On the local property of $\varphi - |A, p_n|_k$ summability of factored Fourier series

H. ÖZARSLAN, Ş. YILDIZ

Abstract- In this paper, a more general theorem concerning the local property of $\varphi - |A, p_n|_k$ summability of factored Fourier series has been proved. Also some new results have been obtained.

Keywords- Summability factors, absolute matrix summability, Fourier series, Hölder inequality

I. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \qquad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [12]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \tag{3}$$

where

$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{4}$$

H. Özarslan is with the Mathematics Department, Erciyes University, 38038, Kayseri, Turkey
(e-mail:seyhan@erciyes.edu.tr).
Ş. Yıldız is with the Mathematics Department, Ahi Evran University, 40100, Kırşehir, Turkey
(e-mail:sebnemyildiz@ahievran.edu.tr).

In the special case, when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|\ C, 1\ |_k$ (resp. $|\bar{N}, p_n|$) summability. Also if we take $k = 1$ and $p_n = 1/n + 1$, summability $|\bar{N}, p_n|_k$ is equivalent to the summability $|R, logn, 1|$.
Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{5}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [23])

$$\sum_{n=1}^{\infty} n^{k-1} \ |\bar{\Delta} A_n(s)|^k < \infty, \tag{6}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [22])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{7}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [19])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{8}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability. Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\varphi - |A, p_n|_k$ reduces to $|C, 1|_k$ summability. Finally, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [5]).

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.
Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{9}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t), \tag{10}$$

where (a_n) and (b_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f (see [24]).
Before stating the main theorem, we must first introduce some further notations.
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{11}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{12}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{13}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{14}$$

II. KNOWN RESULTS

Mohanty [17] demonstrated that the $|R, logn, 1|$ summability of the factored Fourier series

$$\sum \frac{C_n(t)}{log(n+1)} \tag{15}$$

at $t = x$, is a local property of the generating function of $\sum C_n(t)$. Later on Matsumoto [15] improved this result by replacing the series (15) by

$$\sum \frac{C_n(t)}{\{loglog(n+1)\}^{1+\epsilon}}, \quad \epsilon > 0. \tag{16}$$

Generalizing the above result Bhatt [2] proved the following theorem.

Theorem 1 If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, logn, 1|$ of the series $\sum C_n(t)\lambda_n logn$ at a point can be ensured by a local property.
The local property problem of the factored Fourier series have been studied by several authors (see [1], [4], [6]-[7], [9]-[10], [13]-[14], [16], [18], [21]). Few of them are given above.
Furthermore, Bor [8] proved the following theorem in which the conditions on the sequence (λ_n) are somewhat more general than Theorem 1.

Theorem 2 Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $|\bar{N}, p_n|_k$ of the series $\sum C_n(t)\lambda_n P_n$ at a point is a local property of the generating function f.

III. MAIN RESULTS

The aim of this paper is to prove a more general theorem which includes of the above results as special cases.
Now, we shall prove the following theorem.

Theorem 3 Let $k \geq 1$. If $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{17}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{18}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{19}$$

and $\left(\frac{\varphi_n p_n}{P_n}\right)$ be a non-increasing sequence. If all the conditions of Theorem 2 are satisfied and (φ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} p_v\lambda_v = O(1) \quad as \quad m \to \infty, \tag{20}$$

$$\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} P_v\Delta\lambda_v = O(1) \quad as \quad m\to\infty, (21)$$

then the summability $\varphi - |A,p_n|_k$ of the series $\sum C_n(t)P_n\lambda_n$ at a point is a local property of the generating function f.

We need the following lemmas for the proof of our theorem.

Lemma 4 If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n\to\infty$, then $P_n\lambda_n = O(1)$ as $n\to\infty$ and $\sum P_n\Delta\lambda_n < \infty$ ([8]).

Lemma 5 Let $k \geq 1$ and $s_n = O(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent and the conditions (17)-(21) of Theorem 3 are satisfied, then the series $\sum a_n\lambda_n P_n$ is summable $\varphi - |A,p_n|_k$.

Proof of Lemma 5
Let (I_n) denote the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n P_n$. Then, by (13) and (14), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v\lambda_v \\
&= \sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v P_v)\sum_{r=1}^{v} a_r + \hat{a}_{nn}\lambda_n P_n\sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v P_v)s_v + a_{nn}\lambda_n P_n s_n \\
&= \sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv})\lambda_v P_v s_v + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v P_v s_v \\
&- \sum_{v=1}^{n-1}\hat{a}_{n,v+1}p_{v+1}\lambda_{v+1}s_v + a_{nn}\lambda_n P_n s_n \\
&= I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.
\end{aligned}$$

Since

$$\begin{aligned}
&|I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}|^k \\
&\leq 4^k(|I_{n,1}|^k+|I_{n,2}|^k+|I_{n,3}|^k+|I_{n,4}|^k),
\end{aligned}$$

to complete the proof of Lemma 5, it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}|I_{n,r}|^k < \infty, \quad for \quad r=1,2,3,4. \quad (22)$$

First, by applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$,
we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,1}|^k \\
\leq\ & \sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,\lambda_v P_v|s_v|\right\}^k \\
=\ & O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\lambda_v^k P_v^k|s_v|^k\right\} \\
&\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1} \\
=\ & O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\lambda_v^k P_v^k \\
=\ & O(1)\sum_{v=1}^{m}\lambda_v^k P_v^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})| \\
=\ & O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\lambda_v^k P_v^k a_{vv} \\
=\ & O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\lambda_v^k P_v^k\frac{p_v}{P_v} \\
=\ & O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}(\lambda_v P_v)^{k-1}p_v\lambda_v \\
=\ & O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}p_v\lambda_v \\
=\ & O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Lemma 5 and Lemma 4.
Now, using Hölder's inequality we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,2}|^k \\
\leq\ & \sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v P_v|s_v|\right\}^k \\
=\ & O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k\Delta\lambda_v P_v|s_v|^k\right\} \\
&\times\left\{\sum_{v=1}^{n-1}\Delta\lambda_v P_v\right\}^{k-1} \\
=\ & O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\hat{a}_{n,v+1}|^{k-1}\Delta\lambda_v P_v
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v P_v\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v P_v\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v P_v\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\Delta\lambda_v P_v\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 5 and Lemma 4.
Again, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,3}|^k\\
\leq\ &\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|p_{v+1}\lambda_{v+1}s_v\right\}^k\\
=\ &O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k p_v\lambda_v\\
\times\ &\left\{\sum_{v=1}^{n-1}p_v\lambda_v\right\}^{k-1}\\
=\ &O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^{k-1}|\hat{a}_{n,v+1}|p_v\lambda_v\\
=\ &O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|p_v\lambda_v\\
=\ &O(1)\sum_{v=1}^{m}p_v\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{a}_{n,v+1}|\\
=\ &O(1)\sum_{v=1}^{m}p_v\lambda_v\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\\
=\ &O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}p_v\lambda_v\\
=\ &O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

Finally, since $P_n\lambda_n = O(1)$ as $n\to\infty$, we have that

$$
\begin{aligned}
&\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,4}|^k\\
=\ &\sum_{n=1}^{m}\varphi_n^{k-1}a_{nn}^{k}\lambda_n^k P_n^k|s_n|^k\\
=\ &O(1)\sum_{n=1}^{m}\varphi_n^{k-1}a_{nn}^{k-1}\left(\lambda_n\right)^{k-1}\lambda_n P_n^k\frac{p_n}{P_n}\\
=\ &O(1)\sum_{n=1}^{m}\varphi_n^{k-1}a_{nn}^{k-1}(P_n\lambda_n)^{k-1}p_n\lambda_n\\
=\ &O(1)\sum_{n=1}^{m}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}p_n\lambda_n\\
=\ &O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 5 and Lemma 4.
Therefore, we get

$$
\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,r}|^k = O(1) \quad as \quad m\to\infty, for \quad r=1,2,3,4.
$$

This completes the proof of Lemma 5.

IV. PROOF OF THEOREM 3

The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 3 is a consequence of Lemma 5.

V. CONCLUSIONS

Corollary 1. If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability.

Corollary 2. If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a theorem dealing with $|\bar{N}, p_n|_k$ summability.

Corollary 3. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another result dealing with $\varphi - |\bar{N}, p_n|_k$ summability.

Corollary 4. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result dealing with $\varphi - |C, 1|_k$ summability (see [20]).

Corollary 5. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability (see [11]).

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