Birkhoff weak integrability of multifunctions

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***Abstract*—We define and study Birkhoff weak integrability of multifunctions (taking values in the family of nonempty subsets of a real Banach space) relative to a non-negative set function. We obtain some classic integral properties. Results regarding the continuity properties of the set-valued integral are also presented.**

Keywords– Birkhoff weak integral, integrable multifunction, monotone measure, non-negative set function.

I. INTRODUCTION

The theory of set-valued integrals began to develop in the decade 1956-1965 motivated by solving/modeling problems in mathematical economy, statistics or control theory. The first definition of a set-valued integral was given by Dinghas [26] in 1956 by extending the Riemann integral to the set-valued case (the Riemann-Minkowski integral). After Aumann [3] defined in 1965 his integral using selection method, the theory of set-valued integrals has been intensively studied due to its interesting and important theoretical or practical applications (e.g. [11], [12], [17], [20], [24], [33], [41], [43], [44], [46], [53], [54]).

After Aumann, set-valued integrals have been defined and studied by many authors using different techniques: by Aumann selections ([9], [36], [42], [53]), by embedding theorems ([2], [23]), via Pettis method ([1], [14], [15], [25], [45]), by Dunford way using defining sequences ([10], [18], [19], [48]), using finite or infinite Riemann type sums ([4], [5], [6], [7], [8], [13], [26], [28-31], [34], [37], [38], [40], [47], [49], [51], [55], [56], [57], [58]), via Gelfand [35] method ([16], [43], [59]), by Sugeno way ([21, 22], [60]), by Choquet way ([45], [62]). A survey on different set-valued integrals can be found for example in [17], [54].

In this paper we define and study a Birkhoff type (called Birkhoff weak) integral of multifunctions F (taking values in the family of nonempty subsets of a real Banach space) relative to a non-negative set function μ. Unlike other definitions that assume μ to be a measure or finitely additive, in our definition and in some properties of the set-valued integral, μ is an arbitrary non-negative set function and this is a great advantage.

The paper is organized as follows: Section 1 is for introduction. The second section contains some basic concepts and results. In Section 3 we define the Birkhoff weak integral of multifunctions relative to a non-negative set function and present some classic integral properties. In Section 4 we provide some continuity properties of the set-valued integral.

II. PRELIMINARIES

Let be T be a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T, $\mathcal{A}$ a σ-algebra of subsets of T, $(X, \|\cdot\|)$ a real Banach space with the metric d induced by its norm, $\mathcal{P}_0(X)$ the family of all nonempty subsets of X, $\mathcal{P}_c(X)$ the family of all nonempty convex subsets of X and $\mathcal{P}_f(X)$ the family of all nonempty closed subsets of X.

For every $M, N \in \mathcal{P}_0(X)$ and every $\alpha \in \mathbb{R}$, let $M + N = \{x + y | x \in M, y \in N\}$ and $\alpha M = \{\alpha x | x \in M\}$. We denote by $\overline{M}$ the closure of M with respect to the topology induced by the norm of X.

By " $\stackrel{\bullet}{+}$ " we mean the Minkowski addition on $\mathcal{P}_0(X)$, that is,

$$M \stackrel{\bullet}{+} N = \overline{M + N}, \ \ \forall M, N \in \mathcal{P}_0(X).$$

Let h be the Hausdorff metric given by

$$h(M, N) = \max\{e(M, N), e(N, M)\}, \ \ \forall M, N \in \mathcal{P}_0(X),$$

where $e(M, N) = \sup\limits_{x \in M} d(x, N)$ and $d(x, N) = \inf\limits_{y \in N} d(x, y)$.

We denote $|M| = h(M, \{0\}) = \sup\limits_{x \in M} ||x||$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X.

By $i = \overline{1, n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{N} = \{0, 1, 2 \ldots\}$. In the following proposition we recall some properties regarding the excess and the Hausdorff metric [39].

Proposition 1: [39] Let $A, B, C, D, A_i, B_i \in \mathcal{P}_0(X)$, for every $i = \overline{1, n}$ and $n \in \mathbb{N}^*$. Then:

(i) $h(A, B) = h(\overline{A}, \overline{B})$.

(ii) $e(A, B) = 0$ if and only if $A \subseteq \overline{B}$.

(iii) $h(A, B) = 0$ if and only if $\overline{A} = \overline{B}$.

(iv) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$, $\forall \alpha \in \mathbb{R}$.

(v) $h(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i) \leq \sum_{i=1}^n h(A_i, B_i)$.

(vi) $h(\alpha A, \beta A) \leq |\alpha - \beta| \cdot |A|$, $\forall \alpha, \beta \in \mathbb{R}$.

(vii) $h(\alpha A + \beta B, \gamma A + \delta B) \leq |\alpha - \gamma| \cdot |A| + |\beta - \delta| \cdot |B|$, $\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$.

(viii) $h(A + C, B + C) = h(A, B)$, for every $A, B \in \mathcal{P}_{bfc}(X)$ and $C \in \mathcal{P}_b(X)$.

(ix) $\alpha(A + B) = \alpha A + \alpha B$, $\forall \alpha \in \mathbb{R}$.

(x) $(\alpha+\beta)A = \alpha A+\beta A$, for every $\alpha, \beta \in \mathbb{R}$, with $\alpha\beta \geq 0$ and every convex $A \in \mathcal{P}_0(X)$.

(xi) $\alpha A \subseteq \beta A$, for every $\alpha, \beta \in \mathbb{R}_+$, with $\alpha \leq \beta$ and every convex $A \in \mathcal{P}_0(X)$, with $0 \in A$.

Definition 2: (i) A finite (countable, respectively) partition of T is a finite (countable, respectively) family of nonempty sets $P = \{A_i\}_{i=\overline{1,n}}$ ($\{A_n\}_{n\in\mathbb{N}}$, respectively) $\subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup\limits_{i=1}^{n} A_i = T$ ($\bigcup\limits_{n\in\mathbb{N}} A_n = T$, respectively).

(ii) If P and P' are two finite (or countable) partitions of T, then P' is said to be *finer than* P, denoted by $P \leq P'$ (or, $P' \geq P$), if every set of P' is included in some set of P.

(iii) The *common refinement* of two finite or countable partitions $P = \{A_i\}$ and $P' = \{B_j\}$ is the partition $P \wedge P' = \{A_i \cap B_j\}$. We denote by $\mathcal{P}$ the class of all partitions of T and if $\mathrm{A} \in \mathcal{A}$ is fixed, by $\mathcal{P}_{\mathrm{A}}$ we denote the class of all partitions of A.

All over the paper, $\mu : \mathcal{A} \to [0, +\infty)$ will be a non-negative function, with $\mu(\emptyset) = 0$.

Definition 3: [27, 46] μ is said to be:

(i) *monotone* if $\mu(A) \leq \mu(B)$, $\forall A, B \in \mathcal{A}$, with $A \subseteq B$.

(ii) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

(iii) *a submeasure* if μ is monotone and subadditive.

(iv) *σ-subadditive* if $\mu(A) \leq \sum\limits_{n=0}^{\infty} \mu(A_n)$, for every sequence of (pairwise disjoint) sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A = \bigcup\limits_{n=0}^{\infty} A_n \in \mathcal{A}$.

(v) *a (σ-additive) measure* if $\mu(\bigcup\limits_{n=0}^{\infty} A_n) = \sum\limits_{n=0}^{\infty} \mu(A_n)$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

(vi) *finitely additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every disjoint $A, B \in \mathcal{A}$.

(vii) *increasing convergent* if $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ (i.e. $A_n \subset A_{n+1}$, for every $n \in \mathbb{N}$), with $\bigcup\limits_{n=0}^{\infty} A_n = A \in \mathcal{A}$ (denoted by $A_n \nearrow A$).

(viii) *decreasing convergent* if $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ (i.e. $A_{n+1} \subset A_n$, for every $n \in \mathbb{N}$), with $\bigcap\limits_{n=0}^{\infty} A_n = A \in \mathcal{A}$ (denoted by $A_n \searrow A$).

(ix) *order-continuous* (shortly, *o-continuous*) if $\lim\limits_{n\to\infty} \mu(A_n) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A_n \searrow \emptyset$.

(x) *exhaustive* if $\lim\limits_{n\to\infty} \mu(A_n) = 0$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Definition 4: [32] Let $\varphi : \mathcal{A} \to \mathcal{P}_0(\mathrm{X})$ be a set-valued set function. φ is called:

(i) *monotone* if $\varphi(A) \subseteq \varphi(B)$, $\forall A, B \in \mathcal{A}$, with $A \subseteq B$.

(ii) *finitely additive* if $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for every disjoint $A, B \in \mathcal{A}$.

(If φ is $\mathcal{P}_f(\mathrm{X})$-valued, then in the right side we will have the Minkowski addition).

(iii) *an h-multimeasure* if $\lim\limits_{n\to\infty} h(\varphi(A), \sum\limits_{k=0}^{n} \varphi(A_k)) = 0$, for every sequence of mutual disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A = \bigcup\limits_{n=0}^{\infty} A_n \in \mathcal{A}$.

(iv) *increasing convergent* if $\lim\limits_{n\to\infty} h(\varphi(A_n), \varphi(A)) = 0$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $\bigcup\limits_{n=0}^{\infty} A_n = A \in \mathcal{A}$.

(v) *decreasing convergent* if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $\bigcap\limits_{n=0}^{\infty} A_n = A \in \mathcal{A}$.

(vi) *order-continuous* (shortly, *o-continuous*) if $\lim\limits_{n\to\infty} |\varphi(A_n)| = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A_n \searrow \emptyset$.

(vii) *exhaustive* if $\lim\limits_{n\to\infty} |\varphi(A_n)| = 0$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Definition 5: I. [27] Let $\mu : \mathcal{A} \to [0, +\infty)$ be a non-negative set function.

(i) *The variation* $\overline{\mu}$ of μ is the set function $\overline{\mu} : \mathcal{P}(T) \to [0, +\infty]$ defined by $\overline{\mu}(E) = \sup\{\sum\limits_{i=1}^{n} \mu(A_i)\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, with $A_i \subseteq E$, for every $i = \overline{1, n}$.

(ii) μ is said to be *of finite variation on* $\mathcal{A}$ if $\overline{\mu}(T) < \infty$.

(iii) $\widetilde{\mu} : \mathcal{P}(T) \to [0, +\infty]$ is defined for every $A \subseteq T$, by

$$\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}.$$

II. [32] Let $\varphi : \mathcal{A} \to \mathcal{P}_0(X)$ be a set-valued set function.

(i) *The variation* $\overline{\varphi}$ of φ is the set function $\overline{\varphi} : \mathcal{P}(T) \to [0, +\infty]$ defined by $\overline{\varphi}(E) = \sup\{\sum\limits_{i=1}^{n} |\varphi(A_i)|\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, with $A_i \subseteq E$, for every $i = \overline{1, n}$.

(ii) φ is said to be *of finite variation on* $\mathcal{A}$ if $\overline{\varphi}(T) < \infty$.

(iii) $\widetilde{\varphi} : \mathcal{P}(T) \to [0, +\infty]$ is defined for every $A \subseteq T$, by

$$\widetilde{\varphi}(A) = \inf\{\overline{\varphi}(B); A \subseteq B, B \in \mathcal{A}\}.$$

Remark 6: I. In vector or set-valued measure/integral theory, the real functions $\overline{\mu}, \widetilde{\mu}, \overline{\varphi}, \widetilde{\varphi}$ play an important role since various problems in vector or set-valued frame can be thus reduced to the real case.

II. If $E \in \mathcal{A}$, then in the definition of $\overline{\mu}$ we may consider the supremum over all finite partitions $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, of E.

III. $\overline{\mu}$ is monotone and super-additive on $\mathcal{P}(T)$, that is $\overline{\mu}(\bigcup\limits_{i\in I} A_i) \geq \sum\limits_{i\in I} \overline{\mu}(A_n)$, for every finite or countable partition $\{A_i\}_{i\in I}$ of T.

IV. If μ is subadditive ($\sigma-$subadditive, respectively), then $\overline{\mu}$ is finitely additive (σ-additive, respectively).

V. If μ is a finitely additive set function, then μ is o-continuous if and only if $\overline{\mu}$ is o-continuous on $\mathcal{A}$.

Definition 7: A property *(P)* about the points of T *holds almost everywhere* (denoted μ*-a.e.*) if there exists $A \in \mathcal{P}(T)$ so that $\widetilde{\mu}(A) = 0$ and *(P)* holds on $T \backslash A$.

Definition 8: A multifunction $F : T \to \mathcal{P}_0(X)$ is called *bounded* if there exists M $\in [0, +\infty)$ such that $|F(t)| \leq$ M, for every $t \in T$.

III. Birkhoff weak integrability of multifunctions

In this section, we define and study Birkhoff weak integrability of multifunctions and establish some classic integral properties.

In the sequel, suppose $(X, \|\cdot\|)$ is a Banach space, T is infinite, $\mathcal{A}$ is a σ-algebra of subsets of T and $\mu : \mathcal{A} \to [0, +\infty)$ is a non-negative set function such that $\mu(\emptyset) = 0$.

Definition 9: Let be $\emptyset \neq \mathcal{E} \subseteq \mathcal{P}_0(X)$. A multifunction $F : T \to \mathcal{P}_0(\mathrm{X})$ is said to be *Birkhoff weak μ-integrable in $\mathcal{E}$(on* T) (shortly μ-integrable) if there exists $\overline{\mathrm{E}} \in \mathcal{P}_0(X)$ with $\overline{\mathrm{E}} \in \mathcal{E}$ having the property that for every $\varepsilon > 0$, there exist a countable partition P_ε of T and $n_\varepsilon \in \mathbb{N}$ such that for every other countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of T, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$, we have $h(\sum\limits_{k=0}^{n} F(t_k)\mu(A_k), E) < \varepsilon$, for every $n \geq n_\varepsilon$.

The set $\overline{E}$ is called *the Birkhoff weak μ-integral of F on* T and is denoted by $(Bw)\int_T F d\mu$ or simply $\int_T F d\mu$. If $\mathcal{E} = \mathcal{P}_0(X)$, then F is called simply μ-integrable.

Remark 10: If it exists, the integral is unique.

Example 11: I. If $F(t) = \{0\}$, for every $t \in T$, then F is μ-integrable and $\int_T F d\mu = \{0\}$.

II. Suppose $T = \{t_n | n \in \mathbb{N}\}$ is countable, $\{t_n\} \in \mathcal{A}$ and let $F : T \to \mathcal{P}_0(\mathrm{X})$ be such that the series $\sum\limits_{n=0}^{\infty} F(t_n)\mu(\{t_n\})$ is unconditionally convergent. Then F is μ-integrable and $\int_T F d\mu = \overline{\sum\limits_{n=0}^{\infty} F(t_n)\mu(\{t_n\})}$.

III. Suppose $F : T \to \mathcal{P}_0(\mathbb{R})$ is the multifunction defined by $F(t) = [0, f(t)]$, for every $t \in T$, where $f : T \to [0, +\infty)$ is a non-negative function. If f is *μ-integrable in* $\mathcal{E} = \{\{\mathrm{a}\}; \mathrm{a} \in [0, +\infty)\}$ and $\int_T f d\mu = \{a\}$, with a $\in [0, +\infty)$, then F is *μ-integrable and* $\int_T F d\mu =$[0,a].

Proposition 12: Suppose $F : T \to \mathcal{P}_0(\mathrm{X})$ is bounded. *If* $F = \{0\}$ *μ-ae, then F is μ-integrable and* $\int_T F d\mu = \{0\}$.

Proof. Since F is bounded, there exists $M \in [0, \infty)$ so that $|F(t)| \leq M$, for every $t \in T$.
If $M = 0$, then the conclusion is obvious.
Suppose $M > 0$. Denoting $A = \{t \in T; F(t) \neq \{0\}\}$ and since $F = 0$ μ-ae, we have $\widetilde{\mu}(A) = 0$. Then, for every $\varepsilon > 0$, there exists $B_\varepsilon \in \mathcal{A}$ so that $A \subseteq B_\varepsilon$ and $\overline{\mu}(B_\varepsilon) < \varepsilon / M$. Let us take the partition $P_\varepsilon = \{C_i\}_{i \in \mathbb{N}}$ of T, such that $C_0 = T \setminus B_\varepsilon$, $\bigcup\limits_{i=1}^{\infty} C_i = B_\varepsilon$.

Let us consider an arbitrary partition P of T so that $P \geq P_\varepsilon$. Let $t_i \in D_i, i \in \mathbb{N}$ be arbitrarily chosen. Without any loss of generality, we suppose that $P = \{D_i, E_i\}_{i \in \mathbb{N}}$, with pairwise disjoint D_i, E_i such that $\bigcup\limits_{i \in \mathbb{N}} D_i = C_0$ and $\bigcup\limits_{i \in \mathbb{N}} E_i = B_\varepsilon$.

Let be $t_i \in D_i, s_i \in E_i$, for every $i \in \mathbb{N}$.
Now, we have for every $n \in \mathbb{N}$:

$$|\sum_{i=0}^{n} F(t_i)\mu(D_i) + \sum_{i=0}^{n} F(t_i)\mu(E_i)| = |\sum_{i=0}^{n} F(t_i)\mu(E_i)| \leq$$

$$\leq \sum_{i=0}^{n} |F(t_i)|\mu(E_i) \leq M \cdot \overline{\mu}(B_\varepsilon) < \varepsilon.$$

Hence, F is μ-integrable and $\int_T F d\mu = \{0\}$. □

Theorem 13: Let $F : T \to \mathcal{P}_0(X)$ be a μ-integrable multifunction. Then F is μ-integrable on $A \in \mathcal{A}$ if and only if $F\chi_A$ is μ-integrable on T, where χ_A is the characteristic function of A. In this case, $\int_A F d\mu = \int_T F\chi_A d\mu$.

Proof. I. Let us suppose that F is μ-integrable on $A \in \mathcal{A}$. Then for every $\varepsilon > 0$ there exist a partition $P_A^\varepsilon = \{D_n\}_{n \in \mathbb{N}} \in \mathcal{P}_A$ and $n_\varepsilon \in \mathbb{N}$ so that for every partition $P_A = \{B_m\}_{m \in \mathbb{N}}$ of A with $P_A \geq P_A^\varepsilon$ and for every $s_m \in B_m, m \in \mathbb{N}$, we have

$$h(\sum_{i=0}^{m} F(s_i)\mu(B_i), \int_A F d\mu) < \varepsilon, \forall m \geq n_\varepsilon. \quad (1)$$

Let us consider $P_\varepsilon = P_A^\varepsilon \cup \{T \setminus A\}$, which is a partition of T. If P is a partition of T with $P \geq P_\varepsilon$, then without any loss of generality we can suppose that $P = \{C_i, D_i\}_{i \in \mathbb{N}}$ with pairwise disjoint C_i, D_i such that $A = \cup_{i=0}^{\infty} C_i$ and $\cup_{i=0}^{\infty} D_i = T \setminus A$. Now, for every $t_i \in C_i, s_i \in D_i, i \in \mathbb{N}$ we get by (1):

$$h(\sum_{i=0}^{n} F(\chi_A)(t_i)\mu(C_i) + \sum_{i=0}^{n} F(\chi_A)(s_i)\mu(D_i), \int_A F d\mu) =$$

$$= h(\sum_{i=0}^{n} F(t_i)\mu(C_i), \int_A F d\mu) < \varepsilon,$$

for every $n \geq n_\varepsilon$, which says that $F\chi_A$ is μ-integrable on T and $\int_T F\chi_A d\mu = \int_A F d\mu$.

II. Suppose that $F\chi_A$ is μ-integrable on T. Then for every $\varepsilon > 0$ there exist $P_\varepsilon = \{B_n\}_{n \in \mathbb{N}} \in \mathcal{P}$ and $n_\varepsilon \in \mathbb{N}$ so that for every $P = \{E_n\}_{n \in \mathbb{N}}$ partition of T with $P \geq P_\varepsilon$ and every $s_n \in E_n, n \in \mathbb{N}$, we have

$$h(\sum_{k=0}^{n} (F\chi_A)(s_k)\mu(E_k), \int_T (F\chi_A) d\mu) < \varepsilon, \forall n \geq n_\varepsilon. \quad (2)$$

Let us consider $P_A^\varepsilon = \{B_n \cap A\}_{n \in \mathbb{N}}$, which is a partition of A. Let us take $P_A = \{D_n\}_{n \in \mathbb{N}}$ an arbitrary partition of A with $P_A \geq P_A^\varepsilon$ and $P = P_A \cup \{T \setminus A\}$. Then $P \in \mathcal{P}$ and $P \geq P_\varepsilon$. Let us take $t_n \in D_n, n \in \mathbb{N}$ and $s \in T \setminus A$. By (2) we obtain

$$h(\sum_{k=0}^{n} F(t_k)\mu(D_k), \int_T F\chi_A d\mu) =$$

$$= h(\sum_{k=0}^{n} (F\chi_A)(t_k)\mu(D_k) + (F\chi_A)(s)\mu(T \setminus A), \int_T F\chi_A d\mu) < \varepsilon,$$

$\forall n \geq n_\varepsilon$, which assures that F is μ-integrable on A. □

Theorem 14: Let $F, G : T \to \mathcal{P}_0(X)$ be μ-integrable multifunctions. Then $F + G$ is μ-integrable and

$$\int_T (F + G) d\mu = \overline{\int_T F d\mu + \int_T G d\mu}. \quad (3)$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since F is μ-integrable, then there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P \in \mathcal{P}, P = \{A_n\}_{n\in\mathbb{N}}$, with $P \geq P_1$ and every $t_n \in A_n, n \in \mathbb{N}$, we have

$$h\left(\sum_{k=0}^{n} F(t_k)\mu(A_k), \int_T F d\mu\right) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1. \quad (4)$$

Analogously, because G is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P \in \mathcal{P}, P = \{B_n\}_{n\in\mathbb{N}}$, with $P \geq P_2$ and every $t_n \in B_n, n \in \mathbb{N}$, we have

$$h\left(\sum_{k=0}^{n} G(t_k)\mu(B_k), \int_T G d\mu\right) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2. \quad (5)$$

Let be $P_0 = P_1 \wedge P_2$ and $n_0 = \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Then, for every partition $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_0$ and $t_n \in C_n, n \in \mathbb{N}$, by (4) and (5), we get

$$\begin{aligned} &h(\sum_{k=0}^{n}(F+G)(t_k)\mu(C_k), \int_T F d\mu + \int_T G d\mu) \\ &= h(\sum_{k=0}^{n} F(t_k)\mu(C_k) + + \sum_{k=0}^{n} G(t_k)\mu(C_k), \int_T F d\mu \\ &+ \int_T G d\mu) \leq h(\sum_{k=0}^{n} F(t_k)\mu(C_k), \int_T F d\mu) + \\ &+ h(\sum_{k=0}^{n} G(t_k)\mu(C_k), \int_T G d\mu) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $F + G$ is μ-integrable and (3) is satisfied. $\square$

Theorem 15: If $F, G : T \to \mathcal{P}_0(X)$ are μ-integrable multifunctions, then

$$h\left(\int_T F d\mu, \int_T G d\mu\right) \leq \sup_{t\in T} h(F(t), G(t)) \cdot \overline{\mu}(T).$$

Proof. If $\sup_{t\in T} h(F(t), G(t)) = +\infty$, then the conclusion is obvious.

Suppose $\sup_{t\in T} h(F(t), G(t)) < +\infty$. Let $\varepsilon > 0$ be arbitrary. Since F is μ-integrable, then there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_1$ and $t_n \in A_n, n \in \mathbb{N}$, we have

$$h(\int_T F d\mu, \sum_{k=0}^{n} F(t_k)\mu(A_k)) < \frac{\varepsilon}{4}, \forall n \geq n_\varepsilon^1. \quad (6)$$

Analogously, because G is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_2$ and $t_n \in B_n, n \in \mathbb{N}$, we have

$$h(\int_T G d\mu, \sum_{k=0}^{n} G(t_k)\mu(B_k)) < \frac{\varepsilon}{4}, \forall n \geq n_\varepsilon^2. \quad (7)$$

Let be $P_1 \wedge P_2 \in \mathcal{P}$, $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_1 \wedge P_2$ and $t_n \in C_n, n \in \mathbb{N}$ arbitrarily. Consider a fixed $n \in \mathbb{N}, n \geq \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Then from (6) and (7) it results

$$\begin{aligned} &h(\int_T F d\mu, \int_T G d\mu) \leq h(\int_T F d\mu, \sum_{k=0}^{n} F(t_k)\mu(C_k)) + \\ &+ h(\sum_{k=0}^{n} F(t_k)\mu(C_k), \sum_{k=0}^{n} G(t_k)\mu(C_k)) \\ &+ h(\sum_{k=0}^{n} G(t_k)\mu(C_k), \int_T G d\mu) < \\ &< \frac{\varepsilon}{2} + h(\sum_{k=0}^{n} F(t_k)\mu(C_k), \sum_{k=0}^{n} G(t_k)\mu(C_k)) \leq \\ &\leq \frac{\varepsilon}{2} + \sum_{k=0}^{n} h(F(t_k), G(t_k))\mu(C_k) < \frac{\varepsilon}{2} \\ &+ \sup_{t\in T} h(F(t), G(t)) \cdot \overline{\mu}(T), \end{aligned}$$

for every $\varepsilon > 0$. This implies $h(\int_T F d\mu, \int_T G d\mu) \leq \sup_{t\in T} h(F(t), G(t)) \cdot \overline{\mu}(T)$. $\square$

As a consequence of the previous theorem we obtain:

Corollary 16: If $F : T \to \mathcal{P}_0(X)$ is a μ-integrable multifunction, then

$$|\int_T F d\mu| \leq \sup_{t\in T} |F(t)| \cdot \overline{\mu}(T).$$

The next result easily follows by the definition.

Theorem 17: Let $F : T \to \mathcal{P}_0(X)$ be a μ-integrable multifunction and $\alpha \in \mathbb{R}$. Then:

I) αF is μ-integrable and

$$\int_T \alpha F d\mu = \alpha \int_T F d\mu.$$

II) F is $\alpha\mu$-integrable (for $\alpha \in [0, +\infty)$) and

$$\int_T F d(\alpha\mu) = \alpha \int_T F d\mu.$$

The next two results show that the set-valued integral is monotone with respect to the multifunction and to the set function.

Theorem 18: If $F, G : T \to P_0(X)$ are μ-integrable multifunctions so that $F(t) \subseteq G(t)$, for every $t \in T$, then $\int_T F d\mu \subseteq \int_T G d\mu$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since F is μ-integrable, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \geq P_1$ and every $t_n \in A_n, n \in \mathbb{N}$

$$h(\int_T F d\mu, \sum_{k=0}^{n} F(t_k)\mu(A_k)) < \frac{\varepsilon}{3}, \forall n \geq n_\varepsilon^1.$$

Analogously, since G is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \geq P_2$ and every $t_n \in B_n, n \in \mathbb{N}$

$$h(\int_T G d\mu, \sum_{k=0}^{n} G(t_k)\mu(B_k)) < \frac{\varepsilon}{3}, \forall n \geq n_\varepsilon^2.$$

Consider $P_0 = P_1 \wedge P_2$. Let $P \in \mathcal{P}$ be arbitrarily chosen, with $P = \{C_n\}_{n\in\mathbb{N}} \geq P_0$. Let be $t_n \in C_n$, $n \in \mathbb{N}$ and $n \geq \max\{n_\varepsilon^1, n_\varepsilon^2\}$. We get that $h(\int_T Fd\mu, \sum_{k=0}^{n} F(t_k)\mu(C_k)) < \frac{\varepsilon}{3}$ and $(\int_T Gd\mu, \sum_{k=0}^{n} G(t_k)\mu(C_k)) < \frac{\varepsilon}{3}$, which imply

$$e(\int_T Fd\mu, \int_T Gd\mu) \leq h(\int_T Fd\mu, \sum_{k=0}^{n} F(t_k)\mu(C_k))+$$
$$+ e(\sum_{k=0}^{n} F(t_k)\mu(C_k), \sum_{k=0}^{n} G(t_k)\mu(C_k))$$
$$+ h(\sum_{k=0}^{n} G(t_k)\mu(C_k), \int_T Gd\mu)) <$$
$$< \frac{2\varepsilon}{3} + e(\sum_{k=0}^{n} F(t_k)\mu(C_k), \sum_{k=0}^{n} G(t_k)\mu(C_k)).$$

From the hypothesis, it results $e(\sum_{k=0}^{n} F(t_k)\mu(C_k), \sum_{k=0}^{n} G(t_k)\mu(C_k)) = 0$. Consequently, $e(\int_T Fd\mu, \int_T Gd\mu) < \frac{2\varepsilon}{3}$, for every $\varepsilon > 0$, which implies $\int_T Fd\mu \subseteq \int_T Gd\mu$. □

We analogously obtain the following theorem.

Theorem 19: Let be μ_1, $\mu_2 : \mathcal{A} \to [0, +\infty)$ set functions such that $\mu_1(A) \leq \mu_2(A)$, for every $A \in \mathcal{A}$ and $F : T \to \mathcal{P}_c(X)$ a simultaneously μ_1-integrable and μ_2-integrable multifunction such that $0 \in F(t)$, for every $t \in T$. Then $\int_T Fd\mu_1 \subseteq \int_T Fd\mu_2$.

Theorem 20: Let be $\mu_1, \mu_2 : \mathcal{A} \to [0, +\infty)$, with $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$ and suppose $F : T \to \mathcal{P}_c(X)$ is both μ_1-integrable and μ_2-integrable. If $\mu : \mathcal{A} \to [0, +\infty)$ is the set function defined by $\mu(A) = \mu_1(A) + \mu_2(A)$, for every $A \in \mathcal{A}$, then F is μ-integrable and $\int_T Fd(\mu_1 + \mu_2) = \int_T Fd\mu_1 + \int_T Fd\mu_2$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since F is μ_1-integrable, then there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \geq P_1$ and $t_n \in A_n, n \in \mathbb{N}$ we have

$$h\left(\int_T Fd\mu_1, \sum_{k=0}^{n} F(t_k)\mu_1(A_k)\right) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1. \quad (8)$$

Since F is μ_2-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \geq P_2$ and $t_n \in B_n, n \in \mathbb{N}$ we have

$$h(\int_T Fd\mu_2, \sum_{k=0}^{n} F(t_k)\mu_2(B_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2. \quad (9)$$

Let be $n \geq \max\{n_\varepsilon^1, n_\varepsilon^2\}$, $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \geq P_1 \wedge P_2$ and $t_n \in C_n, n \in \mathbb{N}$. Then, by (8) and (9), we get

$$h(\sum_{k=0}^{n} F(t_k)\mu(C_k), \int_T Fd\mu_1 + \int_T Fd\mu_2) \leq$$
$$\leq h(\sum_{k=0}^{n} F(t_k)\mu_1(C_k), \int_T Fd\mu_1)$$
$$+ h(\sum_{k=0}^{n} F(t_k)\mu_2(C_k), \int_T Fd\mu_2) < \varepsilon,$$

which concludes the proof. □

Theorem 21: Suppose $\mu : \mathcal{A} \to [0, +\infty)$ is finitely additive. *Let $F, G : T \to \mathcal{P}_f(X)$ be multifunctions with* $\sup_{t\in T} h(F(t), G(t)) < +\infty$ such that *F is μ-integrable and $F = G$ μ-ae. Then G is μ-integrable and $\int_T Fd\mu = \int_T Gd\mu$.*

Proof. Let be $M = \sup_{t\in T} h(F(t), G(t))$. If $M = 0$, then $F = G$ and the conclusion is evident.

Suppose $M > 0$ and let $\varepsilon > 0$ be arbitrary.

Since F is μ-integrable, there exist $P_\varepsilon = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$ and $n_\varepsilon \in \mathbb{N}$ so that for every $P = \{B_n\}_{n\in\mathbb{N}}$, with $P \geq P_\varepsilon$ and every $t_n \in B_n, n \in \mathbb{N}$

$$h(\sum_{k=0}^{n} F(t_k)\mu(B_k)), \int_T Fd\mu) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon. \quad (10)$$

Let $E \subset T$ be such that $F = G$ on $T \setminus E$ and $\widetilde{\mu}(E) = 0$. By the definition of $\widetilde{\mu}$, there is $A \in \mathcal{A}$ so that $E \subseteq A$ and $\overline{\mu}(A) < \frac{\varepsilon}{4M}$.

Consider $P_0 = \{A \cap A_n, A_n \setminus A\}_{n\in\mathbb{N}} \in \mathcal{P}$. Let also be the arbitrary partition $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_0$ and $t_n \in B_n, n \in \mathbb{N}$. Then, without any loss of generality we suppose that $B_n = B_n^{'} \cup B_n^{''}$, with $\bigcup_{n\in\mathbb{N}} B_n^{'} = A$ and $\bigcup_{n\in\mathbb{N}} B_n^{''} = T \setminus A$.

Consider a fixed $n \geq n_\varepsilon$. Since μ is finitely additive, by (10) we get

$$h(\int_T Fd\mu, \sum_{k=0}^{n} G(t_k)\mu(B_k)) \leq h(\int_T Fd\mu, \sum_{k=0}^{n} F(t_k)\mu(B_k))+$$
$$+ h(\sum_{k=0}^{n} F(t_k)\mu(B_k), \sum_{k=0}^{n} G(t_k)\mu(B_k)) < \frac{\varepsilon}{2}$$
$$+ h(\sum_{k=0}^{n} F(t_k)\mu(B_k), \sum_{k=0}^{n} G(t_k)\mu(B_k)) \leq$$
$$\leq \frac{\varepsilon}{2} + h(\sum_{k=0}^{n} F(t_k)\mu(B_k^{'}), \sum_{k=0}^{n} G(t_k)\mu(B_k^{'}))$$
$$+ h(\sum_{k=0}^{n} F(t_k)\mu(B_k^{''}), \sum_{k=0}^{n} G(t_k)\mu(B_k^{''})) \leq$$
$$\leq \frac{\varepsilon}{2} + \sum_{k=0}^{n} h(F(t_k), G(t_k))\mu(B_k^{'}) + \sum_{k=0}^{n} h(F(t_k), G(t_k))\mu(B_k^{''}).$$

Since for every $k = \overline{0, n}$, $B_k^{''} \subset T \setminus A \subset T \setminus E$ and $F = G$ on $T \setminus E$, then

$$h(\int_T Fd\mu, \sum_{k=0}^{n} G(t_k)\mu(B_k)) < \frac{\varepsilon}{2}$$
$$+ \sum_{k=0}^{n} h(F(t_k), G(t_k)) \cdot \mu(B_k^{'}) \leq$$
$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k=0}^{n} \mu(B_k^{'}) \leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k=0}^{n} \overline{\mu}(B_k^{'}) =$$
$$= \frac{\varepsilon}{2} + 2M \cdot \overline{\mu}(\bigcup_{k=0}^{n} B_k^{'}) \leq \frac{\varepsilon}{2} + 2M \cdot \overline{\mu}(A) < \varepsilon.$$

This concludes the proof. □

IV. Properties of the set valued integral

In this section we get some results concerning the properties of the set multifunction $\varphi : \mathcal{A} \to \mathcal{P}_f(X)$ defined by $\varphi(A) = \int_A F d\mu$, for every $A \in \mathcal{A}$, where $F : T \to \mathcal{P}_0(X)$ is μ-integrable on every set $A \in \mathcal{A}$.

Theorem 22: Let $F : T \to \mathcal{P}_0(X)$ be a multifunction such that F is μ-integrable on every set $A \in \mathcal{A}$. Then:

I. $\varphi \ll \mu$ (i.e., for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $A \in \mathcal{A}$ with $\overline{\mu}(A) < \delta$, it results $|\varphi(A)| < \varepsilon$).

II. If μ is finitely additive, then φ is finitely additive too.

III. Suppose F *is $\mathcal{P}_c(X)$-valued. If μ is monotone, then the same is φ.*

IV. Suppose F is bounded. *If μ is of finite variation, then φ is of finite variation.*

V. If $\overline{\mu}$ is o-continuous (exhaustive respectively), then φ is also o-continuous (exhaustive respectively).

Proof. I. It results from Corollary 3.8.

II. Evidently, $\varphi(\emptyset) = \{0\}$. Let be $A, B \in \mathcal{A}, A \cap B = \emptyset$ and $\varepsilon > 0$.

Since F is μ-integrable on A, there exist a partition $P_A^\varepsilon = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$, $P \geq P_A^\varepsilon$ and $t_n \in E_n, n \in \mathbb{N}$, we have

$$h(\int_A F d\mu, \sum_{k=0}^{n} F(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1. \tag{11}$$

Analogously, since F is μ-integrable on B, we find a partition $P_B^\varepsilon = \{D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$, with $P \geq P_B^\varepsilon$, and $t_n \in E_n, n \in \mathbb{N}$, we have

$$h(\int_B F d\mu, \sum_{k=0}^{n} F(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2. \tag{12}$$

Now, let be the partition $P_{A\cup B}^\varepsilon = \{C_n, D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{A\cup B}$ and $n_\varepsilon = \max\{n_\varepsilon^1, n_\varepsilon^2\}$. If we consider $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{A\cup B}$ such that $P \geq P_{A\cup B}^\varepsilon$, then from (11) and (12) we have

$$\begin{aligned} &h(\sum_{k=0}^{n} F(t_k)\mu(E_k), \int_A F d\mu \dot{+} \int_B F d\mu) \\ &\leq h(\sum_{k=0}^{n} F(t_k)\mu(E_k \cap A), \int_A F d\mu) + \\ &+ h(\sum_{k=0}^{n} F(t_k)\mu(E_k \cap B), \int_B F d\mu) < \varepsilon, \forall n \geq n_\varepsilon. \end{aligned}$$

So, $\int_{A\cup B} F d\mu = \overline{\int_A F d\mu + \int_B F d\mu}$ and thus φ is finitely additive.

III. The proof is similar to that of Theorem 3.10.

IV. Let $\{A_i\}_{i=\overline{1,n}} \subset \mathcal{P}(T)$ be pairwise disjoint sets and $M = \sup_{t\in T} |F(t)|$. By Corollary 3.8 it follows $\sum_{i=1}^{n} |\varphi(A_i)| \leq M \sum_{i=1}^{n} \overline{\mu}(A_i) \leq M\ \overline{\mu}(T)$. This implies $\overline{\varphi}(T) \leq M\ \overline{\mu}(T)$, for every $A \in \mathcal{A}$, which assures that φ is of finite variation.

V. It results from Corollary 3.8. □

Theorem 23: Suppose $\mu : \mathcal{A} \to [0, +\infty)$ is a finitely additive set function of finite variation. Let $F : T \to \mathcal{P}_0(X)$ be a bounded multifunction such that F is μ-integrable on every set $A \in \mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{P}_f(X)$ defined by $\varphi(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}$. Then the following properties hold:

I. If μ is o-continuous (increasing convergent respectively), then the same is φ.

II. If μ is σ-additive, then φ is an h-multimeasure.

Proof. I. The o-continuity follows from Remark 2.6-IV and Theorem 4.1-V. Now, suppose μ is increasing convergent. Let $\varepsilon > 0$ be arbitrary and let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{A}$ be so that $A_n \nearrow A \in \mathcal{A}$. Let $M = \sup_{t\in T} |F(t)|$. If $M = 0$, then $F(t) = 0$, for every $t \in T$ and the conclusion is evident. Suppose $M > 0$. From Theorem 4.1 - II and Corollary 3.8, we have:

$$\begin{aligned} &h(\varphi(A_n), \varphi(A)) \\ &= h(\int_{A_n} F d\mu, \int_{A_n} F d\mu + \int_{A\setminus A_n} F d\mu) \leq \\ &\leq |\int_{A\setminus A_n} F d\mu| \leq M\overline{\mu}(A\setminus A_n) = M(\overline{\mu}(A) - \overline{\mu}(A_n)). \end{aligned} \tag{13}$$

Now, let $\{B_i\}_{i=\overline{1,m}} \subset \mathcal{A}$ be an arbitrary partition of A. Then $B_i \cap A_n \subset B_i \cap A_{n+1}$, for every $n \in \mathbb{N}^*$, $i = \overline{1,m}$, and $\bigcup_{n=1}^{\infty} (B_i \cap A_n) = B_i \cap A = B_i$, for every $i = \overline{1,m}$. Since μ is increasing convergent, for every $i = \overline{1,m}$, there exists $n_0^i(\varepsilon) \in \mathbb{N}$ so that, for every $n \geq n_0^i(\varepsilon)$, $\mu(B_i) - \mu(B_i \cap A_n) < \frac{\varepsilon}{2^i \cdot M}$.

Consequently,

$$\sum_{i=1}^{m} \mu(B_i) \leq \sum_{i=1}^{m} \mu(B_i \cap A_n) + \sum_{i=1}^{m} \frac{\varepsilon}{2^i \cdot M} < \overline{\mu}(A_n) + \frac{\varepsilon}{M},$$

for every $n \geq n_0 = \max_{i=\overline{1,m}} \{n_0^i(\varepsilon)\}$.

Then $\overline{\mu}(A) \leq \overline{\mu}(A_n) + \frac{\varepsilon}{M}$ and by (13) it results φ is increasing convergent.

II. Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{A}$ be a sequence of pairwise disjoint sets, with $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{A}$. Since μ is σ-additive, then it is o-continuous, so, by I, the same is true for the set multifunction φ. Because $B_n = \bigcup_{k=n+1}^{\infty} A_k \searrow \emptyset$ and $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{A}$, there exists $n_0(\varepsilon) \in \mathbb{N}^*$ so that $|\varphi(B_n)| < \varepsilon$, for every $n \geq n_0(\varepsilon)$. Since φ is finitely additive, we have

$$\begin{aligned} &h(\varphi(A), \sum_{k=1}^{n} \varphi(A_k)) = h(\sum_{k=1}^{n} \varphi(A_k) + \varphi(B_n), \sum_{k=1}^{n} \varphi(A_k)) \\ &\leq |\varphi(B_n)| < \varepsilon, \end{aligned}$$

for every $n \geq n_0$, that is, φ is an h-multimeasure. □

V. Conclusions

We have defined and studied Birkhoff weak integrability of multifunctions (taking values in the family of nonempty subsets of a real Banach space) relative to a non-negative set function. Some properties of the set-valued integral are obtained such as linearity, monotonicity, continuity.

Since the definition of Birkhoff weak integral is similar to the definitions of Birkhoff and Gould integrals, one has to compare these three types of set-valued integrals and this will be the subject of our future works.

As open problems:

1. Integrability is usually related to measurability. So, it has to see if there exists a relationship between Birkhoff weak integrability and some measurability type of multifunctions.

2. As it is known in functions case, Birkhoff integrability lies strictly between Bochner and Pettis integrability. Thus, it has to compare this Birkhoff weak set-valued integral with other types of set-valued integrals: Pettis, Dunford, Aumann, McShane, Henstock-Kurzweil etc.

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