SPECTRAL EQUIVALENCE OF S-DECOMPOSABLE OPERATORS

Cristina Serbanescu
Faculty of Applied Sciences
University Politehnica of Bucharest
E-mail: mserbanescuc@yahoo.com

Ioan Bacalu
Faculty of Applied Sciences
University Politehnica of Bucharest
Email: dragosx@yahoo.com

***Abstract*—One of the essential characteristics of the class of decomposa-ble (spectral, scalar, generalized scalar and spectral, $\mathcal{A}$-scalar) operators is the transfer of spectral proprieties from one operator to another using quasinilpotent equivalence ([14]). The family of S-decomposable operators, although larger than the class of decomposable operators studied in several papers (about 40), preserves the most interesting properties of the class of decomposable ope-rators. In this paper we make the link between S-decomposable operators and spectral equivalence (respectively, S-spectral equi-valence). As is known, for two decomposable operators T_1 and T_2 which are spectral equivalent, the spectral properties of T_1 transfer to T_2 (Theorems II.1, II.2, II.3 and Consequence II.1, [14]). We prove that this fact remains partially true for S-decomposable operators, because these operators behave differently and distinctly with respect to spectral equivalence; in this case, the spectral equivalence is not "equivalent" to equality of spectral maximal spaces $X_{T_1}(F) = X_{T_2}(F)$; this equality involves only a weaker property called S-spectral equivalence, which is natural in this case.**

To show the relevance and the necessity of studying the above stated property for the family of S-decomposable operators, we emphasize the consistency of this class, in the sense of how many and varied are the subfamilies that compose it: the restrictions and the quotients (with respect to an invariant subspace) of decomposable (unitary, self-adjoint, normal, spectral (scalar), genera-lized spectral (scalar), $\mathcal{A}$-scalar, $\mathcal{A}$-unitary) operators; the perturbations and the direct sums composed by one decomposable operator and another operator; the subscalar (subnormal, subdecomposable) operators are S-decomposable (practically, S-scalar, S-normal), as restrictions of scalar (respectively, normal, decomposable) operators. Putinar showed that the hiponormal ope-rators are subscalar, hence S-decomposable. The quasinormal operators (i.e. T commutes with T^*T), being subnormal, are S-decomposable; for cosubnormal operators (i.e. T^* is subnormal), the adjointable operators T^* are S-decomposable. Cesaro operators are subscalar, hence S-scalar and S-decomposable; the operators which admit scalar dilatations (extensions) (C. Ionescu-Tulcea) or $\mathcal{A}$-scalar dilatations (El. Stroescu) are S-decomposable. In fact, Albrecht and Eschmeier showed that any operator is the quotient of a restriction or the restriction of a quotient of decomposable operators ([3]), thus any operator is S-decomposable or similar to an S-decomposable operator.

AMS 2000 Mathematics Subject Classification: 47B47, 47B40. **Keywords:** decomposable (strongly decomposable); S-decomposable (strongly S-decomposable); spectral equivalence (S-spectral equivalence).

I. Introduction

Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathbb{C}$ be the field of complex numbers. An operator $T \in \mathbf{B}(X)$ is said to have the *single-valued extension proper-ty* if for any analytic function $f\colon D_f \to X$ (where $D_f \subset \mathbb{C}$ open) with $(\lambda I - T) f(\lambda) \equiv 0$ it results that $f(\lambda) \equiv 0$ ([14], [18]).

For an operator $T \in \mathbf{B}(X)$ having the single-valued extension proper-ty and for $x \in X$, we consider the set $\rho_T(x)$ of all elements $\lambda_0 \in \mathbb{C}$ such that there is a X-valued analytic function $\lambda \to x(\lambda)$ defined on a neighborhood of λ_0 which verifies $(\lambda I - T) x(\lambda) \equiv x$; $x(\lambda)$ is unique, $\rho_T(x)$ is open and $\rho(T) \subset \rho_T(x)$. We take $\sigma_T(x) = \complement\rho_T(x) = \mathbb{C} \setminus \rho_T(x)$ and

$$X_T(F) = \{x \in X; \sigma_T(x) \subset F\},$$

where $F \subset \mathbb{C}$ is closed.
$\rho_T(x)$ is called the *local resolvent set of x with respect to T* and $\sigma_T(x)$ is the *local spectrum of x with respect to T*.

If $T \in \mathbf{B}(X)$ and Y is a (closed) subspace of X invariant to T, let us denote by $T|Y$ the restriction of T to Y. In what follows, by subspace of X we understand a closed linear manifold of X. Recall that $Y \subset X$ is a *spectral maximal space* of T if it is an invariant subspace to T such that for any other subspace Z of X, invariant to T, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies the inclusion $Z \subset Y$ ([14]).

An operator $T \in \mathbf{B}(X)$ is *decomposable* if for any finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that $\sigma(T|Y_i) \subset G_i$ $(i = 1, 2, ..., n)$ and $X = Y_1 + Y_2 + ... + Y_n$ ([14], [19]). An operator T is *strongly decomposable* if $T|Y$ is decomposable, for any spectral maximal space Y of T.

In order to study the link between S-decomposable operators and spectral equivalence, we need several notions from the theory of residually spectral decompositions brought up by F.H. Vasilescu in [25], [26], [27].

An open set $\Omega \subset \mathbb{C}$ is said to be a *set of analytic uniqueness* for $T \in \mathbf{B}(X)$ if for any open set $\omega \subset \Omega$ and any analytic function $f_0\colon \omega \to X$ satisfying the equation $(\lambda I - T) f_0(\lambda) \equiv 0$, it follows that $f_0(\lambda) \equiv 0$ in ω. For $T \in \mathbf{B}(X)$, there is a unique *maximal open set Ω_T of analytic uniqueness* (2.1., [25]). We denote by $S_T = \complement\Omega_T = \mathbb{C} \setminus \Omega_T$ and call it the *analytic spectral residuum of T*.
For $x \in X$, a point λ is in $\delta_T(x)$ if in a neighborhood V_λ of λ there is at least an analytic function f_x (called *T- associated to x*) such that $(\mu I - T) f_x(\mu) \equiv x$, for all $\mu \in V_\lambda$. We shall put

$$\gamma_T(x) = \complement\delta_T(x), \rho_T(x) = \delta_T(x) \cap \Omega_T,$$

$$\sigma_T(x) = \complement\rho_T(x) = \gamma_T(x) \cup S_T$$

and

$$X_T(F) = \{x \in X; \sigma_T(x) \subset F\},$$

where $S_T \subset F \subset \mathbb{C}$ ([25]).

$T \in \mathbf{B}(X)$ has the single-valued extension property if and only if $S_T = \emptyset$; then we have $\sigma_T(x) = \gamma_T(x)$ and there is in $\rho_T(x) = \delta_T(x)$ a unique analytic function $x(\lambda)$, T-associated to x, for any $x \in X$. We shall recall that if $T \in \mathbf{B}(X)$, $S_T \neq \emptyset$ and $X_T(F)$ is closed, for $F \subset \mathbb{C}$ closed, $S_T \subset F$, then $X_T(F)$ is a spectral maximal space of T and $\sigma(T|X_T(F)) \subset F$ ([25], Propositions 2.4 and 3.4).

II. Preliminaries

Definition II.1. A finite family of open sets $G_S \cup \{G_i\}_{i=1}^n$ is said to be an *S-covering* of the closed set $\sigma \subset \mathbb{C}$ if

$$G_S \cup \left(\bigcup_{i=1}^n G_i\right) \supset \sigma \cup S \quad \text{and} \quad \overline{G}_i \cap S = \emptyset \quad (i = 1, 2, ..., n)$$

($S \subset \mathbb{C}$ also closed) ([25]).

Definition II.2. Let $T \in \mathbf{B}(X)$ and let $S \subset \sigma(T)$ be a compact set. T is called *S-decomposable* (see also [11]) if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $Y_S \cup \{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that

(i) $\sigma(T|Y_S) \subset G_S$, $\sigma(T|Y_i) \subset G_i$ $(i = 1, 2, \ldots, n)$

(ii) $X = Y_S + \sum_{i=1}^n Y_i$.

T is *strongly S-decomposable* if the condition (ii) is replaced by

(ii') $Z = (Z \cap Y_S) + \sum_{i=1}^n (Z \cap Y_i)$

where Z is any spectral maximal space of T and we shall say that T is *weakly S-decomposable* if the same condition (ii) is replaced by

(ii") $X = \overline{Y_S + \sum_{i=1}^n Y_i}$.

If in the definition of S-decomposability, Y_S is not necessarily a spectral maximal space of T and $\sigma(T|Y_S) \subset \widetilde{G}_S$, then we say that $T \in \mathbf{D}_S$ (if $A \subset \mathbb{C}$ is bounded, we denote $\widetilde{A} = \mathbb{C} \setminus D^\infty$, where D^∞ is the unbounded component of $\mathbb{C} \setminus A$).

An operator $T \in \mathbf{B}(X)$ is called *(m, S)-decomposable* if in the defi-nition of S-decomposability we consider the S-covering composed by exactly $m + 1$ sets; if $m = 1$ we have $(1, S)$-decomposability. Recall that T is S-decomposable if and only if T is $(1, S)$-decomposable.

Definition II.3. We say that T_1, $T_2 \in \mathbf{B}(X)$ are *spectral equivalent* (or *quasi-nilpotent equivalent*, see [14], Definition 2.1.) and write $T_1 \sim T_2$, if

$$\lim_{n\to\infty} \left\|(T_1 - T_2)^{[n]}\right\|^{\frac{1}{n}} = 0 \text{ and } \lim_{n\to\infty} \left\|(T_2 - T_1)^{[n]}\right\|^{\frac{1}{n}} = 0,$$

where

$$(T_1 - T_2)^{[n]} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} T_1^k T_2^{n-k}.$$

This relation is reflexive, symmetric and transitive.

Theorem II.1. ([14]) *Let T_1, $T_2 \in \boldsymbol{B}(X)$. If $T_1 \sim T_2$, then $\sigma(T_1) = \sigma(T_2)$.*

Theorem II.2. ([14]) *Let $T_1, T_2 \in \boldsymbol{B}(X)$. If T_1 has the single-valued extension property and $T_1 \sim T_2$, then T_2 has also the single-valued extension property and $\sigma_{T_1}(x) = \sigma_{T_2}(x)$, for every $x \in X$.*

Theorem II.3. ([14]) *If $T_1 \in \boldsymbol{B}(X)$ is decomposable and $T_1 \sim T_2$, then $T_2 \in \boldsymbol{B}(X)$ is also decomposable and*

$$X_{T_1}(F) = X_{T_2}(F), \ \ F \subset \mathbb{C} \ \ closed.$$

Conversely, if T_1, $T_2 \in \boldsymbol{B}(X)$ are decomposable operators such that $X_{T_1}(F) = X_{T_2}(F)$, for every $F \subset \mathbb{C}$ closed, then $T_1 \sim T_2$.

Consequence II.1. ([14]) *Let T_1, $T_2 \in \boldsymbol{B}(X)$ be decomposable operators. Then T_1 is spectral equivalent with T_2 if and only if*

$$X_{T_1}(F) = X_{T_2}(F)$$

for every $F \subset \mathbb{C}$ closed.

III. S-decomposable operators spectral equivalent

Definition III.1. Let $T \in \mathbf{B}(X)$ and let $S \subset \mathbb{C}$ be a compact set. T is said to verify *condition α_S* if $X_T(F)$ is closed for any closed $F \supset S$. T is also said to verify *condition β_S* if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$ and for any $x \in X$ we have

$$x = x_S + x_1 + x_2 + ... + x_n$$

where

$$\gamma_T(x_S) \subset G_S, \gamma_T(x_i) \subset G_i \ \ (i = 1, 2, ..., n) \ \ ([11], [26]).$$

T is said to verify *strongly condition β_S* if for any spectral maximal space Y of T, the restriction $T|Y$ verifies condition β_{S_1}, where $S_1 = S \cap \sigma(T|Y)$.

Lemma III.1. *An operator $T \in \boldsymbol{B}(X)$ is S-decomposable if and only if it verifies conditions α_S and β_S.*

Proof: Let $G_S \cup \{G_i\}_{i=1}^n$ be a finite open S-covering of $\sigma(T)$. Since $\overline{G}_i \cap S = \emptyset$ $(1 \le i \le n)$ we have

$$X_T(\overline{G}_i \cup S) = Y_i \oplus Y_S$$

where $X_T(\overline{G}_i \cup S)$, Y_i, Y_S are spectral maximal spaces of T (see Propositions 2.4. and 3.4., [25]). Also, if Y is a spectral maximal space of T we have $\delta_{T|Y}(x) \subset \delta_T(x)$, for any $x \in Y$, hence $\gamma_T(x) \subset \gamma_{T|Y}(x) \subset \sigma(T|Y)$, for any $x \in Y$. If we considering these remarks, our assertion is obvious. ■

Proposition III.1. *Let T_1, $T_2 \in \boldsymbol{B}(X)$. If T_1 is S-decomposable with $S_{T_1} = \emptyset$ (i.e. T_1 has the single-valued extension property; particularly $\dim S \le 1$) and T_1, T_2 are spectral equivalent, then T_2 is also S-decomposable.*

Proof: If T_1, T_2 are spectral equivalent, then

$$\sigma(T_1) = \sigma(T_2), \sigma_{T_1}(x) = \sigma_{T_2}(x), \ \ x \in X \ \ (S_{T_1} = \emptyset \Rightarrow S_{T_2} = \emptyset)$$

and

$$X_{T_1}(F) = X_{T_2}(F)$$

for any $F \subset \mathbb{C}$ closed ([14]). Therefore T_2 also verifies conditions α_S and β_S, hence T_2 is also S-decomposable. ■

Proposition III.2. *Let $T_1, T_2 \in \boldsymbol{B}(X)$ with $S_{T_1} \neq \emptyset$, $S_{T_2} \neq \emptyset$. If T_1 is spectral equivalent with T_2, then*

$$\gamma_{T_1}(x) = \gamma_{T_2}(x),$$

for any $x \in X$.

Proof: We observe that the proof of this proposition is similar to the proof of Theorem 2.4. ([14], Chapter 1) and we present briefly the main steps. Let $\lambda_0 \in \delta_{T_1}(x)$; then there is an analytic function $x_1(\lambda)$ defined on a neighborhood ω of λ_0 such that $(\lambda I - T_1)x_1(\lambda) \equiv x$, for any $\lambda \in \omega$.

Let $\Delta_1 = \{\lambda; |\lambda - \lambda_0| \leq r_1\} \subset \Delta_2 = \{\lambda; |\lambda - \lambda_0| < r_2\} \subset \delta_{T_1}(x)$, with $r_1 < r_2$ and $M = \sup \|x_1(\lambda)\|$ on $\{\lambda; |\lambda - \lambda_0| = r_2\}$; then for $\lambda \in \Delta_1$ we have

$$\left\| \frac{x_1^{(n)}(\lambda)}{n!} \right\| = \left\| \frac{1}{2\pi i} \int_{|\mu - \lambda_0| = r_2} \frac{x_1(\mu)}{(\mu - \lambda)^{n+1}} \, \mathrm{d}\mu \right\|$$

$$\leq \frac{M \cdot r_2}{(r_2 - r_1)^{n+1}}.$$

From the fact that $\lim\limits_{n\to\infty} \left\| (T_2 - T_1)^{[n]} \right\|^{\frac{1}{n}} = 0$, it results that for every $\varepsilon > 0$, there is $M(\varepsilon) > 0$ such that

$$\left\| (T_2 - T_1)^{[n]} \right\| < M(\varepsilon) \cdot \varepsilon^n \ \ (n \geq 0)$$

and by taking $\varepsilon = \dfrac{r_2 - r_1}{2}$ we obtain

$$\left\| (T_2 - T_1)^{[n]} \frac{x_1^{(n)}(\lambda)}{n!} \right\| < \frac{r_2 \cdot M \cdot M(\varepsilon)}{r_2 - r_1} \cdot \left(\frac{\varepsilon}{r_2 - r_1} \right)^n \leq \frac{M}{2^n},$$

hence the series $x_2(\lambda) = \sum\limits_{n=0}^{\infty} (-1)^n (T_2 - T_1)^{[n]} \dfrac{x_1^{(n)}(\lambda)}{n!}$ is absolutely uniformly convergent in Δ_1; therefore, $\Delta_1 \subset \delta_{T_1}(x)$ being arbitrary, it converges absolutely and uniformly in every compact $K \subset \delta_{T_1}(x)$. It follows that $x_2(\lambda)$ is analytic on $\delta_{T_1}(x)$ and we show that $(\lambda I - T_2)x_2(\lambda) \equiv x$. If we take the n-times derivative of the identity $(\lambda I - T_1)x_1(\lambda) \equiv x$ we obtain

$$(\lambda I - T_1)x_1^{(n)}(\lambda) \equiv -n x_1^{(n-1)}(\lambda)$$

and

$$(\lambda I - T_2)x_2(\lambda) = \sum_{n=0}^{\infty} (-1)^n (\lambda I - T_2)(T_2 - T_1)^{[n]} \frac{x_1^{(n)}(\lambda)}{n!} =$$
$$= \sum_{n=0}^{\infty} (\lambda I - T_2)[(\lambda I - T_2) - (\lambda I - T_1)]^{[n]} \frac{x_1^{(n)}(\lambda)}{n!} =$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} (T_2 - T_1)^{[n+1]} \frac{x_1^{(n)}(\lambda)}{n!} + (\lambda I - T_1)x_1(\lambda) -$$
$$- \sum_{n=1}^{\infty} (-1)^n (T_2 - T_1)^{[n]} \frac{x_1^{(n-1)}(\lambda)}{(n-1)!} =$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} (T_2 - T_1)^{[n+1]} \frac{x_1^{(n)}(\lambda)}{n!} + (\lambda I - T_1)x_1(\lambda) -$$
$$- \sum_{n=0}^{\infty} (-1)^{n+1} (T_2 - T_1)^{[n+1]} \frac{x_1^{(n)}(\lambda)}{n!} = (\lambda I - T_1)x_1(\lambda) \equiv x,$$

i.e. $\delta_{T_1}(x) \subset \delta_{T_2}(x)$. In a similar way, we will show that $\delta_{T_2}(x) \subset \delta_{T_1}(x)$, thus $\delta_{T_1}(x) = \delta_{T_2}(x)$ and $\gamma_{T_1}(x) = \gamma_{T_2}(x)$. ■

Theorem III.1. *Let $T_1, T_2 \in \boldsymbol{B}(X)$ be two operators spectral equivalent. Then the analytic spectral residuum of T_1 is equal to the analytic spectral residuum of T_2, i.e. $S_{T_1} = S_{T_2}$.*

Proof: Let $f \colon \omega \to X$ be an analytic function that verifies the property $(\lambda I - T_2)f(\lambda) \equiv 0$, where $\omega \subset \Omega_{T_1}$. Then $(T_1 - T_2)^{[n]} f(\lambda) = (T_1 - \lambda I)^n f(\lambda)$, because $(\lambda I - T_2)^p f(\lambda) \equiv 0$ for $p \geq 1$ and

$$(T_1 - T_2)^{[n]} f(\lambda) = [(T_1 - \lambda I) - (T_2 - \lambda I)]^{[n]} f(\lambda) =$$
$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (T_1 - \lambda I)^k (T_2 - \lambda I)^{n-k} f(\lambda) = (T_1 - \lambda I)^n f(\lambda).$$

By the Cauchy's root criterion, the series $S = \sum\limits_{n=0}^{\infty} (T_1 - T_2)^{[n]} (\mu - \lambda)^{-n-1}$ is absolutely convergent in the uniform topology of $\mathbf{B}(X)$, for every $\mu \neq \lambda$, because

$$\lim_{n\to\infty} \left(\frac{\left\| (T_1 - T_2)^{[n]} \right\|}{|\mu - \lambda|^{n+1}} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left\| (T_1 - T_2)^{[n]} \right\|^{\frac{1}{n}}}{|\mu - \lambda|} = \frac{0}{|\mu - \lambda|} = 0.$$

According to relations (4.8.4.) and (5.2.3.) ([23]) we have

$$R(\mu, T_1) = \sum_{n=0}^{\infty} (T_1 - \lambda I)^n (\mu - \lambda)^{-n-1}$$

for every μ such that $|\mu - \lambda| > \|T_1 - \lambda I\|$, therefore

$$(\mu I - T_1) \left(\sum_{n=0}^{\infty} \frac{(T_1 - T_2)^{[n]}}{(\mu - \lambda)^{n+1}} \right) f(\lambda) =$$
$$(\mu I - T_1) \left(\sum_{n=0}^{\infty} \frac{(T_1 - \lambda I)^n}{(\mu - \lambda)^{n+1}} \right) f(\lambda) =$$
$$= (\mu I - T_1) R(\mu, T_1) f(\lambda) = f(\lambda), \ \text{ for any } \ |\mu - \lambda| > \|T_1 - \lambda I\|.$$

It follows that the analytic function (in $\mathbb{C} \setminus \{\lambda\} = \complement\{\lambda\}$)

$$g_\lambda(\mu) = Sf(\lambda) = \sum_{n=0}^{\infty}(T_1 - T_2)^{[n]}(\mu - \lambda)^{-n-1}f(\lambda)$$

verifies the relation

$$(\mu I - T_1)g_\lambda(\mu) = f(\lambda) \qquad (\alpha)$$

on a open set $\{\mu; |\mu - \lambda\| > \|T_1 - \lambda I\|\}$ and, by analytic extension, for every $\mu \neq \lambda$. Thus $\complement\{\lambda\} \subset \delta_{T_1}(f(\lambda))$, i.e. $\gamma_{T_1}(f(\lambda)) \subset \{\lambda\}$.

Now we have to show that $\gamma_{T_1}(f(\lambda)) = \emptyset$. Let $\lambda_0 \in \omega$, $\omega_0 = \{\lambda \in \mathbb{C}; |\lambda - \lambda_0| \leq r_0\} \subset \omega$ and λ, $\mu \in \mathrm{Int}\omega_0$. By integrating the identity (α) we find

$$(\mu I - T_1)\frac{1}{2\pi i}\int_{|\xi-\lambda_0|=r_0}\frac{g_\xi(\mu)}{\xi - \lambda}\,\mathrm{d}\xi = \frac{1}{2\pi i}\int_{|\xi-\lambda_0|=r_0}\frac{f(\xi)}{\xi - \lambda}\,\mathrm{d}\xi = f(\lambda)$$

$(\mu \longrightarrow \frac{1}{2\pi i}\int_{|\xi-\lambda_0|=r_0}\frac{g_\xi(\mu)}{\xi - \lambda}\,\mathrm{d}\xi$ is analytic on ω_0).

We obviously have $\mathrm{Int}\omega_0 = \{\mu \in \mathbb{C}; |\mu - \lambda_0| < r_0\} \subset \delta_{T_1}(f(\lambda))$, so $\{\lambda\} \subset \delta_{T_1}(f(\lambda))$.
It results that $\gamma_{T_1}(f(\lambda)) \subset \complement\{\lambda\}$, therefore $\gamma_{T_1}(f(\lambda)) \subset \{\lambda\} \cap \complement\{\lambda\} = \emptyset$. Then we have $f(\lambda) \equiv 0$ on ω, so $\Omega_{T_1} \subset \Omega_{T_2}$. In a similar way, we prove that $\Omega_{T_2} \subset \Omega_{T_1}$, hence $S_{T_1} = S_{T_2}$. ■

Theorem III.2. *Let $T_1, T_2 \in \boldsymbol{B}(X)$. If T_1 is S-decomposable and $T_1 \sim T_2$, then T_2 is also S-decomposable and*

$$X_{T_1}(F) = X_{T_2}(F),$$

for every closed $F \subset \mathbb{C}$, $F \supset S$.

Proof: According to the previous propositions, we have $S_{T_1} = S_{T_2}$, $\gamma_{T_1}(x)$
$= \gamma_{T_2}(x)$, $\sigma_{T_1}(x) = \sigma_{T_2}(x)$, for any $x \in X$, $X_{T_1}(F) = X_{T_2}(F)$, for any closed $F \subset \mathbb{C}$, $F \supset S$. Therefore T_2 also verifies conditions α_S and β_S. Then T_2 is S-decomposable. ■

Proposition III.3. *Let $T \in \boldsymbol{B}(X)$ and let σ be a separated part of $\sigma(T)$. Let*

$$E(\sigma, T) = \frac{1}{2\pi i}\int_\Gamma R(\lambda, T)\,\mathrm{d}\lambda$$

be the spectral projection corresponding to σ ([18], *VII p.3*) *(where Γ is a system of curves containing in $\rho(T)$ and surrounding σ). Then $E(\sigma, T)X$ is a spectral maximal space of T and $\sigma(T|E(\sigma, T)X) = \sigma$.*

Proof: The proof is presented in 1.3.10., [23]. ■

Proposition III.4. *Let $T \in \boldsymbol{B}(X)$ be S-decomposable and let F be a closed set, $F \subset \mathbb{C}, F \cap S = \emptyset$. Then the space Y_F from the equality*

$$X_T(S \cup F) = X_T(S) \oplus Y_F$$

given by

$$Y_F = E(F, T|X_T(S \cup F))X_T(S \cup F)$$

is spectral maximal space of T and $\sigma(T|Y_F) \subset F$, where $E(F, T|X_T(S \cup F))$ is one of the previous proposition.

Proof: Because $X_T(S \cup F)$ is a spectral maximal space of T and Y_F, according to previous proposition, it is a spectral maximal space of $T|X_T(S \cup F)$ and it results that Y_F is spectral maximal space of T. ■

Proposition III.5. *Let T_1, $T_2 \in \boldsymbol{B}(X)$ be two S-decomposable operators enjoying the same spectral maximal spaces, $X_{T_1}(F) = X_{T_2}(F)$, for any $F \supset S$ closed. Then the spectra of T_1 and T_2 are equal, $\sigma(T_1) = \sigma(T_2)$.*

Proof: $S_{T_1} \subset S$, $S_{T_2} \subset S$, $X_{T_1}(\sigma(T_1)) = X = X_{T_2}(\sigma(T_2))$, so we have:

$$\begin{aligned}\sigma(T_1) &= \sigma(T_1|X) = \sigma(T_1|X_{T_2}(\sigma(T_2))) \\ &= \sigma(T_1|X_{T_1}(\sigma(T_2))) \subset \sigma(T_2) \\ \sigma(T_2) &= \sigma(T_2|X) = \sigma(T_2|X_{T_1}(\sigma(T_1))) \\ &= \sigma(T_2|X_{T_2}(\sigma(T_1))) \subset \sigma(T_1).\end{aligned}$$

■

Proposition III.6. *An operator $T \in \boldsymbol{B}(X)$ is strongly S-decomposable if and only if $T|Y$ is strongly S_1-decomposable for any spectral maximal space Y of T, where $S_1 = S \cap \sigma(T|Y)$. Particularly, if $\sigma(T|Y) \cap S = \emptyset$, then $T|Y$ is strongly decomposable* (see 2.6.3., 2.6.4., 2.6.5., [12]).

Definition III.2. Let $T \in \mathbf{B}(X)$ and let $\sigma \subset \sigma(T)$ be a compact set. σ is a *set-spectrum* for T if there is an invariant subspace Y to T such that $\sigma(T|Y) = \sigma$.

Proposition III.7. *Let $T \in \boldsymbol{B}(X)$ be S-decomposable and let $\sigma \subset \sigma(T)$ such that $\sigma = \overline{\mathrm{Int}\sigma}$ (in the topology of $\sigma(T)$), with $\sigma \cap S = \emptyset$ or $\sigma \supset S$. Then σ and $\sigma' = \overline{\sigma(T) \setminus \sigma}$ are sets-spectrum for T and*

$$\sigma(T|Y_\sigma) = \sigma, \;\; \sigma(T|X_T(\sigma')) = \sigma' \;\; or$$
$$\sigma(T|Y_{\sigma'}) = \sigma', \;\; \sigma(T|X_T(\sigma)) = \sigma$$

($X_T(S \cup \sigma) = X_T(S) \oplus Y_\sigma$ or $X_T(S \cup \sigma') = X_T(S) \oplus Y_{\sigma'}$, $\sigma' \cap S = \emptyset$).

Proof: The proof of these assertions is similar to the proof from the case of decomposable operators ([12], Proposition 1.3.2.). ■

Proposition III.8. *According to the previous proposition, it results that an operator $T \in \boldsymbol{B}(X)$ is S-decomposable if and only if there is a system $Y_S \cup \{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that*

$$\sigma(T|Y_S) = \overline{G}_S, \;\; \sigma(T|Y_i) = \overline{G}_i \;\; (i = 1, 2, ..., n),$$
$$X = Y_S + Y_1 + Y_2 + ... + Y_n,$$

for any open finite S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$, where G_i, $G_S \subset \sigma(T)$ are open $(i = 1, 2, ..., n)$.

Proof: Indeed, if $G'_S \cup \{G'_i\}_{i=1}^n$ is a finite open S-covering of $\sigma(T)$ and $Y'_S \cup \{Y'_i\}_{i=1}^n$ is a corresponding system of spectral maximal spaces of T such that

$$\sigma(T|Y'_S) \subset G'_S, \;\; \sigma(T|Y'_i) \subset G'_i \;\; (i = 1, 2, ..., n),$$
$$X = Y'_S + Y'_1 + Y'_2 + ... + Y'_n,$$

then

$$\overline{G}_S = \overline{G'_S \cap \sigma(T)},\ \ \overline{G}_i = \overline{G'_i \cap \sigma(T)}\ \ (i = 1, 2, ..., n)$$

are sets-spectrum for T and

$$Y_S = X_T(\overline{G}_S),\ \ X_T(S \cup \overline{G}_i) = X_T(S) + Y_{\overline{G}_i}$$

Y_S, $Y_{\overline{G}_i} = Y_i$ being spectral maximal spaces of T with $\sigma(T|Y_S) = \overline{G}_S$, $\sigma(T|Y_i) = \overline{G}_i$.
But

$$\sigma(T|Y'_S) \subset G'_S \cap \sigma(T) \subset \overline{G}_S = \sigma(T|Y_S)$$

$$\sigma(T|Y'_i) \subset G'_i \cap \sigma(T) \subset \overline{G}_i = \sigma(T|Y_i) \quad (i = 1, 2, ..., n)$$

hence $Y'_S \subseteq Y_S$, $Y'_i \subseteq Y_i$ $(i = 1, 2, ..., n)$ and $X = Y'_S + Y'_1 + Y'_2 + ... + Y'_n \subseteq Y_S + Y_1 + Y_2 + ... + Y_n = X$. Conversely is obvious. ■

For the case of decomposable operators, according to Theorem II.3 and Consequence II.1, two operators T_1, $T_2 \in \mathbf{B}(X)$ are spectral equivalent, $T_1 \sim T_2$, if and only if the spectral maximal spaces $X_{T_1}(F)$, $X_{T_2}(F)$ are equal, for any $F \subset \mathbb{C}$ closed. This fact seems not be true in the case of S-decomposable operators; according to Theorem III.2, the spectral equivalence transfers the property of S-decomposability from one operator to another and the equality of the spectral spaces, $X_{T_1}(F) = X_{T_2}(F)$; but conversely, the equality of the spectral spaces does not really involve the spectral equivalence. Using direct sums composed by one decomposable operator and two operators which are not spectral equivalent, it can be created several examples in this sense.

Because we want also to fit these cases into a coherent theory, let us impose the concept of *S-spectral equivalence* (*residually spectral equivalence or spectral equivalence modulo S*).

Definition III.3. Let T_1, $T_2 \in \mathbf{B}(X)$ be S-decomposable with $\sigma(T_1) = \sigma(T_2)$. We say that T_1, T_2 are *S-spectral equivalent* if for any spectral maximal space Y of T_1 (or T_2), with $\sigma(T_1|Y) \cap S = \emptyset$ (or $\sigma(T_2|Y) \cap S = \emptyset$), the restrictions $T_1|Y$ and $T_2|Y$ are spectral equivalent.

Theorem III.3. *Let T_1, $T_2 \in \boldsymbol{B}(X)$ be strongly S-decomposable and $X_{T_1}(F) = X_{T_2}(F)$, for any $F \supset S$ closed. Then T_1 and T_2 are S-spectral equivalent.*

Proof: According to Proposition III.5, because the spectral spaces $X_{T_1}(F)$, $X_{T_2}(F)$ are equal, for any $F \supset S$ closed, it results that $\sigma(T_1) = \sigma(T_2)$.

We observe that any spectral maximal space Y of T_1 with the spectrum $\sigma(T_1|Y) \cap S = \emptyset$ is given by the relations

$$X_{T_1}(S \cup \sigma) = X_{T_1}(S) + Y_\sigma,\ \ Y = Y_\sigma, \sigma = \sigma(T_1|Y)$$

and it is also a spectral maximal space of T_2.

We recall that a spectral maximal space Y_1 of $T|Y$, where Y is spectral maximal space of T, is also a spectral maximal space of T.

T_1, T_2 being strongly S-decomposable, than the restrictions $T_1|Y$ and $T_2|Y$ are decomposable and, from Theorem III.2 (see also Preliminaries and Theorem II.3), are also spectral equivalent, hence T_1 and T_2 are S-spectral equivalent. ■

Corollary III.1. *Let T_1, $T_2 \in \boldsymbol{B}(X)$. If T_1 is strongly S-decomposable and T_1 is spectral equivalent with T_2, then T_2 is also strongly S-decomposable.*

Proof: According to Theorem III.2, T_2 is S-decomposable and the spectral maximal spaces $X_{T_1}(F)$, $X_{T_2}(F)$ are equal, with $F \supset S$ closed; it follows that any spectral maximal space Y of T_1, with $\sigma(T_1|Y) \cap S = \emptyset$, is also a spectral maximal space of T_2 (see Proposition III.4)
$(X_{T_1}(S \cup \sigma(T_1|Y)) = X_{T_2}(S \cup \sigma(T_1|Y)) = X_{T_1}(S) + Y = X_{T_2}(S) + Y)$.

Because $T_1|Y$ is decomposable (Proposition III.6) and $T_1|Y$ is spectral equivalent with $T_2|Y$, according to Theorem III.2, it results that $T_2|Y$ is S-decomposable, hence T_2 is strongly S-decomposable. ■

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