Some parametric inequalities on strongly regular graphs spectra

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***Abstract*—**In this work we deal with the problem of finding suitable admissibility conditions for the parameter sets of strongly regular graphs. To address this problem we analyze the regularity of these graphs through the introduction of a parameter and deduce some parametric admissibility conditions. Applying an asymptotic analysis, the conditions obtained enabled us to extract some spectral conclusions over the class of strongly regular graphs.

***Keywords*-**Strongly regular graph, Euclidean Jordan algebra, Matrix analysis.

I. Introduction

In this paper we obtain admissibility conditions on the parameters and spectra of strongly regular graphs. The conditions are deduced by the analysis of a parameter such that the regularity of the graph is given as a function of its order. Also, the relationship between this class of graphs and Euclidean Jordan algebras is explored. We consider a generalized binomial series regarding an element of the Jordan frame of an Euclidean Jordan algebra associated to the strongly regular graph, in an analogous manner as it was done in [16].

The paper is organized as follows. A survey on the main concepts regarding Euclidean Jordan algebras and Strongly regular graphs is presented in Section II and in Section III, respectively. Next, in Section IV, we present the algebraic environment of our work, that is, an Euclidean Jordan algebra spanned by the adjacency matrix of a strongly regular graph. Our main results are presented in Section V, were the asymptotic parameters of the generalized binomial Hadamard series of a strongly regular graph are introduced, in a similar manner than in [16]. Analyzing these parameters, we establish some asymptotic admissibility conditions over the parameters and the spectra of strongly regular graphs. Finally, in Section VI, some conclusions and experimental results are presented.

II. Real Euclidean Jordan Algebras

In the following paragraphs we introduce the fundamental concepts on Euclidean Jordan algebras as well as some notation. Such algebras were introduced in 1934 by Pascual Jordan, John von Neumann and Eugene Wigner, [10]. These algebras have applications in several branches of Mathematics such as statistics, [13], interior point methods, [6], [7] and combinatorics [3], [15], [16].

Euclidean Jordan algebras are vector spaces, $\mathcal{V}$, over the field $\mathbb{R}$, where it is defined an inner product, $< \cdot, \cdot >$, and a bilinear form $(u, v) \mapsto u \bullet v$, from $\mathcal{V} \times \mathcal{V}$ into $\mathcal{V}$, such that:

(i) $u \bullet v = v \bullet u$;
(ii) $u \bullet (u^2 \bullet v) = u^2 \bullet (u \bullet v)$, with $u^2 = u \bullet u$;
(iii) $\exists e \in \mathcal{V}$: $\forall u \in \mathcal{V}$, $e \bullet u = u \bullet$ [(iv)]$e = u$;
(v) $< u \bullet v, w >=< v, u \bullet w >$.

The element e is usually called the *unit element* of $\mathcal{V}$. Along this paper we only deal with real finite dimensional Euclidean Jordan algebras with unit element.

An element u in $\mathcal{V}$ is an *idempotent* if $u^2 = u$. Two idempotents u and v of $\mathcal{V}$ are *orthogonal* if $u \bullet v = 0$. A *complete system of orthogonal idempotents* of $\mathcal{V}$ is a set $\{u_1, u_2, \ldots, u_k\}$ of orthogonal idempotents such that $u_1 + u_2 + \cdots + u_k = e$.

For every u in $\mathcal{V}$, there are unique distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, and an unique complete system of orthogonal idempotents $\{u_1, u_2, \ldots, u_k\}$ such that

$$u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k, \tag{1}$$

(see [5], Theorem III.1.1). These λ_j's are usually called the eigenvalues of u and (1) is called the *first spectral decomposition* of u.

The *rank* of an element u in $\mathcal{V}$ is the least natural number l, such that the set $\{e, u, \ldots, u^l\}$ is linearly dependent (where $u^k = u \bullet u^{k-1}$) and we write $\text{rank}(u) = l$. This concept is expanded by defining the rank of the algebra $\mathcal{V}$ as the natural number $\text{rank}(\mathcal{V}) = \max\{\text{rank}(u) : u \in \mathcal{V}\}$. The elements of $\mathcal{V}$ with rank equal to the rank of $\mathcal{V}$ are called *regular* elements of $\mathcal{V}$. From now on, it is assumed that $\mathcal{V}$ has rank r.

An idempotent is called *primitive* if it is non-zero and it cannot be written as the sum of two other distinct non-zero orthogonal idempotents. A *Jordan frame* is a complete system of orthogonal idempotents such that each idempotent is primitive.

For every u in $\mathcal{V}$, where V is an Euclidean Jordan algebra with rank r, there exists a Jordan frame $\{u_1, u_2, \ldots, u_r\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

$$u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_r u_r. \tag{2}$$

The decomposition (2) is called the *second spectral decomposition* of u.

Further information on real Euclidean Jordan algebras can be found in Koecher's lecture notes [11] and in the monograph by Faraut and Korányi [5].

III. Strongly Regular Graphs

In this section we present the family of strongly regular graphs. The basic concepts of Graph Theory are assumed to be known by the reader and can be found in [8].

Strongly regular graphs is a family of regular graphs in which each pair of vertices have a fixed number of neighbors. This family was introduced in [2] in the context of partial geometries and partially balanced designs. One problem suggested when studying these graphs is to obtain feasibility conditions on their parameter set.

A simple, non-null, not complete and undirected graph X is *strongly regular* with *parameter set* (n, k, a, c) if it is a k-regular graph with n vertices such that each pair of vertices has a common neighbors if they are adjacent and c common neighbors, otherwise.

The parameters of a strongly regular graph are related by the following equality:

$$k(k-a-1) = (n-k-1)c. \tag{3}$$

If X is a strongly regular graph with parameters (n, k, a, c), then its complement graph, $\overline{X}$, that is, the graph with the same vertex set and such that two distinct vertices are adjacent in $\overline{X}$ if and only if they are not adjacent in X, is also strongly regular with parameters $(n, \bar{k}, \bar{a}, \bar{c})$, where

$$\bar{k} = n-k-1, \tag{4}$$
$$\bar{a} = n-2-2k+c, \tag{5}$$
$$\bar{c} = n-2k+a. \tag{6}$$

It is also well known (see, for instance, [8]) that the eigenvalues of a strongly regular graph X with parameter set (n, k, a, c) are k and

$$\theta = \frac{a-c+\sqrt{(a-c)^2+4(k-c)}}{2} \text{ and} \tag{7}$$
$$\tau = \frac{a-c-\sqrt{(a-c)^2+4(k-c)}}{2}. \tag{8}$$

The eigenvalues θ and τ are usually called the *restricted eigenvalues* of X. The multiplicities m_θ of θ and m_τ of τ are (see the equalities for the multiplicities presented in [12] and simplify):

$$m_\theta = \frac{-\tau n+\tau-k}{\theta-\tau},$$
$$m_\tau = \frac{\theta n+k-\tau}{\theta-\tau}.$$

Taking into account the information presented above, the following conditions are deduced:

1) The nontrivial Krein conditions obtained in [14]:

$$(\theta+1)(k+\theta+2\theta\tau) \leq (k+\theta)(\tau+1)^2 \tag{9}$$
$$(\tau+1)(k+\tau+2\theta\tau) \leq (k+\tau)(\theta+1)^2 \tag{10}$$

2) The Seidel's absolute bounds (see [4]):

$$n \leq \frac{m_\theta(m_\theta+3)}{2} \tag{11}$$
$$n \leq \frac{m_\tau(m_\tau+3)}{2}. \tag{12}$$

The conditions presented are used as feasibility conditions for parameter sets (n, k, a, c), that is, if (n, k, a, c) is a parameter set of a strongly regular graph, then inequalities (9)-(12) must hold. With these conditions many parameter sets are discarded as possible parameter sets of strongly regular graphs. However, there are still many parameter sets for which it is unknown if there exists a corresponding strongly regular graph.

The Krein conditions (9)-(10) and the Seidel's absolute bounds (11)-(12) are special cases of general inequalities obtained for association schemes, which constitute more general combinatorial structures, since strongly regular graphs correspond to the particular case of symmetric association schemes with two classes (see, for instance, [1]).

IV. An Euclidean Jordan algebra spanned by the adjacency matrix of a strongly regular graph

From now on, we consider the Euclidean Jordan algebra of real symmetric matrices of order n, $\mathcal{V}$, with the Jordan product defined by

$$\forall A, B \in \mathcal{V},\ A \bullet B = \frac{AB+BA}{2}, \tag{13}$$

where AB is the usual product of matrices. Furthermore, the inner product of $\mathcal{V}$ is defined as $< A, B >= \text{tr}(AB)$, where $\text{tr}(\cdot)$ is the classical trace of matrices, that is, the sum of its eigenvalues.

From now on, we will, we only consider (n, k, a, c) strongly regular graphs such that $0 < c \leq k < n-1$. Let X be a (n, k, a, c)-strongly regular graph and let A be the adjacency matrix of X. Then A has three distinct eigenvalues, namely the degree of regularity k of X, and the restricted eigenvalues θ and τ, given in (7) and (8). Recall that k and θ are the nonnegative eigenvalues and τ is the negative eigenvalue of A. Now we consider the Euclidean Jordan subalgebra of $\mathcal{V}$, $\mathcal{V}'$, spanned by the identity matrix of order n, I_n, and the powers of A. Since A has three distinct eigenvalues, then $\mathcal{V}'$ is a three dimensional Euclidean Jordan algebra with $\text{rank}(\mathcal{V}') = 3$.

Let $S = \{E_0, E_1, E_2\}$ be the unique complete system of orthogonal idempotents of $\mathcal{V}'$ associated to A, with

$$E_0 = \frac{A^2-(\theta+\tau)A+\theta\tau I_n}{(k-\theta)(k-\tau)} = \frac{J_n}{n},$$
$$E_1 = \frac{A^2-(k+\tau)A+k\tau I_n}{(\theta-\tau)(\theta-k)},$$
$$E_2 = \frac{A^2-(k+\theta)A+k\theta I_n}{(\tau-\theta)(\tau-k)},$$

where J_n is the matrix whose entries are all equal to 1. Since $\mathcal{V}'$ is an Euclidean Jordan algebra that is closed for the *Hadamard product* of matrices, denoted by $\circ$ (see the definition of Hadamard product, for instance, in [9]), and S is a basis of $\mathcal{V}'$, then there exist real numbers $q^i_{\alpha 2}$ and $q^i_{\alpha\beta 11}$, $0 \leq i, \alpha, \beta \leq 2$, $\alpha \neq \beta$, such that

$$E_\alpha \circ E_\alpha = \sum_{i=0}^{2} q^i_{\alpha 2} E_i, \tag{14}$$

$$E_\alpha \circ E_\beta = \sum_{i=0}^{2} q^i_{\alpha\beta 11} E_i. \tag{15}$$

These numbers $q^p_{\alpha 2}$ and $q^p_{\alpha\beta 11}$, $0 \leq \alpha, \beta \leq 2$, $\alpha \neq \beta$, are called the *Krein parameters* of the graph X (see [12]), since $q^1_{11} \geq 0$ and $q^2_{22} \geq 0$ yield the *Krein admissibility conditions*, (9) and (10), presented in [8, Theorem 21.3].

In what follows, we introduce some important notation. Firstly, considering $S = \{E_0, E_1, E_2\}$ and rewriting the idempotents under the new basis $\{I_n, A, J_n - A - I_n\}$ of $\mathcal{V}'$ we obtain

$$\begin{aligned} E_0 &= \frac{\theta-\tau}{n(\theta-\tau)} I_n + \frac{\theta-\tau}{n(\theta-\tau)} A + \\ &+ \frac{\theta-\tau}{n(\theta-\tau)} \left(J_n - A - I_n\right), \\ E_1 &= \frac{|\tau| n + \tau - k}{n(\theta-\tau)} I_n + \frac{n+\tau-k}{n(\theta-\tau)} A + \\ &+ \frac{\tau-k}{n(\theta-\tau)} \left(J_n - A - I_n\right), \\ E_2 &= \frac{\theta n + k - \theta}{n(\theta-\tau)} I_n + \frac{-n+k-\theta}{n(\theta-\tau)} A + \\ &+ \frac{k-\theta}{n(\theta-\tau)} \left(J_n - A - I_n\right). \end{aligned} \tag{16}$$

Now, we introduce another matrix product, the *Kronecker product* (see [9]), denoted by $C \otimes D$, for matrices $C = [C_{ij}] \in \mathcal{M}_{m,n}(\mathbb{R})$ and $D = [D_{ij}] \in \mathcal{M}_{p,q}(\mathbb{R})$, with $m, n, p, q \in \mathbb{N}$, defined by

$$C \otimes D = \begin{pmatrix} C_{11}D & \cdots & C_{1n}D \\ \vdots & \ddots & \vdots \\ C_{m1}D & \cdots & C_{mn}D \end{pmatrix}.$$

Note that $\mathcal{M}_{m,n}(\mathbb{R})$ is the usual notation for the space of real $m \times n$ matrices.

Finally, we introduce the following compact notation for the Hadamard and the Kronecker powers of the elements of S. Let x, y, z, α and β be natural numbers such that $0 \leq \alpha, \beta \leq 2$, $x \geq 2$ and $\alpha < \beta$. Then, we define

$$E^{\circ x}_\alpha = (E_\alpha)^{\circ x} \text{ and } E^{\otimes x}_\alpha = (E_\alpha)^{\otimes x},$$
$$E^{\circ yz}_{\alpha\beta} = (E_\alpha)^{\circ y} \circ (E_\beta)^{\circ z} \text{ and } E^{\otimes yz}_{\alpha\beta} = (E_\alpha)^{\otimes y} \otimes (E_\beta)^{\otimes z},$$
$$E^{\circ x}_{\alpha\oplus\beta} = (E_\alpha + E_\beta)^{\circ x} \text{ and } E^{\otimes x}_{\alpha\oplus\beta} = (E_\alpha + E_\beta)^{\otimes x}.$$

V. Asymptotic Parametric Conditions for Strongly Regular Graphs

Let X be a strongly regular graph with parameter set (n, k, a, c) and adjacency matrix A. Let $S = \{E_0, E_1, E_2\}$ be the unique complete system of orthogonal idempotents of the Euclidean Jordan algebra $\mathcal{V}'$ associated to A. We will make use of the same notation introduced in the paper [16].

Let x and ϵ be real positive numbers such that $\epsilon \in]0,1[$. Since $|(\theta n + k - \theta)/(n(\theta-\tau))| < 1, |(-n+k-\theta)/(n(\theta-\tau))| < 1$ and $|(k-\theta)/(n(\theta-\tau))| < 1$, then the series $\sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \epsilon \left(E^{\circ 2}\right)^i$ converges to a real number, $S_{x\epsilon}$, and we can write

$$\begin{aligned} S_{x\epsilon} &= \sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \left(\epsilon E^{\circ 2}\right)^i \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \epsilon^i \left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^{2i} I_n + \\ &+ \sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \epsilon^i \left(\frac{-n + k - \theta}{n(\theta-\tau)}\right)^{2i} A + \\ &+ \sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \epsilon^i \left(\frac{k - \theta}{n(\theta-\tau)}\right)^{2i} (J_n - A - I_n) \\ &= \frac{1}{\left(1 - \epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x} I_n + \\ &+ \frac{1}{\left(1 - \epsilon\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x} A + \\ &+ \frac{1}{\left(1 - \epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x} (J_n - A - I_n). \end{aligned} \tag{17}$$

Note that the eigenvalues of $S_{x\epsilon}$ are real positive numbers.

Consider the first spectral decomposition of $S_{x\epsilon}$, $S_{x\epsilon} = q^0_{x\epsilon} E_0 + q^1_{x\epsilon} E_1 + q^2_{x\epsilon} E_2$. We call the parameters $q^i_{x\epsilon}$, for $i = 0, \ldots, 2$, the *asymptotic parameters* of the strongly regular graph X.

Proceeding in an asymptotical analysis of the parameter $q^2_{x\epsilon}$, we establish Theorem V.1 which presents a feasibility condition for strongly regular graphs whose regularity is less than a half of its order.

Theorem V.1. *Let X be a strongly regular graph with parameter set (n, k, a, c) and distinct restricted eigenvalues θ and τ. If $k = (1/2 - \beta)n$, with $\beta \in]0, 1/2[$, then*

$$|\tau| < \frac{\theta(\theta + 1 - 2\beta)}{2\beta}. \tag{18}$$

Proof: Since $q^2_{x\epsilon} \geq 0$, for all x in $\mathbb{R}$ and for all ϵ in $]0,1[$, then, for any real numbers x and $\epsilon \in]0,1[$, we have

$$\begin{aligned} 0 &\leq \frac{1}{\left(1 - \epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x} + \frac{1}{\left(1 - \epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x} \tau + \\ &+ \frac{1}{\left(1 - \epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x} (-\tau - 1). \end{aligned} \tag{19}$$

After some algebraic manipulation of (19) we obtain

$$|\tau| \leq \frac{\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x - \left(1-\epsilon\left(\frac{\theta*n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x}{\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x - \left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x} \cdot \cdot \frac{\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x}{\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x}. \quad (20)$$

Making $x \to 0$ in inequality (20) and noting that

$$\lim_{x\to 0} \frac{\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x}{\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x} = 1,$$

we obtain

$$|\tau| \leq \frac{\ln\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right) - \ln\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)}{\ln\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right) - \ln\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)}. \quad (21)$$

Applying the Mean value Theorem to the function f such that $f(u) = \ln(1-u), \forall u \in]0,1[$ on the intervals $I_1 = \left[\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2, \epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right]$ and $I_2 = \left[\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2, \epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right]$, by (21) we deduce

$$|\tau| \leq \frac{\frac{1}{1-u_1}}{\frac{1}{1-u_2}} \frac{\epsilon\frac{\theta}{\theta-\tau}\frac{\theta n+2(k-\theta)}{n(\theta-\tau)}}{\epsilon\left(\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2 - \left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)}, \quad (22)$$

with $u_1 \in I_1$ and $u_2 \in I_2$.

After some straightforward simplifications of (22), we obain

$$|\tau| \leq \frac{1-u_2}{1-u_1} \frac{\frac{\theta}{\theta-\tau}\frac{\theta n+2(k-\theta)}{n(\theta-\tau)}}{\frac{1}{(\theta-\tau)^2} - \frac{2(k-\theta)}{n(\theta-\tau)^2}}. \quad (23)$$

Since $k = (1-2\beta)n$, inequality (23) can be written as

$$|\tau| \leq \frac{1-u_2}{1-u_1} \frac{\frac{\theta}{\theta-\tau}\frac{\theta n+(1-2\beta)n-2\theta}{n(\theta-\tau)}}{\frac{1}{(\theta-\tau)^2} - \frac{1-2\beta-2\theta}{(\theta-\tau)^2}}. \quad (24)$$

Finally, from (24) we have

$$|\tau| \leq \frac{1-u_2}{1-u_1} \frac{\theta(\theta+1-2\beta)}{2\beta}.$$

Therefore,

$$|\tau| \leq \lim_{\epsilon\to 0} \frac{1-u_2}{1-u_1} \frac{\theta+1-2\beta)\theta}{2\beta}$$

and (18) from Theorem V.1 is attained ■

Remark V.1. *Let X be a strongly regular graph with parameter set (n,k,a,c) and eigenvalues k, θ and τ. The inequality (18) presented in Theorem V.1 provides us the following information. The absolute value of τ is conditioned by the value of θ and we can say that this restriction is much more perceptible when the regularity k is much smaller than $n/2$.*

Due to Remark V.1, we establish a restriction of inequality (18) when $k < \alpha$, with $\alpha < n/2$. Accordingly, Corollary V.1 presents an inequality for parameter sets with $k < \alpha = n/3$, that can be easily deduced from Theorem V.1.

Corollary V.1. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k < n/3$, then*

$$|\tau| < \theta(3\theta+2). \quad (25)$$

Applying Theorem V.1 to the complement graph $\overline{X}$, we establish inequality (26) in Corollary V.2 for strongly regular graphs with parameter set (n,k,a,c) such that $k = n(\frac{1}{2}+\beta)-1$, with $\beta \in]0, \frac{1}{2}[$.

Corollary V.2. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k = n(1/2+\beta)-1$, with $\beta \in]0,1/2[$, then*

$$\theta < \frac{(|\tau|-1)(|\tau|-2\beta)}{2\beta} - 1. \quad (26)$$

Remark V.2. *As we did for Theorem V.1 with Remark V.1, we can conclude that the value of β in inequality (26) of Corollary V.2 must not be near zero, in order for the inequality to be useful. Therefore, k must be a natural number bigger, but not close, than $n/2-1$.*

In the following Corollary, we present a restriction of the inequality (26) for $\beta > 1/6$, which is the same of considering $k > 2n/3 - 1$.

Corollary V.3. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k > 2n/3-1$, then*

$$\theta < (|\tau|-1)(3|\tau|-1) - 1. \quad (27)$$

From now on we will compose the idempotent E_2 with the matrix $S_{x\epsilon}$ and analyze its eigenvalues. Observe that all the eigenvalues of $E_2 \circ S_{x\epsilon}$ are positive since $E_2 \circ S_{x\epsilon}$ is a principal submatrix of $E_2 \otimes S_{x\epsilon}$ and the eigenvalues of E_2 and $S_{x\epsilon}$ are all positive (here we apply the Eigenvalues Interlacing Theorem, see [9]).

Considering the following spectral decomposition $E_2 \circ S_{x\epsilon} = q^0_{2,x\epsilon}E_0 + q^1_{2,x\epsilon}E_1 + q^2_{2,x\epsilon}E_2$ and analyzing the parameter $q^0_{2,x\epsilon}$ we deduce, in Theorem V.2, another admissibility condition for strongly regular graphs with parameter set (n,k,a,c).

Theorem V.2. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k = (1/2-\beta)n$, with $\beta \in]0,1/2[$, then*

$$k < \frac{(2\theta+1-2\beta)\theta(\theta+1-2\beta)}{2(1+2\beta)\beta}. \quad (28)$$

Proof: Firstly, we note that the following inequality (29) is verified for any real numbers x and $\epsilon \in]0,1[$.

$$\begin{aligned} 0 &\leq \frac{\theta n+k-\theta}{n(\theta-\tau)} \frac{1}{\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x} + \\ &+ \frac{-n+k-\theta}{n(\theta-\tau)} \frac{1}{\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x} k + \\ &+ \frac{k-\theta}{n(\theta-\tau)} \frac{1}{\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x} (n-k-1). \quad (29) \end{aligned}$$

Since the elements of S are orthogonal, then $E_2 E_0 = 0$ and we can conclude that

$$\frac{k-\theta}{(n(\theta-\tau)}(n-k-1) = -\frac{\theta n+k-\theta}{n(\theta-\tau)} - \frac{-n+k-\theta}{n(\theta-\tau)}k$$

and by some algebraic manipulation of (29) we obtain the inequality (30).

$$\begin{aligned} k &\leq \frac{\theta n+k-\theta}{n-k+\theta} \frac{\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x}{\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x} \cdot \\ &\cdot \frac{\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x - \left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^x}{\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^x - \left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)^x} \quad (30) \end{aligned}$$

Making $x \to 0$ on the right hand side of (30) we obtain the following:

$$\begin{aligned} k &\leq \frac{\theta n+k-\theta}{n-k+\theta} \cdot \\ &\cdot \frac{\ln\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right) - \ln\left(1-\epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)}{\ln\left(1-\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right) - \ln\left(1-\epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right)}. \end{aligned} \quad (31)$$

Applying the Mean-Value Theorem to the real function $f(u) = \ln(1-u)$, for all u in $]0,1[$, on the intervals $I_1 = \left[\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2, \epsilon\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right]$ and $I_2 = \left[\epsilon\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2, \epsilon\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2\right]$, we obtain the inequality (32).

$$k \leq \frac{\theta n+k-\theta}{n-k+\theta} \frac{\frac{1}{1-u_1}}{\frac{1}{1-u_2}} \frac{\frac{\theta}{\theta-\tau}\frac{\theta n+2(k-\theta)}{n(\theta-\tau)}}{\left(\left(\frac{(n-k+\theta)}{n(\theta-\tau)}\right)^2 - \left(\frac{(k-\theta)}{n(\theta-\tau)}\right)^2\right)}, \quad (32)$$

with $u_1 \in I_1$ and $u_2 \in I_2$. After some algebraic manipulation of the inequality (32) we obtain

$$k \leq \frac{(1-u_2)(\theta n+k-\theta)\theta(\theta n+2(k-\theta))}{(1-u_1)(n-k+\theta)(n-2k+2\theta)}. \quad (33)$$

Recalling that $k = ((1-2\beta)n)/2$, with $\beta \in]0, \frac{1}{2}[$, and after neglecting some terms in the multiplicative factors of both the numerator and denominator of the right hand side of (33) we obtain (34)

$$k < \frac{(1-u_2)(2\theta+1-2\beta)\theta(\theta+1-2\beta)}{2(1-u_1)(1+2\beta)\beta}. \quad (34)$$

Finally, making $\epsilon \to 0$, then $u_1, u_2 \to 0$ and inequality (28) follows. ■

Considering $\beta > \frac{1}{6}$, which implies that $k < \frac{n}{3}$, we deduce the following consequence of Theorem V.2.

Corollary V.4. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k < n/3$, then*

$$k < \frac{(3\theta+1)\theta(3\theta+2)}{2}. \quad (35)$$

Applying Corollary V.4 to the complement graph $\overline{X}$, we establish Corollary V.5.

Corollary V.5. *Let X be a strongly regular graph with parameter set (n,k,a,c) and distinct eigenvalues k, θ and τ. If $k > 2n/3 - 1$, then*

$$n-k-1 < (2|\tau|-1)(|\tau|-1)(3|\tau|-1). \quad (36)$$

VI. Conclusions and Experimental Results

We conclude our paper with some experimental results which test the inequalities presented in Section V and deduce some spectral conclusions.

Firstly we analyze inequality (25) from Corollary V.1. In Table I we consider the parameter sets $X_1 = (184, 48, 2, 16)$, $X_2 = (256, 66, 2, 22)$ and $X_3 = (1275, 378, 57, 135)$ and we test them for the parameter $Q^1 = \theta(3\theta+2) - |\tau|$, obtained from the inequality (25). Note that each parameter set considered satisfies the condition of Corollary V.1, that is, $k < n/3$.

	X_1	X_2	X_3
θ	2.0	2.0	3.0
τ	−16.0	−22.0	−81.0
Q^1	0	−6.0	−48.0

TABLE I
Experimental results for the inequality (25).

The results presented in Table I confirm the conclusion that one can deduce from the analysis of inequality (25), that is, when $k < n/3$, the value of $|\tau|$ cannot be too big relatively to the value of θ.

Next we analyze inequality (27) from Corollary V.3. In Table II we consider the parameter sets from the complement graphs considered before, that is, $\overline{X_1} = (184, 135, 102, 90)$, $\overline{X_2} = (256, 189, 144, 126)$ and $\overline{X_3} = (1275, 896, 652, 576)$, and we test them for the parameter $Q^2 = (|\tau|-1)(3|\tau|-1) - 1 - \theta$, obtained from the inequality (27). Note that each

	$\overline{X_1}$	$\overline{X_2}$	X_3
θ	15.0	21.0	80.0
τ	-3.0	-3.0	-4.0
Q^2	0	-6.0	-48.0

TABLE II
EXPERIMENTAL RESULTS FOR THE INEQUALITY (27).

parameter set considered satisfies the condition of Corollary V.3, that is, $k > 2n/3 - 1$.

In this case we can draw a similar conclusion than before. The results presented in Table II confirm that, when $k > 2n/3 - 1$, the value of θ cannot be too big relatively to the value of $|\tau|$.

We proceed our analyzes with inequality (35) from Corollary V.4. In Table III we consider the parameter sets $X_4 = (300, 92, 10, 36)$, $X_5 = (400, 114, 8, 42)$ and $X_6 = (441, 128, 10, 48)$, which comply with the conditions of Corollary V.4, and we also consider the parameter $Q^3 = (3\theta + 1)\theta(3\theta + 2)/2 - k$, obtained from inequality (35).

	X_4	X_5	X_6
θ	2.0	2.0	2.0
τ	-28.0	-36.0	-40.0
Q^3	-36.0	-58.0	-72.0

TABLE III
EXPERIMENTAL RESULTS FOR THE INEQUALITY (35).

The results presented in Table III confirm the conclusions one can draw from the inequality (35) of Corollary V.4, that is, if $k < n/3$, then k cannot be much bigger than the value of θ, since inequality (35) establishes that k is smaller than a polynomial in θ of degree 3 with positive coefficients.

Finally, in Table IV, we analyze the complement parameter sets of the graphs previously considered, that is, $\overline{X_4} = (300, 207, 150, 126)$, $X_5 = (400, 285, 212, 180)$ and $X_6 = (441, 312, 231, 195)$, which comply with the conditions of Corollary V.5, and we also consider the parameter $Q^4 = (2|\tau| - 1)(|\tau| - 1)(3|\tau| - 1) - n + k + 1$, obtained from inequality (36).

	$\overline{X_4}$	$\overline{X_5}$	$\overline{X_6}$
θ	27.0	35.0	39.0
τ	-3.0	-3.0	-3.0
Q^4	-12.0	-34.0	-48.0

TABLE IV
EXPERIMENTAL RESULTS FOR THE INEQUALITY (36).

Our last experiments presented in Table IV are analogous to the ones from our previous experiments. Here, one can say that the regularity of the complement graph, $n - k - 1$, cannot be much bigger than the absolute value of the negative restricted eigenvalue, since it is restricted by a polynomial in $|\tau|$ of degree 3 which is always positive.

In this paper we have established parametric admissibility conditions which enlightened some relations between the elements of the spectrum of a strongly regular graph. These relations were highlighted in the previous paragraphs.

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