational nature to describe different families of Laguerre polynomials, ordinary and generalized, and to introduce the Legendre polynomials through a special class of Laguerre polynomials themselves. Many of the identities presented, involving families of different polynomials were derived by using the structure of the operators who satisfy the rules of a Weyl group.

In this paper, we first present the Laguerre and Legendre polynomials, and their generalizations, from an operational point of view, we discuss some operational identities and further we derive some interesting relations involving an exotic class of orthogonal polynomials in the description of Legendre polynomials.

Keywords— Laguerre polynomials, Legendre polynomials, Weyl group, Monomiality Principle, Generating Functions.

I. INTRODUCTION

In this section we will present the concepts and the related aspects of the Monomiality Principle [1,2] to explore different approaches for Laguerre and Legendre polynomials [3]. The associated operational calculus introduced by the Monomiality Principle allows us to reformulate the theory of the generalized Laguerre and Legendre polynomials from a unified point of view. In fact, these are indeed shown to be particular cases of more general polynomials and can be also used to derive classes of isospectral problems, an interesting example is given by the generalized Bessel functions [4]. Many properties of conventional and generalized orthogonal polynomials have been shown to be derivable, in a straightforward way, within an operational framework, which is a consequence of the Monomiality Principle. Before to investigate the case of the generalized Laguerre and Legendre polynomials, let us briefly discuss about the Monomiality Principle.

By *quasi-monomial* [1] we mean any expression characterized by an integer *n*, satisfying the relations:

$$\begin{aligned} \hat{M} f_n &= f_{n+1} \\ \hat{P} f_n &= n f_{n-1} \end{aligned} \tag{1}$$

where $\hat{M}$ and $\hat{P}$ play the role of multiplicative and derivative operators. An example of quasi-monomial is provided by:

$$_{\delta}x_n = \prod_{m=0}^{n}\left(x - m\delta\right), \tag{2}$$

whose associated multiplication and derivative operators read:

$$\begin{aligned} \hat{M} &= x e^{\delta \frac{d}{dx}} \\ \hat{P} &= \frac{e^{\delta \frac{d}{dx} - 1}}{\delta}. \end{aligned} \tag{3}$$

It is worth noting that, when $\delta = 0$ then:

$$\begin{aligned} \hat{M} &= x \\ \hat{P} &= \frac{d}{dx}. \end{aligned} \tag{4}$$

More in general, a given polynomial

$p_n(x)$, $n \in \mathbb{N}, x \in \mathbb{C}$

can be considered a *quasi-monomial* if two operators

$\hat{M}$ and $\hat{P}$

called multiplicative and derivatives operator respectively, can be defined in such a way that:

$$\begin{aligned} \hat{M}\, p_n(x) &= p_{n+1}(x) \\ \hat{P}\, p_n(x) &= n p_{n-1}(x) \end{aligned} \tag{5}$$

with:

$$\left[\hat{M}, \hat{P}\right] = \hat{M}\hat{P} - \hat{P}\hat{M} = \hat{1}, \tag{6}$$

that is $\hat{M}$, $\hat{P}$ and $\hat{1}$ satisfy a Weyl group structure [5] with respect commutation operation. The *rules* we have just established can be exploited to completely characterize the family of polynomials; we note indeed that, if $\hat{M}$ and $\hat{P}$ have

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a differential realization, the polynomial $p_n(x)$ satisfy the differential equation:

$$\hat{M}\,\hat{P}\,p_n(x) = p_n(x). \tag{7}$$

If $p_0(x) = 1$, then $p_n(x)$ can be explicit constructed as:

$$\hat{M}^n(1) = p_n(x). \tag{8}$$

If $p_0(x) = 1$, then the generating function of $p_n(x)$ can always be cast in the form:

$$e^{t\hat{M}}(1) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} p_n(x) \tag{9}$$

where $t \in \mathbb{R}, |t| < +\infty$.

In some previous papers [6-9], we have shown that different families of Hermite polynomials are an example of *quasi-monomial*. In the case of two-variable Kampé de Fériet polynomials [10,11]:

$$H_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{10}$$

we note that the associated multiplication and derivative operators, are identified as:

$$\begin{aligned} \hat{M} &= x + 2y\frac{\partial}{\partial x} \\ \hat{P} &= \frac{\partial}{\partial x}. \end{aligned} \tag{11}$$

According to what has been discussed above, since $H_0(x,y) = 1$ then $H_n(x,y)$ can be explicitly constructed as:

$$\begin{aligned} &\left(x + 2y\frac{\partial}{\partial x}\right)^n (1) = \sum_{r=0}^{n} \binom{n}{r} (2y)^r H_{n-r}(x,y) \left(\frac{\partial}{\partial x}\right)^r (1) = \\ &= H_n(x,y). \end{aligned} \tag{12}$$

The above identity is essentially the Burchnall operational formula, whose proof can be found in the paper [12].

The Laguerre polynomials too, can be viewed as *quasi-monomial* [1,13] and the relevant characteristic operators are:

$$\begin{aligned} \hat{M} &= 1 - \hat{D}_x^{-1} \\ \hat{P} &= -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \end{aligned} \tag{13}$$

with $\hat{D}_x^{-1}$ being a negative derivative operator defined as:

$$\hat{D}_x^{-n} a = a\frac{x^n}{n!}, \text{ where } a \equiv const. \tag{14}$$

and:

$$\hat{D}_x^{-1} \frac{d}{dx} = \hat{1}.$$

By following an analogous procedure, it is possible to derive the explicit forms of the Laguerre polynomials, directly by using the concepts and the formalism of the Monomiality Principle. In fact, as for the generalized Hermite polynomials, since $L_0(x) = 1$, we have [1,7,13]:

$$L_n(x) = \left(1 - \hat{D}_x^{-1}\right)^n (1) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \hat{D}_x^{-s} (1), \tag{15}$$

that is:

$$L_n(x) = n! \sum_{s=0}^{n} \frac{(-1)^s x^s}{(n-s)!(s!)^2}. \tag{16}$$

It is possible to generalize the Laguerre polynomials by using the formalism of the Monomiality Principle; in fact, by starting from the operational relations [1,13], we can set:

$$L_n(x,y) = \left(y - \hat{D}_x^{-1}\right)^n (1) \tag{17}$$

to define the generalized two-variable Laguerre polynomials.

By multiplying both sides of above expression with t^n, $t \in \mathbb{R}, |t| < +\infty$, and then summing up over *n*, we can state the link between the two-variable Laguerre polynomials and their generating function:

$$\sum_{n=0}^{+\infty} \left[t\left(y - \hat{D}_x^{-1}\right)\right]^n = \sum_{n=0}^{+\infty} t^n L_n(x,y)$$

and, finally:

$$\frac{1}{1-yt} \exp\left(-\frac{xt}{1-yt}\right) = \sum_{n=0}^{+\infty} t^n L_n(x,y) \tag{18}$$

which immediately gives the following relation:

$$L_n(x,y) = y^n L_n\left(\frac{x}{y}\right). \tag{19}$$

It is important to emphasize the powerful tool represented by the formalism of the Monomiality Principle. Moreover, we can

derive other important relations, linking the ordinary Laguerre polynomials and their generalizations of two variables.
It is easy to obtain the following recurrence relations regarding the generalized two-variable Laguerre polynomials [1,3]:

$$\frac{\partial}{\partial y} L_n(x,y) = nL_{n-1}(x,y) \,, \tag{20}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) L_n(x,y) = nL_n(x,y) \,, \tag{21}$$

and finally:

$$-\left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y} \right) L_n(x,y) = \frac{\partial}{\partial y} L_n(x,y). \tag{22}$$

The last relation could be seen as a first order linear differential equation in the variable y of the polynomials function $L_n(x,y)$. By noting that:

$$\sum_{n=0}^{+\infty} t^n L_n(x,0) = \exp(-xt) = \sum_{n=0}^{+\infty} t^n \frac{(-1)^n}{n!} x^n \,,$$

we get:

$$L_n(x,0) = \frac{(-1)^n}{n!} x^n \tag{23}$$

and so that, we can write then solution of the Cauchy's problem of equation (22) with initial condition $y = 0$:

$$L_n(x,y) = e^{\left[-y\left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y} \right) \right]} \frac{(-1)^n}{n!} x^n \,.$$

It is also possible to define the so-called associate generalized Laguerre polynomials through the use of the Monomiality Principle. By following the procedure followed to introduce the ordinary Laguerre polynomials (see eq. (15)), we have [3,13]:

$$L_n^{(\alpha)}(x) = \left(1 - \frac{d}{dx} \right)^{\alpha} \left(1 - \hat{D}_x^{-1} \right)^n (1) = \left(1 - \frac{d}{dx} \right)^{\alpha+n} \frac{(-1)^n x^n}{n!} \tag{24}$$

with α being not necessarily an integer.
The use of the above operational identities could be used to simplify the derivation of many properties concerning the Laguerre and generalized Laguerre polynomials. In the next section we will derive different relations involving these families of polynomials and we also start to introduce the Legendre polynomials through a special class of generalized Laguerre polynomials.

II. Operational identities for Laguerre polynomials and Legendre polynomials

In the previous section, we have introduced the ordinary Laguerre polynomials $L_n(x)$ by using the formalism of the Monomiality Principle (see eq. (15)). It is immediately to note that:

$$L_{2n}(x) = \left(1 - \hat{D}_x^{-1} \right)^n \left(1 - \hat{D}_x^{-1} \right)^n (1) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \hat{D}_x^{-s} L_n(x) \tag{25}$$

and by applying the identities outlined in equations (15) and (16) and by observing the explicit form of the associate Laguerre polynomials, presented in equation (24), we easily obtain that:

$$\hat{D}_x^{-s} L_n(x) = \hat{D}_x^{-s} \left(n! \sum_{k=0}^{n} \frac{(-1)^k x^k}{(n-k)!(k!)^2} \right) = \frac{n! x^s}{(n+s)!} L_n^{(s)}(x) \tag{26}$$

which allows us to conclude with:

$$L_{2n}(x) = (n!)^2 \sum_{s=0}^{n} \frac{(-1)^s x^s}{(n-s)!(n+s)!s!} L_n^{(s)}(x) \,. \tag{27}$$

It is also possible to write the ordinary Laguerre polynomials of index *2n* in a different form. We begin to note that:

$$L_{2n}(x) = \left(1 - 2\hat{D}_x^{-1} + \hat{D}_x^{-2} \right)^n (1) \,,$$

which, after exploited the above formal identity, we get:

$$L_{2n}(x) = \sum_{s=0}^{n} \binom{n}{s} \hat{D}_x^{-2s} \left(1 - 2\hat{D}_x^{-1} \right)^{n-s} (1) \,.$$

It is easy to recognize that, starting from the operational definition of the ordinary Laguerre polynomials (see equation (16)), we can obtain:

$$\left(1 - \mu\hat{D}_x^{-1} \right)^n (1) = L_n(\mu x) \,, \tag{28}$$

where μ is a continuous variable. The expression of the Laguerre polynomials $L_{2n}(x)$, state in relation (27), can be now written in the following form:

$$L_{2n}(x) = n! \sum_{s=0}^{n} \frac{x^{2s}}{(n+s)!s!} L_{n-s}^{(2s)}(2x) \,. \tag{29}$$

The procedure shown above could be generalized to derive further interesting operational relations involving the ordinary Laguerre polynomials, in particular to obtain the addition and multiplication formulae. We have, in fact, by using newly the operational definition of the polynomials $L_n(x)$, given in equation (16):

$$L_{n+m}(x)=\left(1-\hat{D}_x^{-1}\right)^n\left(1-\hat{D}_x^{-1}\right)^m(1)=\sum_{s=0}^{n}\binom{n}{s}(-1)^s\hat{D}_x^{-s}L_m(x),$$

and then:

$$L_{n+m}(x)=n!m!\sum_{s=0}^{n}\frac{(-1)^s x^s}{(n-s)!(m-s)!s!}L_m^{(s)}(x), \tag{30}$$

that is the summation formula. Similarly, we can easily state the multiplication formula, by following the same procedure. We have, in fact:

$$L_{nm}(x)=\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s\left[(m-1)n\right]!x^s}{\left[n(m-1)+s\right]!}L_{n(m-1)}^{(s)}(x). \tag{31}$$

The operational treatment shown in the previous sections, regarding the Hermite and Laguerre polynomials, ordinaries and in some cases generalized, can be also used to investigate, under an operational point of view, different families of classical polynomials, as for example the Legendre and Jacobi polynomials. Moreover, this approach allows us to derive new identities for these classes of polynomials and, by using the formalism of the Monomiality Principle, helps us to recognize many other families of polynomial as *quasi-monomial*.

We remind that the generalized Hermite polynomials $H_n(x,y)$ and the two-variable Laguerre polynomials $L_n(x,y)$ can be viewed as quasi-monomials with the following operators:

$$\begin{array}{ll}\hat{M}_H=x+2y\dfrac{\partial}{\partial x} & \hat{M}_L=y-\hat{D}_x^{-1}\\ \hat{P}_H=\dfrac{\partial}{\partial x} & \hat{P}_L=-\dfrac{\partial}{\partial x}x\dfrac{\partial}{\partial x}\end{array} \tag{32}$$

It has been shown that the Legendre polynomials [14] can be introduced directly by using the multiplication and derivative operators acting on the Hermite and Laguerre polynomials, in the sense that we can combine the above operators (eq. (32)) to define new operators that can be used to view the Legendre polynomials as *quasi-monomials* and at same time to define them. Let the operators:

$$\begin{aligned}\hat{M}&=y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\\ \hat{P}&=\frac{\partial}{\partial y}\end{aligned} \tag{33}$$

by using the formalism of the Monomiality Principle [1,7-9], after noting that:

$$\left[\hat{M},\hat{P}\right]=1$$

we can immediately obtain the explicit form of the polynomials defined by the above operators

$$\hat{M}^n(1)=\left(y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)^n(1)=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{x^r y^{n-2r}}{(n-2r)!(r!)^2} \tag{34}$$

where again we have used the identity:

$$\hat{D}_x^{-1}=\frac{x^n}{n!}. \tag{35}$$

The polynomials obtained in relation (34) can be recognized as a special class of Laguerre polynomials [13-15] and then we can set:

$${}_2L_n(x,y)=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{x^r y^{n-2r}}{(n-2r)!(r!)^2}. \tag{36}$$

A fortiori, we note that the condition prescribed by the Monomiality Principle is respected, in fact:

$${}_2L_0(x,y)=1. \tag{37}$$

These generalized Laguerre polynomials can be reduced to the ordinary Legendre polynomials:

$$P_n(x)=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^{n-2r}x^r(1-x^2)^{n-2r}}{(n-2r)!(r!)^2 2^{2(n-2r)}} \tag{38}$$

by setting:

$${}_2L_n\left(-\frac{1}{4}(1-x^2),x\right)=P_n(x). \tag{39}$$

We can also derive the generating function of the above polynomials. Since we have observed that ${}_2L_0(x,y)=1$, we immediately get:

$$\exp\left(t\hat{M}\right)(1)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}{}_2L_n(x,y), \tag{40}$$

where again $t\in\mathbb{R},|t|<+\infty$. By exploiting the exponential on the above relation, we have:

$$\exp\left(t\hat{M}\right)(1)=\exp\left[t\left(y+2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)\right](1)=\exp\left(ty+2t\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)(1)$$

Since the operators do not commute:

$$\left[ty, 2t\,\hat{D}_x^{-1}\frac{\partial}{\partial y}\right] = -2t^2\,\hat{D}_x^{-1},$$

and then we obtain:

$$\exp\left[t\left(y + 2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)\right](1) = \exp(ty)\exp\left(2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)\exp\left(t^2\hat{D}_x^{-1}\right)(1)$$

that is:

$$\exp\left[t\left(y + 2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)\right](1) = \exp(ty)\exp\left(t^2\hat{D}_x^{-1}\right)\exp\left(2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)(1)$$

From the fact that:

$$\exp\left(2\hat{D}_x^{-1}\frac{\partial}{\partial y}\right)(1) = 1,$$

we finally obtain:

$$e^{yt}e^{\hat{D}_x^{-1}t^2} = \sum_{n=0}^{+\infty}\frac{t^n}{n!}\,{}_2L_n(x,y). \tag{41}$$

We can observe that the Tricomi function of order zero [16] of a suitable variable, reads:

$$C_0\left(-xt^2\right) = \sum_{n=0}^{+\infty}\frac{t^{2n}}{n!}\frac{x^n}{n!} \tag{42}$$

so, we can write:

$$e^{yt}C_0\left(-xt^2\right) = \sum_{n=0}^{+\infty}\frac{t^n}{n!}\,{}_2L_n(x,y) \tag{43}$$

and then, relatively to the Legendre polynomials, by using equation (39), we can state:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}P_n(x) = e^{xt}C_0\left(\frac{1}{4}(1-x^2)t^2\right). \tag{44}$$

The use of the structure of Laguerre polynomials, recognized as *quasi-monomials*, allowed us to define the Legendre polynomials, in this case through a special class of Laguerre. We can follow a different approach, by considering the ordinary Laguerre polynomials introduced in the previous section. From equation (16), let us introduce this family of polynomials [14,15]:

$$R_n(x,y) = n!\left(-\hat{D}_x^{-1} + \hat{D}_y^{-1}\right)^n = (n!)^2\sum_{s=0}^{n}\frac{(-1)^{n-s}y^s x^{n-s}}{\left[(n-s)!\right]^2(s!)^2}, \tag{45}$$

and by following the same procedure adopted in the case of polynomials $L_n(x)$, we get:

$$R_n(x,y) = n!\sum_{s=0}^{n}(-1)^s\binom{n}{s}L_{n-s}(x)L_s(y). \tag{46}$$

These polynomials could be also recognized as *quasi-monomial* if we consider the following form:

$$\frac{R_n(x,y)}{n!} = \sum_{s=0}^{n}(-1)^s\binom{n}{s}L_{n-s}(x)L_s(y), \tag{47}$$

and the operators:

$$\begin{aligned}\hat{M} &= -\hat{D}_x^{-1} + \hat{D}_y^{-1}\\ \hat{P}_x &= -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\\ \hat{P}_y &= \frac{\partial}{\partial y}y\frac{\partial}{\partial y}\end{aligned} \tag{48}$$

where is immediately verify the *rules* of Monomiality Principle, presented in previous section. By following the above procedure, we can also derive the related generating function:

$$\begin{aligned}&\sum_{n=0}^{+\infty}\frac{t^n}{(n!)^2}R_n(x,y) = \sum_{n=0}^{+\infty}\frac{t^n}{n!}\left(-\hat{D}_x^{-1} + \hat{D}_y^{-1}\right)^n =\\ &= e^{t\hat{D}_y^{-1}}e^{-t\hat{D}_x^{-1}} = C_0(-yt)C_0(xt).\end{aligned} \tag{49}$$

We note that the generating function of the special polynomials defined in equation (45) could be expressed in terms of Tricomi functions of zero order as for the generalized Laguerre polynomials of type ${}_2L_n(x,y)$ (see equation (43)). The procedure followed to introduce the polynomials:

$$R_n(x,y) \text{ or } \frac{R_n(x,y)}{n!} \tag{50}$$

and the similarity of their generating function with the polynomials ${}_2L_n(x,y)$, lead us to observe that it is possible to derive the Legendre polynomials in a different way. From equation (18), we can write:

$$P_n(x) = (n!)^2\sum_{r=0}^{n}\frac{(-1)^{n-r}\left(\frac{1-x}{2}\right)^{n-r}\left(\frac{1+x}{2}\right)^r}{\left[(n-r)!\right]^2(r!)^2} \tag{51}$$

and then, we find:

$$P_n(x) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right). \tag{52}$$

These two sections have been shown how the operational methods, inducted by the concepts of the Monomiality Principle, provide to determine a synthetic and computationally powerful view on the theory of one variable and multi-variable polynomials, in particular in the treatment of some generalizations of the polynomials belonging to the Laguerre family and in the description of the special nature of the Legendre polynomials. In the next section, we will derive some useful identities regarding the Legendre polynomials, by proceeding as done before for the Laguerre polynomials.

III. Operational rules for Legendre polynomials

The method based on the *rules* of Monomiality Principle, described in the previous sections, is devoted to simplify the operational relations of many families of classical polynomials and, from the other hand, to recognized generalizations of well-known classes of polynomials or special functions, by using the identities derived in the descriptions of those polynomials themselves. This method could be used in the future to explore many interesting topics such as those explored for damage mechanics [17], for the dynamics of contact [18] and for electromagnetic field problems [19-20], were series representations are important solution methods. In this section, we will extend the results obtained for the Hermite and Laguerre polynomials to the family of Legendre polynomials by using the characterization introduced in Section II and will also discuss some useful relations, linking the above mentioned classes of orthogonal polynomials.

We have introduced in Section I, the associate ordinary Laguerre polynomials, by acting directly on the operational realization. It is possible to extend such a generalization regarding the polynomials $R_n(x,y)$ defined in equation (45), directly on their explicit forms.:

$$R_n^{(p,q)}(x,y) = (n+p)!(n+q)!\sum_{s=0}^{n}\frac{(-1)^s y^{n-s} x^s}{(n-s)!(n-s+q)!s!(s+p)!}. \tag{53}$$

Let us remind that the inverse derivative operator acts in the following way:

$$\hat{D}_x^{-n} a = a\frac{x^n}{n!}, \text{ where } a \equiv const.$$

and then, we can state the following identity:

$$\hat{D}_x^{-p}\hat{D}_y^{-q} R_n(x,y) = \frac{(n!)^2 x^p y^q}{(n+p)!(n+q)!} R_n^{(p,q)}(x,y). \tag{54}$$

It is possible to obtain similar relations outlined for the ordinary Laguerre polynomials, as done in Section I, for the polynomials $R_n(x,y)$. In fact, by using their operational definition, exploited in equations (45) and (46), we can, for example, derive a summation formula. We note that:

$$R_{2n}(x,y) = \frac{(2n)!}{n!}\left(-\hat{D}_x^{-1} + \hat{D}_y^{-1}\right)^n R_n(x,y), \tag{55}$$

by exploiting the r.h.s of the above identity, we get:

$$R_{2n}(x,y) = \frac{(2n)!}{n!}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\hat{D}_y^{-(n-s)}\hat{D}_x^{-s} R_n(x,y), \tag{56}$$

and finally, after noted the expression of the associate polynomials of the form $R_n^{(p,q)}(x,y)$, presented in equation (53), we have:

$$R_{2n}(x,y) = (2n)!n!\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{y^{n-s}x^s}{(2n-s)!(n+s)!}R_n^{(s,n-s)}(x,y).$$

In Section II, we have defined the generalized two-variable Laguerre polynomials of the type ${}_2L_n(x,y)$ to derive the Legendre polynomials. It is immediately to exploit a different generating function:

$$\sum_{n=0}^{+\infty} t^n\, {}_2L_n(x,y) = \frac{1}{\sqrt{1-2yt+\left(y^2-4x\right)t^2}} \tag{57}$$

where is always $t \in \mathbb{R}, |t| < +\infty$. By noting that the Legendre polynomials are linked to the following generating function:

$$\sum_{n=0}^{+\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}, \tag{58}$$

we immediately can conclude that:

$${}_2L_n(x,y) = \left(y^2-4x\right)^{\frac{n}{2}} P_n\left(\frac{y}{\sqrt{y^2-4x}}\right). \tag{59}$$

The above relations between the generalized Laguerre polynomials and Legendre polynomials, allows us to derive other operational relations involving the polynomials $R_n(x,y)$ and links with the Legendre polynomials too. By comparing the previous equations (39) and (59),we have in fact:

$$R_n(x,y) = \left(x+y\right)^n P_n\left(\frac{y-x}{y+x}\right), \tag{60}$$

which stresses the link shown in equation (52). From equation (57), we can use the above identity to deriving the generating function of the polynomials $R_n(x,y)$. We have in fact:

$$\sum_{n=0}^{+\infty} t^n R_n(x,y) = \frac{1}{\sqrt{1-2(x-y)t+\left(x^2+y\right)t^2}}. \quad (61)$$

The relations presented in Section II, regarding the generating function of the Laguerre polynomials of type ${}_2L_n(x,y)$ (eq. (43)), can be used to obtain a further generating function for the polynomials $R_n(x,y)$, that is:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} R_n(x,y) = e^{(y-x)t} C_0\left(xyt^2\right), \quad (61)$$

which confirms the strong link with the special class of Laguerre polynomials ${}_2L_n(x,y)$ and then with the Legendre polynomials.

IV. Concluding Remarks

The results shown in the previous sections have described the Legendre polynomials through different classes of polynomials. The Laguerre-like polynomials of the type ${}_2L_n(x,y)$ and the exotic polynomials $R_n(x,y)$. It is interesting to note that many of the properties obtained were derived by the use of the concepts and the related formalism of the Monomiality Principle, in the sense that we have stated the identities related to the classes of polynomials discussed, by exploiting their nature of *quasi-monomials*. In this concluding section, we will present some relations which could be deeper investigated in forthcoming papers and that could be used in modeling advanced materials in continuum mechanics [21-24], and in electromagnetic field propagation problems [25-27] and in transmission lines [28-30].

We note that, by using the explicit form of the generalized two-variable Hermite polynomials (see equation (10)) and by the definition of the Laguerre-like polynomials ${}_2L_n(x,y)$, we can write [14]:

$${}_2L_n(x,y) = H_n\left(y, \hat{D}_x^{-1}\right), \quad (62)$$

by implying that the action of the operator contained in the argument of the Hermite polynomials is referred to the unit. The use of the inverse derivative operator allows us to state useful operational identities related to the Laguerre-like polynomials. We have, in fact:

$$\hat{D}_x^{-s}\ {}_2L_n(x,y) = \frac{n!x^s}{(n+s)!}\ {}_2L_n^{(s)}(x,y), \quad (63)$$

where we have considered the *associate* to the Laguerre-like polynomials, as in equation (24), by setting:

$${}_2L_n^{(s)}(x,y) = (n+s)!\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^{n-2r}x^r}{(n-2r)!r!(r+s)!}.$$

The above relations are useful to state the following important identity:

$${}_2L_n^{(s)}(x,y) = n!\sum_{s=0}^{n} \frac{2^s}{(n-s)!s!} H_{n-s}\left(y, \hat{D}_x^{-1}\right) x^s\ {}_2L_{n-s}^{(s)}(x,y), \quad (64)$$

which can be written in a more convenient form:

$$H_m\left(y, \hat{D}_x^{-1}\right) {}_2L_n^{(s)}(x,y) = = m!n!\sum_{s=0}^{\left[\frac{m}{2}\right]} \frac{y^{m-2s}x^s}{(m-2s)!(n+s)!s!}\ {}_2L_n^{(s)}(x,y). \quad (65)$$

These last identities as well as those previously discussed in Section II, involving polynomials of two indices, suggest that these families of polynomials can be viewed formally as two-index polynomials and then we expect that their generating functions will be defined on a couple of indices instead of a single index.

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