The Birkhoff weak integral of real functions with respect to a multimeasure

Anca Croitoru
"Al. I. Cuza" University,
Faculty of Mathematics,
Iaşi, Romania
Email: croitoru@uaic.ro

Alina Gavriluţ
University "Alexandru Ioan Cuza",
Faculty of Mathematics,
Bd. Carol I, No. 11, Iaşi, 700506,
ROMANIA
Email: gavrilut@uaic.ro

Alina Iosif
Petroleum-Gas University of Ploieşti,
Department of Computer Science,
Information Technology, Mathematics and Physics,
Bd. Bucureşti, No. 39,
Ploieşti 100680, ROMANIA
Email: emilia.iosif@upg-ploiesti.ro

***Abstract*—In this paper we define and study a new Birkhoff type integral $(Bw)\int_A f d\mu$ (called Birkhoff weak) for a real function f with respect to a set multifunction μ taking values in the family of all nonempty subsets of a real Banach space. Some classical properties are presented, such as heredity, monotonicity (relative to the function f, to the set multifunction μ and to the set A), homogeneity (with respect to f and μ) and additivity (relative to f, μ and A). Birkhoff weak integrability properties on atoms are also established.**

Keywords– Birkhoff weak integral, integrable function, multimeasure, monotone set multifunction, atom.

I. Introduction

Beginning with the work of Choquet [7], the theory of set-valued integrals started to develop due to its remarkable applications in statistics, evidence theory, data mining problems, decision making theory, subjective evaluations, medicine.

Different types of set-valued integrals have been defined and studied by many authors (e.g., [1], [3], [4], [5], [6], [8], [10], [12-16], [17], [20], [21], [22-25], [26], [27], [28], [29], [30]).

The Birkhoff integral [2] was defined for a vector function $f : T \to X$ with respect to a complete finite measure $m : \mathcal{A} \to [0, +\infty)$, using series of type $\sum_{n=1}^{\infty} f(t_n)m(A_n)$ determined by a countable partition $\{A_n\}_{n\in\mathbb{N}^*}$ of T and $t_n \in A_n$, for every $n \in \mathbb{N}^*$. This definition was generalized (for example in [11]) for the case of a vector multifunction F and a complete finite measure $m : \mathcal{A} \to [0, +\infty)$ using series of type $\sum_{n=1}^{\infty} F(t_n)m(A_n)$.

In [18] Gould defined an integral for a vector function $f : T \to X$ relative to a complete finite measure $m : \mathcal{A} \to [0, +\infty)$ using finite sums of type $\sum_{k=1}^{n} f(t_k)m(A_k)$ determined by a finite partition $\{A_k\}_{k=1}^{n}$ of T and $t_k \in A_k$ for every $k \in \{1, 2, \dots, n\}$.

Considering countable partitions and finite sums instead of series, in this paper we define and study a new Birkhoff type integral for real functions with respect to set multifunctions taking values in the family of all nonempty subsets of a real Banach space. This definition is more simple, easier handle and may be placed between the Birkhoff integral and the Gould integral.

The paper is organized as follows: Section 1 is for introduction. In the second section we give some basic concepts and results. In Section 3 we define a new Birkhoff type integral $(Bw)\int_A f d\mu$ (called Birkhoff weak) for a real function f with respect to a set multifunction μ taking values in the family of all nonempty subsets of a real Banach space. We present some classical properties of this integral, such as heredity, monotonicity (relative to the function f, to the set multifunction μ and to the set A), homogeneity (with respect to f and μ) and additivity (relative to f, μ and A). Section 4 contains some particular cases concerning Birkhoff weak integrability on atoms. The final Section 5 highlights some conclusions.

II. Preliminaries

Let be T a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathbb{R}^T$ the set of all real functions defined on T. Let also be $(X, \|\cdot\|)$ a real Banach space with the metric d induced by its norm, $\mathcal{P}_0(X)$ the family of all nonempty subsets of X, $\mathcal{P}_c(X)$ the family of all nonempty convex subsets of X, $\mathcal{P}_f(X)$ the family of all nonempty closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonempty bounded closed subsets of X, $\mathcal{P}_{bfc}(X)$ the family of all nonempty bounded closed convex subsets of X and $\mathcal{P}_{kc}(X)$ the family of all nonempty compact convex subsets of X.

For every $M, N \in \mathcal{P}_0(X)$ and every $\alpha \in \mathbb{R}$, let $M + N = \{x + y | x \in M, y \in N\}$ and $\alpha M = \{\alpha x | x \in M\}$. We denote by $\overline{M}$ the closure of M with respect to the topology induced by the norm of X.

By "$\dot{+}$" we mean the Minkowski addition on $\mathcal{P}_0(X)$, that is,

$$M \dot{+} N = \overline{M + N}, \ \ \forall M, N \in \mathcal{P}_0(X).$$

Let h be the Hausdorff metric given by

$$h(M, N) = \max\{e(M, N), e(N, M)\}, \ \ \forall M, N \in \mathcal{P}_0(X),$$

where $e(M, N) = \sup_{x\in M} d(x, N)$ and $d(x, N) = \inf_{y\in N} d(x, y)$.

It is well-known that $(\mathcal{P}_{bf}(X), h)$ and $(\mathcal{P}_{kc}(X), h)$ are complete metric spaces ([19]).

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X.

By $i = \overline{1,n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\mathbb{N} = \{0, 1, 2 \ldots\}$. We also denote $\mathbb{R}_+ = [0, \infty)$. In the following proposition we recall some properties regarding the excess and the Hausdorff metric ([19]).

Proposition 1: Let $A, B, C, D, A_i, B_i \in \mathcal{P}_0(X)$, for every $i = \overline{1,n}$ and $n \in \mathbb{N}^*$. Then:
(i) $h(A, B) = h(\overline{A}, \overline{B})$.
(ii) $e(A, B) = 0$ if and only if $A \subseteq \overline{B}$.
(iii) $h(A, B) = 0$ if and only if $\overline{A} = \overline{B}$.
(iv) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$, $\forall \alpha \in \mathbb{R}$.
(v) $h(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i) \le \sum_{i=1}^n h(A_i, B_i)$.
(vi) $h(\alpha A, \beta A) \le |\alpha - \beta| \cdot |A|$, $\forall \alpha, \beta \in \mathbb{R}$.
(vii) $h(\alpha A \dot{+} \beta B, \gamma A \dot{+} \delta B) \le |\alpha - \gamma| \cdot |A| + |\beta - \delta| \cdot |B|$, $\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$.
(viii) $h(A + C, B + C) = h(A, B)$, for every $A, B \in \mathcal{P}_{bfc}(X)$ and $C \in \mathcal{P}_b(X)$.
(ix) $\alpha(A + B) = \alpha A + \alpha B$, $\forall \alpha \in \mathbb{R}$.
(x) $(\alpha + \beta)A = \alpha A + \beta A$, for every $\alpha, \beta \in \mathbb{R}$, with $\alpha\beta \ge 0$ and every convex $A \in \mathcal{P}_0(X)$.
(xi) $\alpha A \subseteq \beta A$, for every $\alpha, \beta \in \mathbb{R}_+$, with $\alpha \le \beta$ and every convex $A \in \mathcal{P}_0(X)$, with $\{0\} \subseteq A$.
(xii) If $X = \mathbb{R}$, then $h([a, b], [c, d]) = \max\{|a - c|, |b - d|\}$, for every $a, b, c, d \in \mathbb{R}$, $a \le b$, $c \le d$.

In the sequel, let $\mathcal{A}$ be a σ-algebra of subsets of T.

Definition 2: (i) A finite (countable, respectively) partition of T is a finite (countable, respectively) family of nonempty sets $P = \{A_i\}_{i=\overline{1,n}}$ ($\{A_n\}_{n\in\mathbb{N}}$, respectively) $\subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \ne j$ and $\bigcup\limits_{i=1}^{n} A_i = T$ ($\bigcup\limits_{n\in\mathbb{N}} A_n = T$, respectively).

(ii) If P and P' are two finite (or countable) partitions of T, then P' is said to be *finer than* P, denoted by $P \le P'$ (or, $P' \ge P$), if every set of P' is included in some set of P.

(iii) The *common refinement* of two finite or countable partitions $P = \{A_i\}$ and $P' = \{B_j\}$ is the partition $P \wedge P' = \{A_i \cap B_j\}$.

Obviously, $P \wedge P' \ge P$ and $P \wedge P' \ge P'$.
We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$ we denote the class of all partitions of A.

All over the paper, $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ will be a set multifunction, with $\mu(\emptyset) = \{0\}$.

Definition 3: μ is said to be:
(i) *monotone* if $\mu(A) \subseteq \mu(B)$, $\forall A, B \in \mathcal{A}$, with $A \subseteq B$.
(ii) *subadditive* if $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.
(iii) *a multisubmeasure* if μ is monotone and subadditive.
(iv) *finitely additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every disjoint $A, B \in \mathcal{A}$.
(v) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{A}$, with $\mu(B) = \{0\}$.
(vi) *σ-null-null-additive* if $\mu(\bigcup\limits_{n=0}^{\infty} A_n) = \{0\}$, $\forall A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with $\mu(A_n) = \{0\}$.

Definition 4: Let $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ be a set-valued set function.
(i) *The variation* $\overline{\mu}$ of μ is the set function $\overline{\mu} : \mathcal{P}(T) \to [0, +\infty]$ defined by $\overline{\mu}(E) = \sup\{\sum\limits_{i=1}^{n} |\mu(A_i)|\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^n \subset \mathcal{A}$, with $A_i \subseteq E$, for every $i = \overline{1,n}$.
(ii) μ is said to be *of finite variation on* $\mathcal{A}$ if $\overline{\mu}(T) < \infty$.
(iii) $\widetilde{\mu}$, defined, for every $A \subseteq T$, by

$$\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}.$$

Remark 5:
I. If $E \in \mathcal{A}$, then in definition of $\overline{\mu}$ we may consider the supremum over all finite partitions $\{A_i\}_{i=1}^n \subset \mathcal{A}$, of E.
II. $|\mu(A)| \le \overline{\mu}(A)$, for every $A \in \mathcal{A}$;
III. $\overline{\mu}$ is monotone and super-additive on $\mathcal{P}(T)$, that is $\overline{\mu}(\bigcup\limits_{i\in I} A_i) \ge \sum\limits_{i\in I} \overline{\mu}(A_n)$, for every finite or countable partition $\{A_i\}_{i\in I}$ of T.
IV. If μ is finitely additive, then $\overline{\mu}$ is finitely additive.
V. If μ is a multisubmeasure, then μ is null-additive.

Remark 6: Suppose X is a Banach lattice and we denote by Λ the positive cone of X, i. e. $\Lambda = \{x \in X; x \ge 0\}$. If $m : \mathcal{A} \to \Lambda$ is a set function, we consider *the induced set multifunction* (see [13]) $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$. Then:
I. $|\mu(A)| = \|m(A)\|, \forall A \in \mathcal{A}$;
II. $\overline{\mu} = \overline{m}$ on $\mathcal{P}(T)$;
III. $\widetilde{\mu} = \widetilde{m}$ on $\mathcal{P}(T)$;
IV. If m is monotone (σ-subadditive, σ-additive, respectively), then μ is monotone (σ-subadditive, σ-additive set-valued measure, respectively).

Definition 7: A property *(P)* about the points of T *holds almost everywhere* (denoted μ-*a.e.*) if there exists $A \in \mathcal{P}(T)$ so that $\widetilde{\mu}(A) = 0$ and *(P)* holds on $T \backslash A$.

Definition 8: I. A set $A \in \mathcal{A}$ is said to be an *atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{A}$, with $B \subset A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

II. μ is said to be *finitely (countably*, respectively) *purely atomic* if there is a finite (countable, respectively) disjoint family $\{A_i\}_{i=1}^n(\{A_n\}_{n\in\mathbb{N}}$, respectively)$\subset \mathcal{A}$ of atoms of μ so that $T = \bigcup\limits_{i=1}^{n} A_i (T = \bigcup\limits_{n=0}^{\infty} A_n$, respectively).

Lemma 9: Let $\mu : \mathcal{A} \to \mathcal{P}_0(X)$, with $\mu(\emptyset) = \{0\}$ and let $A \in \mathcal{A}$ be an atom of μ.

I. If μ is monotone and the set $B \in \mathcal{A}$ is so that $B \subseteq A$ and $\mu(B) \supsetneq \{0\}$, then B is also an atom of μ and $\mu(A \backslash B) = \{0\}$. Moreover, if μ is null-additive, then $\mu(B) = \mu(A)$.

II. If μ is monotone and null-additive, then for every finite partition $\{B_i\}_{i=1}^n$ of A, there exists only one $i_0 = \overline{1, n}$ so that $\mu(B_{i_0}) = \mu(A)$ and $\mu(B_i) = \{0\}$ for every $i = \overline{1, n}$, $i \neq i_0$.

III. Suppose μ is monotone, null-additive and σ-null- null-additive. Then for every countable partition $\{B_n\}_{n \in \mathbb{N}}$ of A, there is an unique $n_0 \in \mathbb{N}$ so that $\mu(B_{n_0}) = \mu(A)$ and $\mu(B_n) = \{0\}$ for every $n \in \mathbb{N}$, $n \neq n_0$.

Proof. I. Since A is an atom and $\mu(B) \neq \{0\}$, it results $\mu(A \backslash B) = \{0\}$. Let be $C \in \mathcal{A}$, $C \subseteq B$. Since $C \subseteq A$ and A is an atom, it follows $\mu(C) = \{0\}$ or $\mu(A \backslash C) = \{0\}$. If $\mu(A \backslash C) = \{0\}$, by the monotonicity of μ, it results $\mu(B \backslash C) = \{0\}$. So, B is an atom of μ. If moreover μ is null-additive, since $\mu(A) = \mu((A \backslash B) \cup B)$ and $\mu(A \backslash B) = \{0\}$, we obtain $\mu(B) = \mu(A)$.

II. If $\mu(B_i) = \{0\}$ for every $i = \overline{1, n}$, by the null-additivity of μ, it results $\mu(A) = \{0\}$, false! Then there is $i_0 = \overline{1, n}$ such that $\mu(B_{i_0}) \neq \{0\}$. From I, it follows $\mu(B_{i_0}) = \mu(A)$ and $\mu(A \backslash B_{i_0}) = \{0\}$. But $B_i \subseteq A \backslash B_{i_0}$ for every $i = \overline{1, n}$, $i \neq i_0$ and since μ is monotone, it results $\mu(B_i) = \{0\}$, for every $i = \overline{1, n}$, $i \neq i_0$.

III. The proof is analogous to that of II. $\square$

In the sequel let T be a locally compact Hausdorff topological space, $\mathcal{K}$ be the lattice of all compact subsets of T, $\mathcal{B}$ be the Borel σ-algebra (that is the smallest σ-algebra containing $\mathcal{K}$) and τ be the class of all open sets belonging to $\mathcal{B}$.

In order to state our next theorems, some results of Gavriluţ [14] will be presented.

Definition 10: A set multifunction $\mu : \mathcal{B} \to \mathcal{P}_0(X)$ is called *regular* if for each set $A \in \mathcal{B}$ and each $\varepsilon > 0$, there exist $K \in \mathcal{K}$ and $D \in \tau$ such that $K \subseteq A \subseteq D$ and $|\mu(D \backslash K)| < \varepsilon$.

Theorem 11: Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be regular multisubmeasure. If $A \in \mathcal{B}$ is an atom of μ, then there exists an unique point $a \in A$ such that $\mu(A) = \mu(\{a\})$.

Corollary 12: Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be a regular multisubmeasure. If $A \in \mathcal{B}$ is an atom of μ, then there exists an unique point $a \in A$ such that $\mu(A \backslash \{a\}) = \{0\}$.

Remark 13: Suppose $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic regular multisubmeasure. So there exists a finite family $\{A_i\}_{i=1}^n \subset \mathcal{A}$ of pairwise disjoint atoms of μ so that $T = \bigcup\limits_{i=1}^{n} A_i$. By Corollary 12, there are unique $a_1, a_2, \ldots, a_n \in T$ such that $\mu(A_i \backslash \{a_i\}) = \{0\}$, for every $i = \overline{1, n}$. Then we have

$$\mu(T \backslash \{a_1, \ldots, a_n\}) \subset \mu(T \backslash \{a_1\}) + \ldots + \mu(T \backslash \{a_n\}) = \{0\},$$

which implies $\mu(T \backslash \{a_1, \ldots, a_n\}) = \{0\}$. Now, since μ is null-additive, it follows $\mu(T) = \mu(\{a_1, \ldots, a_n\})$.

III. Birkhoff weak integrability of real functions relative to a set multifunction

In this section we define a Birkhoff type integral (named Birkhoff weak) of real functions with respect to a set multifunction and present some of its classical properties.

In the sequel, suppose $(X, \| \cdot \|)$ is a Banach space, T is infinite, $\mathcal{A}$ is a σ-algebra of subsets of T and $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is a set multifunction of finite variation such that $\mu(\emptyset) = \{0\}$.

Definition 14: I. [9] Let $m : \mathcal{A} \to [0, \infty)$ be a non-negative set function. A function $f \in \mathbb{R}^T$ is said to be *Birkhoff weak m-integrable* (*on* T) if there exists $a \in \mathbb{R}$ having the property that for every $\varepsilon > 0$, there exist a countable partition P_ε of T and $n_\varepsilon \in \mathbb{N}$ such that for every other countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of T, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$, it holds $|\sum\limits_{k=0}^{n} f(t_k) m(A_k) - a| < \varepsilon$, for every $n \geq n_\varepsilon$.
The real a is called *the Birkhoff weak m-integral of f* (*on* T) and is denoted by $(Bw) \int_T f dm$ or simply $\int_T f dm$.
II. A function $f \in \mathbb{R}^T$ is said to be *Birkhoff weak μ-integrable on* T (shortly μ-integrable) if there exists $E \in \mathcal{P}_0(X)$ having the property that for every $\varepsilon > 0$, there exist a countable partition P_ε of T and $n_\varepsilon \in \mathbb{N}$ such that for every other countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of T, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$, it holds $h(\sum\limits_{k=0}^{n} f(t_k) \mu(A_k), E) < \varepsilon$, for every $n \geq n_\varepsilon$.
The set E is called *the Birkhoff weak μ-integral of f on T* and is denoted by $(Bw) \int_T f d\mu$ or simply $\int_T f d\mu$.

f is called Birkhoff weak μ-integrable on a set $E \in \mathcal{A}$ if the restriction $f|E$ is Birkhoff weak μ-integrable on $(E, \mathcal{A}_E, \mu)$ and its integral is denoted by $(B) \int_E f d\mu$ or simply $\int_E f d\mu$.

Remark 15: If they exist, the integrals in Definition 14 are unique.

Example 16: I. Suppose $T = \{t_n | n \in \mathbb{N}\}$ is countable, $\{t_n\} \in \mathcal{A}$ and let be $f : T \to \mathbb{R}$ such that the series $\sum\limits_{n=0}^{\infty} f(t_n) \mu(\{t_n\})$ is unconditionally convergent. Then f is Birkhoff weak μ-integrable and $(Bw) \int_T f d\mu = \sum\limits_{n=0}^{\infty} f(t_n) \mu(\{t_n\})$.

II. Suppose $m : \mathcal{A} \to [0, \infty)$ is a non-negative set function and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is the set multifunction induced by m, that is $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$. Let $f : T \to \mathbb{R}_+$ be a function. Then f is Birkhoff weak μ-integrable on T if and only if f is Birkhoff weak m-integrable on T. Moreover, $(Bw) \int_T f d\mu = [0, (Bw) \int_T f dm]$.
This follows by Definition 14 and Proposition 1-(xii).

In the sequel we present some classical integral properties.

Theorem 17: Let $f \in \mathbb{R}^T$ be bounded. If $f = 0$ μ-ae, then f is μ-integrable and $\int_T f d\mu = \{0\}$.

Proof. Since f is bounded, there exists $M > 0$ so that $|f(t)| \le M$, for every $t \in T$.
Denoting $A = \{t \in T; f(t) \neq 0\}$ and since $f = 0$ μ-ae, we have $\widetilde{\mu}(A) = 0$. Then, for every $\varepsilon > 0$, there exists $B_\varepsilon \in \mathcal{A}$ so that $A \subseteq B_\varepsilon$ and $\overline{\mu}(B_\varepsilon) < \varepsilon/M$. Let $P_\varepsilon = \{C_i\}_{i\in\mathbb{N}}$ be a partition of T, such that $C_0 = T \setminus B_\varepsilon$ and $\bigcup_{i=1}^{\infty} C_i = B_\varepsilon$.
Consider now an arbitrary partition $P = \{D_i\}_{i\in\mathbb{N}}$ of T so that $P \ge P_\varepsilon$. Let $t_i \in D_i, i \in \mathbb{N}$ be arbitrarily chosen. Without any loss of generality, we may consider $P = P' \cup P''$, $P' = \{D_i^{'}\}_{i\in\mathbb{N}}$, $P'' = \{D_i^{''}\}_{i\in\mathbb{N}}$, where $\bigcup_{i\in\mathbb{N}} D_i^{'} = C_0$ and $\bigcup_{i\in\mathbb{N}} D_i^{''} = B_\varepsilon$.
Now, for every $n \in \mathbb{N}$ it holds:

$$|\sum_{i=0}^{n} f(t_i)\mu(D_i)| \le |\sum_{i=0}^{n} f(t_i)\mu(D_i^{''})| \le$$
$$\le M \cdot \sum_{i=0}^{n} |\mu(D_i^{''})| \le M \cdot \overline{\mu}(B_\varepsilon) < \varepsilon.$$

Hence, f is μ-integrable and $\int_T f d\mu = \{0\}$. □

Theorem 18: [10] Let $f : T \to \mathbb{R}$ be a real function. Then f is μ-integrable on $A \in \mathcal{A}$ if and only if $f\chi_A$ is μ-integrable on T, where χ_A is the characteristic function of A.

Theorem 19: Let be $\mu : \mathcal{A} \to \mathcal{P}_c(X)$ and $f, g : T \to \mathbb{R}$ μ-integrable functions so that $f(t) \cdot g(t) \ge 0$, for every $t \in T$. Then $f + g$ is μ-integrable and

$$\int_T (f+g) d\mu = \int_T f d\mu \overset{\bullet}{+} \int_T g d\mu. \tag{1}$$

Proof. Since f is μ-integrable, then for every $\varepsilon > 0$, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P \in \mathcal{P}, P = \{A_n\}_{n\in\mathbb{N}}$, with $P \ge P_1$ and every $t_n \in A_n, n \in \mathbb{N}$, we have

$$h\left(\sum_{k=0}^{n} f(t_k)\mu(A_k), \int_T f d\mu\right) < \frac{\varepsilon}{2}, \forall n \ge n_\varepsilon^1. \tag{2}$$

Analogously, because g is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P \in \mathcal{P}, P = \{B_n\}_{n\in\mathbb{N}}$, with $P \ge P_2$ and every $t_n \in B_n, n \in \mathbb{N}$, we have

$$h\left(\sum_{k=0}^{n} g(t_k)\mu(B_n), \int_T g d\mu\right) < \frac{\varepsilon}{2}, \forall n \ge n_\varepsilon^2. \tag{3}$$

Let be $P_0 = P_1 \wedge P_2$ and $n_0 = \max\{n_\varepsilon^1, n_\varepsilon^2\}$.

Then for every partition $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \ge P_0$ and $t_n \in C_n, n \in \mathbb{N}$, by (2) and (3) we get

$$h(\sum_{k=0}^{n}(f+g)(t_k)\mu(C_k), \int_T f d\mu + \int_T g d\mu) =$$
$$= h(\sum_{k=0}^{n} f(t_k)\mu(C_k) + \sum_{k=0}^{n} g(t_k)\mu(C_k), \int_T f d\mu + \int_T g d\mu) \le$$
$$\le h(\sum_{k=0}^{n} f(t_k)\mu(C_k), \int_T f d\mu) +$$
$$+ h(\sum_{k=0}^{n} g(t_k)\mu(C_k), \int_T g d\mu) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f + g$ is μ-integrable and (1) is satisfied. □

Theorem 20: If $f, g : T \to \mathbb{R}$ are μ-integrable bounded functions, then

$$h\left(\int_T f d\mu, \int_T g d\mu\right) \le \sup_{t\in T} |f(t) - g(t)| \cdot \overline{\mu}(T).$$

Proof. Since f is μ-integrable, then for every $\varepsilon > 0$, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \ge P_1$ and $t_n \in A_n, n \in \mathbb{N}$, we have

$$h(\int_T f d\mu, \sum_{k=0}^{n} f(t_k)\mu(A_k)) < \frac{\varepsilon}{4}, \forall n \ge n_\varepsilon^1. \tag{4}$$

Analogously, because g is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \ge P_2$,

$$h(\int_T g d\mu, \sum_{k=0}^{n} g(t_k)\mu(B_k)) < \frac{\varepsilon}{4}, \forall n \ge n_\varepsilon^2. \tag{5}$$

Let be $P_1 \wedge P_2 \in \mathcal{P}$, $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \ge P_1 \wedge P_2$ and $t_n \in C_n, n \in \mathbb{N}$ arbitrarily. Consider a fixed $n \in \mathbb{N}, n \ge \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Then from (4) and (5) it results

$$h(\int_T f d\mu, \int_T g d\mu) \le h(\int_T f d\mu, \sum_{k=0}^{n} f(t_k)\mu(C_k)) +$$
$$+ h(\sum_{k=0}^{n} f(t_k)\mu(C_k), \sum_{k=0}^{n} g(t_k)\mu(C_k)) +$$
$$+ h(\sum_{k=0}^{n} g(t_k)\mu(C_k), \int_T g d\mu) << \frac{\varepsilon}{2}$$
$$+ h(\sum_{k=0}^{n} f(t_k)\mu(C_k), \sum_{k=0}^{n} g(t_k)\mu(C_k)) \le$$
$$\le \frac{\varepsilon}{2} + \sum_{k=0}^{n} |f(t_k) - g(t_k)||\mu(C_k)| < \frac{\varepsilon}{2}$$
$$+ \sup_{t\in T} |f(t) - g(t)| \cdot \overline{\mu}(T),$$

for every $\varepsilon > 0$. This implies $h(\int_T f d\mu, \int_T g d\mu) \le \sup_{t\in T} |f(t) - g(t)| \cdot \overline{\mu}(T)$. □

As a consequence of the previous theorem we obtain:

Corollary 21: If $f : T \to \mathbb{R}$ is a μ-integrable bounded function, then

$$|\int_T f d\mu| \le \sup_{t\in T} |f(t)| \cdot \overline{\mu}(T).$$

The next proposition easily follows from Definition 14-II.

Theorem 22: Let be $f : T \to \mathbb{R}$ a μ-integrable function and $\alpha \in \mathbb{R}$. Then:
I) αf is μ-integrable and

$$\int_T \alpha f d\mu = \alpha \int_T f d\mu;$$

II) f is $\alpha\mu$-integrable and

$$\int_T f d(\alpha\mu) = \alpha \int_T f d\mu.$$

Theorem 23: Suppose $\mu : \mathcal{A} \to \mathcal{P}_c(X)$ is so that 0 is in $\mu(A)$ for every A in $\mathcal{A}$. If $f, g : T \to \mathbb{R}_+$ are μ-integrable functions on T so that $f \le g$ on T, then $\int_T f d\mu \subseteq \int_T g d\mu$.

Proof. Since f is μ-integrable, for every $\varepsilon > 0$, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \ge P_1$ and every $t_n \in A_n, n \in \mathbb{N}$

$$h(\int_T f d\mu, \sum_{k=0}^n f(t_k)\mu(A_k)) < \frac{\varepsilon}{3}, \forall n \ge n_\varepsilon^1.$$

Analogously, because g is μ-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \ge P_2$ and every $t_n \in B_n, n \in \mathbb{N}$

$$h(\int_T g d\mu, \sum_{k=0}^n g(t_k)\mu(B_k)) < \frac{\varepsilon}{3}, \forall n \ge n_\varepsilon^2.$$

Consider $P_0 = P_1 \wedge P_2$. Let $P \in \mathcal{P}$ be arbitrarily chosen, with $P = \{C_n\}_{n\in\mathbb{N}} \ge P_0$. Then $P \ge P_1$ and $P \ge P_2$. Let be $t_n \in C_n$, $n \in \mathbb{N}$ and $n \ge \max\{n_\varepsilon^1, n_\varepsilon^2\}$. We get that $h(\int_T f d\mu, \sum\limits_{k=0}^n f(t_k)\mu(C_k)) < \frac{\varepsilon}{3}$ and $(\int_T g d\mu, \sum\limits_{k=0}^n g(t_k)\mu(C_k)) < \frac{\varepsilon}{3}$, which imply

$$\begin{aligned} & e(\int_T f d\mu, \int_T g d\mu) \le h(\int_T f d\mu, \sum_{k=0}^n f(t_k)\mu(C_k)) + \\ & + e(\sum_{k=0}^n f(t_k)\mu(C_k), \sum_{k=0}^n g(t_k)\mu(C_k)) + \\ & + h(\sum_{k=0}^n g(t_k)\mu(C_k), \int_T g d\mu)) < \\ & < \frac{2\varepsilon}{3} + e(\sum_{k=0}^n f(t_k)\mu(C_k), \sum_{k=0}^n g(t_k)\mu(C_k)). \end{aligned}$$

According to (xi) and (ii) of Proposition 1, it holds $e(\int_T f d\mu, \int_T g d\mu) < \frac{2\varepsilon}{3}$, for every $\varepsilon > 0$, which implies $\int_T f d\mu \subseteq \int_T g d\mu$. □

Theorem 24: Let be $\mu_1, \mu_2 : \mathcal{A} \to \mathcal{P}_0(X)$, with $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$ and suppose $f : T \to [0, +\infty)$ is both μ_1-integrable and μ_2-integrable. If $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is the set multifunction defined by $\mu(A) = \mu_1(A) + \mu_2(A)$, for every $A \in \mathcal{A}$, then f is μ-integrable and

$$\int_T f d(\mu_1 + \mu_2) = \int_T f d\mu_1 \dot{+} \int_T f d\mu_2.$$

Proof. Since f is μ_1-integrable, then for every $\varepsilon > 0$, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \ge P_1$ and $t_n \in A_n, n \in \mathbb{N}$ we have

$$h\left(\int_T f d\mu_1, \sum_{k=0}^n f(t_k)\mu_1(A_k)\right) < \frac{\varepsilon}{2}, \forall n \ge n_\varepsilon^1. \quad (6)$$

Since f is μ_2-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \ge P_2$ and $t_n \in B_n, n \in \mathbb{N}$ we have

$$h(\int_T f d\mu_2, \sum_{k=0}^n f(t_k)\mu_2(B_k)) < \frac{\varepsilon}{2}, \forall n \ge n_\varepsilon^2. \quad (7)$$

Let be $n \ge \max\{n_\varepsilon^1, n_\varepsilon^2\}$, $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \ge P_1 \wedge P_2$ and $t_n \in C_n, n \in \mathbb{N}$.
Then, by (6) and (7), we get

$$\begin{aligned} & h(\sum_{k=0}^n f(t_k)\mu(C_k), \int_T f d\mu_1 + \int_T f d\mu_2) = \\ & = h(\sum_{k=0}^n f(t_k)[\mu_1(C_k) + \mu_2(C_k)], \int_T f d\mu_1 + \int_T f d\mu_2) = \\ & = h(\sum_{k=0}^n f(t_k)\mu_1(C_k) + \sum_{k=0}^n f(t_k)\mu_2(C_k), \int_T f d\mu_1 + \int_T f d\mu_2) \\ & \le h(\sum_{k=0}^n f(t_k)\mu_1(C_k), \int_T f d\mu_1) + \\ & + h(\sum_{k=0}^n f(t_k)\mu_2(C_k), \int_T f d\mu_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which implies that f is μ-integrable and $\int_T f d(\mu_1 + \mu_2) = \int_T f d\mu_1 \dot{+} \int_T f d\mu_2$. □

Theorem 25: Let be μ_1, $\mu_2 : \mathcal{A} \to \mathcal{P}_0(X)$ set multifunctions and $f : T \to \mathbb{R}$ a simultaneously μ_1-integrable and μ_2-integrable function. If $\mu_1(A) \subseteq \mu_2(A)$, for every $A \in \mathcal{A}$, then $\int_T f d\mu_1 \subseteq \int_T f d\mu_2$.

Proof. Let $\varepsilon > 0$ be arbitrarily. Since f is μ_1-integrable, there exist $P_1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \ge P_1$ and $t_n \in A_n, n \in \mathbb{N}$

$$h(\int_T f d\mu_1, \sum_{k=0}^n f(t_k)\mu_1(A_k)) < \frac{\varepsilon}{3}, \forall n \ge n_\varepsilon^1.$$

Analogously, since f is μ_2-integrable, there exist $P_2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}, P \ge \mathcal{P}_2$ and $t_n \in B_n, n \in \mathbb{N}$

$$h(\int_T f d\mu_2, \sum_{k=0}^n f(t_k)\mu_2(B_k)) < \frac{\varepsilon}{3}, \forall n \ge n_\varepsilon^2.$$

Let $P_0 = P_1 \wedge P_2$, and let $P \in \mathcal{P}$ be arbitrarily chosen, with $P = \{C_n\}_{n\in\mathbb{N}} \ge P_0$. Let be $t_n \in C_n$, $n \in \mathbb{N}$ and $n \ge \max\{n_\varepsilon^1, n_\varepsilon^2\}$.

We get that $h(\int_T f d\mu_1, \sum_{k=0}^{n} f(t_k)\mu_1(C_k)) < \frac{\varepsilon}{3}$ and $h(\int_T f d\mu_2, \sum_{k=0}^{n} f(t_k)\mu_2(C_k)) < \frac{\varepsilon}{3}$, which imply

$$e(\int_T f d\mu_1, \int_T f d\mu_2) \leq e(\int_T f d\mu_1, \sum_{k=0}^{n} f(t_k)\mu_1(C_k))+$$
$$+ e(\sum_{k=0}^{n} f(t_k)\mu_1(C_k), \sum_{k=0}^{n} f(t_k)\mu_2(C_k))+$$
$$+ e(\sum_{k=0}^{n} f(t_k)\mu_2(C_k), \int_T f d\mu_2) <$$
$$< \frac{2\varepsilon}{3} + e(\sum_{k=0}^{n} f(\theta_k)\mu_1(C_k), \sum_{k=0}^{n} f(t_k)\mu_2(C_k)).$$

According to Proposition 1-(ii), we have

$$e(\sum_{k=0}^{n} f(t_k)\mu_1(C_k), \sum_{k=0}^{n} f(t_k)\mu_2(C_k)) = 0.$$

Consequently, $e(\int_T f d\mu_1, \int_T f d\mu_2) < \varepsilon$, for every $\varepsilon > 0$, which implies the equality $e(\int_T f d\mu_1, \int_T f d\mu_2) = 0$. Applying again Proposition 1-(ii), it results $\int_T f d\mu_1 \subseteq \int_T f d\mu_2$. □

Theorem 26: Suppose μ is finitely additive. Let $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$. If $f : T \to \mathbb{R}$ is μ-integrable on A and μ-integrable on B, then f is μ-integrable on $A \cup B$, and, moreover,

$$\int_{A\cup B} f d\mu = \int_A f d\mu \dot{+} \int_B f d\mu.$$

Proof. Let be $\varepsilon > 0$. Since f is μ-integrable on A, there exist a partition $P_A^\varepsilon = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$, $P \geq P_A^\varepsilon$ and $t_n \in E_n, n \in \mathbb{N}$, we have

$$h(\int_A f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1.$$

Analogously, since f is μ-integrable on B, we find a partition $P_B^\varepsilon = \{D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$ and $n_\varepsilon^2 \in \mathbb{N}$ so that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$, with $P \geq P_B^\varepsilon$, and $t_n \in E_n, n \in \mathbb{N}$, we have

$$h(\int_B f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2.$$

Consider $P_{A\cup B}^\varepsilon = \{C_n, D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{A\cup B}$ and $n \geq \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Let $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{A\cup B}$ such that $P \geq P_{A\cup B}^\varepsilon$, then we have

$$h(\sum_{k=0}^{n} f(t_k)\mu(E_k), \int_A f d\mu \dot{+} \int_B f d\mu) =$$
$$= h(\sum_{k=0}^{n} f(t_k)[\mu(E_k \cap A) + \mu(E_k \cap B)]), \int_A f d\mu \dot{+} \int_B f d\mu)$$
$$\leq h(\sum_{k=0}^{n} f(t_k)\mu(E_k \cap A), \int_A f d\mu)+$$
$$+ h(\sum_{k=0}^{n} f(t_k)\mu(E_k \cap B), \int_B f d\mu) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is thus finished. □

Theorem 27: Suppose μ is monotone. Let be $A, B \in \mathcal{A}$, with $A \subseteq B$. If $f : T \to \mathbb{R}$ is μ-integrable on A and μ-integrable on B, then

$$\int_A f d\mu \subseteq \int_B f d\mu.$$

Proof. Since f is μ-integrable on A, for every $\varepsilon > 0$, there exist $P_\varepsilon^1 = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$ and $n_\varepsilon^1 \in \mathbb{N}$ so that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_A$, with $P \geq P_\varepsilon^1$, and $t_n \in E_n, n \in \mathbb{N}$ we have

$$h(\int_A f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1. \tag{8}$$

Analogously, there exist $P_\varepsilon^2 = \{D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$, with $P \geq P_\varepsilon^2$, and $t_n \in E_n, n \in \mathbb{N}$

$$h(\int_B f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k)) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2. \tag{9}$$

We consider $\widetilde{P}_\varepsilon^1 = \{C_n, B \setminus A\}_{n\in\mathbb{N}}$. Then $\widetilde{P}_\varepsilon^1 \in \mathcal{P}_B$ and $\widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2 \in \mathcal{P}_B$.

Let also be an arbitrary partition $P = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$, with $P \geq \widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2$.

We observe that $P_\varepsilon'' = \{E_n \cap A\}_{n\in\mathbb{N}}$ is also a partition of A and $P_\varepsilon'' \geq P_\varepsilon^1$. Consider $n_\varepsilon = \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Let $t_n \in E_n \cap A, n \in \mathbb{N}$.

Then by (8) and (9), for a fixed $n \geq n_\varepsilon$, we have

$$h(\int_B f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k)) < \frac{\varepsilon}{2}$$

and

$$h(\int_A f d\mu, \sum_{k=0}^{n} f(t_k)\mu(E_k \cap A)) < \frac{\varepsilon}{2}.$$

According to Proposition 1-(ii), we obtain

$$e(\int_A f d\mu, \int_B f d\mu) \leq h(\int_A f d\mu, \sum_{k=0}^{n} f(t_k)m(E_k \cap A))+$$
$$+e(\sum_{k=0}^{n} f(t_k)\mu(E_k \cap A), \sum_{k=0}^{n} f(t_k)\mu(E_k))+$$
$$+h(\sum_{k=0}^{n} f(t_k)\mu(E_k), \int_B f d\mu) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for every $\varepsilon > 0$. Then $\int_A f d\mu \subseteq \int_B f d\mu$, as claimed. □

Theorem 28: Suppose $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is finitely additive. *Let $f, g : T \to \mathbb{R}$ be bounded functions so that:*

(i) f is μ-integrable and

(ii) $f = g$ μ-ae.

Then g is μ-integrable and $\int_T f d\mu = \int_T g d\mu$.

Proof. Let $M = \max\{\sup_{t\in T} |f(t)|, \sup_{t\in T} |g(t)|\}$. If $M = 0$, then $f = g = 0$ and the conclusion is evident. Suppose $M > 0$. Let $\varepsilon > 0$ be arbitrarily. Since f is μ-integrable, there exist $P_\varepsilon =$

$\{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$ and $n_\varepsilon \in \mathbb{N}$ so that for every $P = \{B_n\}_{n\in\mathbb{N}}$, with $P \geq P_\varepsilon$ and every $t_n \in B_n, n \in \mathbb{N}$

$$h(\sum_{k=0}^{n} f(t_k)\mu(B_k)), \int_T f d\mu) < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon.$$

Let $E \subset T$ be such that $f = g$ on $T\setminus E$ and $\widetilde{\mu}(E) = 0$. By the definition of $\widetilde{\mu}$, there is $A \in \mathcal{A}$ so that $E \subseteq A$ and $\overline{\mu}(A) < \frac{\varepsilon}{4M}$.
Consider $P_0 = \{A \cap A_n, A_n\setminus A\}_{n\in\mathbb{N}} \in \mathcal{P}$. Let also be the arbitrary partition $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_0$ and $t_n \in B_n, n \in \mathbb{N}$. Then, without any loss of generality we suppose that $B_n = B_n^{'} \cup B_n^{''}$, with $\bigcup\limits_{n\in\mathbb{N}} B_n^{'} = A$ and $\bigcup\limits_{n\in\mathbb{N}} B_n^{''} = T \setminus A$. Considering a fixed $n \geq n_\varepsilon$, we prove that $h(\int_T f d\mu, \sum\limits_{k=0}^{n} g(t_k)\mu(B_k)) < \varepsilon$ (then g is μ-integrable on T and $\int_T f d\mu = \int_T g d\mu$).

Indeed,

$$h(\int_T f d\mu, \sum_{k=0}^{n} g(t_k)\mu(B_k)) \leq h(\int_T f d\mu, \sum_{k=0}^{n} f(t_k)\mu(B_k)) + $$
$$+ h(\sum_{k=0}^{n} f(t_k)\mu(B_k), \sum_{k=0}^{n} g(t_k)\mu(B_k)) <$$
$$< \frac{\varepsilon}{2} + h(\sum_{k=0}^{n} f(t_k)\mu(B_k), \sum_{k=0}^{n} g(t_k)\mu(B_k)).$$

Now, since μ is finitely additive, we get

$$h(\sum_{k=0}^{n} f(t_k)\mu(B_k), \sum_{k=0}^{n} g(t_k)\mu(B_k)) =$$
$$= h(\sum_{k=0}^{n} f(t_k)\mu(B_k^{'}) + \sum_{k=0}^{n} f(t_k)\mu(B_k^{''}),$$
$$\sum_{k=0}^{n} g(t_k)\mu(B_k^{'}) + \sum_{k=0}^{n} g(t_k)\mu(B_k^{''}))$$
$$\leq h(\sum_{k=0}^{n} f(t_k)\mu(B_k^{'}), \sum_{k=0}^{n} g(t_k)\mu(B_k^{'})) +$$
$$+ h(\sum_{k=0}^{n} f(t_k)\mu(B_k^{''}), \sum_{k=0}^{n} g(t_k)\mu(B_k^{''}))$$
$$\leq \sum_{k=0}^{n} |f(t_k) - g(t_k)||\mu(B_k^{'}| + \sum_{k=0}^{n} |f(t_k) - g(t_k)||\mu(B_k^{''})|.$$

Therefore,

$$h(\int_T f d\mu, \sum_{k=0}^{n} g(t_k)\mu(B_k)) < \frac{\varepsilon}{2} + \sum_{k=0}^{n} |f(t_j) - g(t_j)| \cdot |\mu(B_k^{'}|$$
$$+ \sum_{k=0}^{n} |f(t_j) - g(t_j)| \cdot |\mu(B_k^{''})|.$$

Since for every $k = \overline{0, n}$, $B_k^{''} \subset T \setminus A \subset T \setminus E$ and $f = g$ on $T\setminus E$, then $f(t_k) = g(t_k)$, for every $k = \overline{0, n}$. Consequently,

$$h(\int_T f d\mu, \sum_{k=0}^{n} g(t_k)\mu(B_k)) <$$
$$< \frac{\varepsilon}{2} + \sum_{k=0}^{n} |f(t_j) - g(t_j)| \cdot |\mu(B_k^{'})| \leq$$
$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k=0}^{n} |\mu(B_k^{'})| \leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k=0}^{n} \overline{\mu}(B_k^{'}) =$$
$$= \frac{\varepsilon}{2} + 2M \cdot \overline{\mu}(\bigcup_{k=0}^{n} B_k^{'}) \leq \frac{\varepsilon}{2} + 2M \cdot \overline{\mu}(A) <$$
$$< \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon,$$

so the proof is finished. □

IV. Birkhoff weak integrability on atoms

In this section we obtain some properties regarding Birkhoff weak integrability on atoms and on finitely purely atomic set-valued measure spaces.

In the sequel, suppose $(X, \|\cdot\|)$ is a Banach space, T is infinite, $\mathcal{A}$ is a σ-algebra of subsets of T and $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is a set multifunction of finite variation such that $\mu(\emptyset) = \{0\}$.

Firstly, we present a characterization result of Birkhoff weak integrability on atoms.

Theorem 29: Suppose $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is a σ-null-null-additive multisubmeasure and $A \in \mathcal{A}$ is an atom of μ. Let $f : T \to \mathbb{R}$ be a real function. Then f is Birkhoff weak μ-integrable on A if and only if there exists $E \in \mathcal{P}_0(X)$ having the property that for every $\varepsilon > 0$ there exist a countable partition $P_\varepsilon = \{A_n\}_{n\in\mathbb{N}}$ of T and $n_\varepsilon \in \mathbb{N}$ such that for every $t_n \in A_n$ we have

$$h(\sum_{k=0}^{n} f(t_k)\mu(A_k), E) < \varepsilon, \forall n \geq n_\varepsilon.$$

Proof. Let $P^{'} = \{B_n\}_{n\in\mathbb{N}}$ be a countable partition of A. Since A is an atom of μ, according to Lemma 9-III, we may suppose without any loss of generality that B_1 is an atom of μ, $\mu(B_1) = \mu(A)$ and $\mu(B_n) = \{0\}$, for every $n \geq 2$. If we consider $P = \{C_n\}_{n\in\mathbb{N}}$ another countable partition of A, with $P \geq P$, then, reasoning as before, we may suppose that C_1 is an atom of μ, $\mu(C_1) = \mu(A)$ and $\mu(C_n) = \{0\}$, for every $n \geq 2$.
Since $P \geq P^{'}$, we discuss two cases:
I. $C_1 \subset B_1$. In this case, $\mu(C_1) = \mu(B_1) = \mu(A)$.
II. $C_1 \subset \bigcup\limits_{n=2}^{\infty} B_n$. We observe that $\mu(C_1) \subset \mu(\bigcup\limits_{n=2}^{\infty} B_n) = \{0\}$. (False!) □

In the sequel, T is a locally compact Hausdorff topological space and $\mathcal{B}$ is the Borel σ-algebra of T.

Theorem 30: Suppose $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a regular σ-null-null-additive multisubmeasure. If $f : T \to \mathbb{R}$ is Birkhoff

μ-integrable on an atom $A \in \mathcal{B}$, then $\int_A f d\mu = \overline{f(a)\mu(A)}$, where $a \in A$ is the single point resulting by Theorem 11.

Proof. Let be $\varepsilon > 0$. Since f is Birkhoff μ-integrable, by Definition 14-II there exists $P_\varepsilon = \{B_n\}_{n\in\mathbb{N}}$ a countable partition of A so that for every $t_n \in B_n$, $n \in \mathbb{N}$, we have

$$h(\sum_{k=0}^{n} f(t_k)\mu(B_k), \int_A f d\mu) < \varepsilon. \tag{10}$$

Suppose (by Lemma 9-III) that $\mu(B_0) = \mu(A)$ and $\mu(B_n) = \{0\}$, for every $n \in \mathbb{N}^*$. According to Theorem 11, there is an unique a in A so that $\mu(A) = \mu(\{a\})$. Suppose $a \notin B_0$. Then there exists an unique $k_0 \in \mathbb{N}^*$ such that $a \in B_{k_0}$. Since μ is monotone and $\mu(B_{k_0}) = \{0\}$, it follows $\mu(\{a\}) = \{0\} = \mu(A)$, false!

So $a \in B_0$. Taking $t_0 = a$, from (10) we obtain

$$h(f(a)\mu(A), \int_A f d\mu) < \varepsilon,$$

for every $\varepsilon > 0$, which shows that $\int_A f d\mu = \overline{f(a)\mu(A)}$. □

Corollary 31: Suppose $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic regular σ-null-null-additive multisubmeasure, with $T = \bigcup_{i=1}^{n} A_i$, where $\{A_i\}_{i=1}^n \subset \mathcal{A}$ are pairwise disjoint atoms of m. If the real function $f : T \to \mathbb{R}$ is m-integrable, then $\int_T f d\mu = \overline{\sum_{i=1}^{n} f(a_i)\mu(A_i)}$, where $a_i \in A_i$ is the single point resulting by Theorem 11, for every $i = \overline{1, n}$.

Corollary 32: Suppose $\mu : \mathcal{B} \to \mathcal{P}_{bf}(X)$ is a regular σ-null-null-additive multisubmeasure and let f, $f_n : T \to \mathbb{R}$ be m-integrable on an atom $A \in \mathcal{B}$ of μ, such that $\lim\limits_{n\to\infty} f_n(a) = f(a)$, where $a \in A$ is the single point resulting from Theorem 11. Then $\lim\limits_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$.

V. Conclusions

In this paper we have defined a new Birkhoff type integral $(Bw) \int_A f d\mu$ (called Birkhoff weak) for a real function f with respect to a set multifunction μ taking values in the family of all nonempty subsets of a real Banach space. Some classical properties of this integral are presented, such as heredity, monotonicity (relative to f, μ and A), homogeneity (with respect to f and μ) and additivity (by f, μ and A). Birkhoff weak integrability properties on atoms are also established.

Our future research on this integral concerns comparative results with other set-valued integrals, such as the integrals of Aumann type, Gould type or Choquet type and a Radon-Nikodym type theorem for Birkhoff weak integrability.

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