# On the $m$-term best approximation of functions and greedy algorithm

Martin Grigoryan

***Abstract-*** It is proved that the trigonometric system possesses the $L^1$-strong and greedy property. Also it is described the class of Lebesgue integrable functions such that the error between function and $m$-term best approximant with respect to the trigonometric system has the following behavior - $o\left(\frac{1}{\ln^\delta m}\right)$, $\delta > 0$.

***Keywords-***$m$-term best approximant, trigonometric system, greedy algorithm.

## I. Introduction

Linear approximations project the function on m vectors selected a priori. The approximation can be made more precisely by choosing the m orthogonal vectors depending on the signal properties.

Non-linear algorithms outperform linear projections by approximating each function with vectors selected adaptively within a basis. Let $\{\varphi_n(x)\}$ be an orthogonal basis in $L^2[0,1]$ , and let $\{f_m(x)\}$ be the projection of $f$ over $m$ vectors whose indices are in $A_m$.

$$f_m(x) = \sum_{k \in A_m} < f, \varphi_k > \varphi_k(x), \quad \text{where}$$

$$< f, \varphi_k >= d_k(f) := \int_0^1 f(t)\varphi_k(t)dt.$$

The approximation error have the form

$$r_m(f) := ||f - f_m||_2 = \left[\int_0^1 |f(x) - f_m(x)|^2 dx\right]^{\frac{1}{2}} =$$

$$= \left(\sum_{k \in A_m} |d_k(f)|^2\right)^{\frac{1}{2}}$$

To minimize this error, the indices in $A_m$ must correspond to the m vectors having the largest inner product amplitude $| < f, \varphi_k > |$. They are the vectors that best correlate $f(x)$ . So they can be interpreted as the "main" features of $f(x)$ . The resulting $r_m(f)$ is necessarily smaller than the error of the linear approximation, which selects the m approximation vectors independently of $f(x)$. Let us sort $\{|d_k(f)|\}_{k \geq 1}$ in decreasing order

$$|d_{\sigma(k)}(f)| \geq |d_{\sigma(k+1)}(f)|, \ k = 1, 2, ... \ ; .$$

M. Grigoryan is with the Department of physics, chair of Higher Mathematics, Yerevan State University, Yerevan, Armenia 0025 e-mail: gmarting@ysu.am

The best non-linear approximation is

$$f_m^{best}(x) = \sum_{k=1}^m d_{\sigma(k)}(f)\varphi_{\sigma(k)}(x).$$

For any $f(x) \in L^1[0,1]$ and $m = 1, 2, ...$ we put

$$c_k(f) = \int_0^1 f(t)e^{-i2\pi kt}dt, \ k = 0, \pm 1, \pm 2, .... \ ;$$

$$S_m(f) = \sum_{|k| \leq m} c_k(f)e^{i2\pi kx}$$

We call a permutation $\sigma = \{\sigma(k)\}_{k=1}^\infty$ of natural numbers decreasing and write $\sigma \in D(f)$, if

$$|c_{\sigma(k)}(f)| \geq |c_{\sigma(k+1)}(f)|, \ k = 1, 2, ... \ ; \ \sigma(-k) = -\sigma(k) \ .$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the $m$-th greedy approximant of $f$ with respect to the trigonometric system $T \equiv \{e^{i2\pi kx}\}_{k=-\infty}^{+\infty}$ corresponding to a permutation $\sigma \in D(f)$ by formula

$$G_m(f) = G_m(f, T, \sigma) = \sum_{1 \leq |k| \leq m} c_{\sigma(k)}(f)e^{i2\pi\sigma(k)x} \quad .$$

This nonlinear method of approximation is known as greedy algorythm (see for example [1], [2]).

The greedy algorithm of a function $f \in L_{[0,1]}$ with respect to the trigonometric system is said to converge to $f$ in the norm of $L^1[0,1]$ if

$$\lim_{m \to \infty} \int_0^1 |G_m(f, T, \sigma) - f(x)|dx = 0 \ ,$$

for some $\sigma \in D(f)$. For more on that algorithm , see [3]-[20].

The above mentioned definitions are given not in the most general form and only in the generality, in which they will be applied in the present paper.

Note that $G_m(x, f, T)$ gives the best m-term approximation in $L^2[0,1]-$ norm

$$\|G_m(f, \Psi, \sigma) - f\|_2 = R_m(f) = \inf_{|n| \in \Lambda} \|\sum a_n e^{i2\pi kx} - f\|_2 =$$

$$= \left(\sum_{n=m+1}^\infty |c_{\sigma(n)}(f)|^2\right)^{\frac{1}{2}}$$

where inf is taken over coefficients $a_n$ and sets of indices

$\Lambda$with cardinality $|\Lambda| = m$, and $\sigma = \{\sigma(n)\}_{n=1}^\infty \in D(f)$

It is clear that for each $f(x) \in L^2[0,1], R_m(f) \to 0 as m \to \infty$ .

V.N.Temlyakov [3] proved the existance of a function $f_0(x) \in L_{[0,1]}$, such that

$$\overline{\lim_{m\to\infty}} \int_0^1 |G_m(f_o, T, \sigma)| dx = +\infty \ .$$

In this paper we prove the following results.

**Theorem 1. (**$L$-strong and greedy property**).** *For any $\epsilon > 0$ there exists a measurable set $E \subset [0,1]$, with measure $|E| > 1-\epsilon$ such that for any function $f(x) \in L_{[0,1]}$ one can find a function $g(x) \in L[0,1]$ equal to $f(x)$ on $E$ such that its Fourier series and greedy algorithm with respect to the trigonometric system converges to $g(x)$ in the $L_{[0,1]}$- norm.*

**Theorem 2.Let** $f \in L^2[0,1]$ be a periodic function with period 1 **and let** $\delta > 0$, if

$$\int_0^1 \int_0^1 \frac{[f(x+t)-f(x-t)]^2}{t} (\ln \frac{1}{t})^\delta dx < \infty,$$

then

$$R_k^2(f) = ( \ \|G_m(f,\Psi,\sigma) - f\|_2)^2 \leq$$
$$\leq (\sum_{k=1}^{\infty} |c_k(f)|^2 (\ln k)^{1+\delta}) \frac{1}{(\ln k - \ln 2)^\delta} \ .$$

and

$$R_k^2(f) = o\left(\frac{1}{\ln^\delta k}\right), (R_k^2(f) \ln k)^\delta) \to 0 ask \to \infty)$$

Theorem 1 is a consequence of the more general theorem, wich is stated as follows.

**Theorem 3.** *For any $\epsilon > 0$ there exists a measurable set $E \subset [0,1]$, with measure $|E| > 1-\epsilon$ such that for any $f(x) \in L_{[0,1]}$, some $g(x) \in L_{[0,1]}$, $g(x) = f(x)$ on $E$ and a rearrangement $\{\sigma_f(k)\}_{k=-\infty}^{+\infty}$ $(\sigma_f(-k) = -\sigma_f(k))$ of integers $0, \pm 1, \pm 2, ...$ can be found, such that*

1) $|c_{\sigma_f(k)}(g)| > |c_{\sigma_f(k+1)}(g)|; \quad \forall k \geq 0$
2) $||G_m(g)|| \leq 3||g|| \leq 12||f|| \quad ; \quad \lim_{m\to\infty} ||G_m(g) - g|| = 0$
3) $||S_m(g)|| \leq 3||g|| \leq 12||f|| \quad ; \quad \lim_{m\to\infty} ||S_m(g) - g|| = 0$

With respect to the theorem 3 the following questions remain open.

**Question 1.** Can one take modified function $g(x)$ and rearrangement $\{\sigma_f(k)\}$ to satisfy conditions 1)-3) as well as series $\sum_{k=-\infty}^{\infty} c_{\sigma_f(k)}(g) e^{i2\pi\sigma(k)x}$ converges almost everywhere?

**Question 2.** *Is it possible to choose the rearrangement $\{\sigma_f(k)\}_{k=-\infty}^{+\infty}$ in the theorem 3 independent of $f$?*

**Question 3.** *Is it possible to choose the function g(x) in the theorem 3 such that*

$$|c_k(g)| > |c_{k+1}(g)|; \quad \forall k \geq 0$$

In connection with questions 2 and 3 we know the following results

**Theorem 4.** Let $T \equiv \{e^{i2\pi kx}\}_{k=-\infty}^{+\infty}$ the trigonometric system.Then its elements can be rearranged so that the resulting system $\{e^{i2\pi\sigma(k)x}\}_{k=-\infty}^{+\infty}$ has the folowing property: *for any* $0 < \epsilon < 1$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\epsilon$, such that for any function $f(x) \in L^1[0,1]$ there exists a function $g(x) \in L^1[0,1]$ coinciding with $f(x)$ on $E$ and such that the *sequence* $\{|c_{\sigma(k)}(g)|\ ,\ k \in spec(g)\}$ *is monotonically decreasing and the series*$\sum_{n=1}^{\infty} c_{\sigma(n)}(g) e^{i2\pi\sigma(k)x}$converges in $L^1[0,1]$ ( where $spec(f)$) is the support of $c_k(f)$, i.e. the set of integers where $c_k(f)$ is non-zero).

Note that in 1939 Men'shov [21] proved the following fundamental theorem.

**Theorem (Men'shov's $C$-strong property).** *For every measurable, almost everywhere finite function $f$ on $[0,2\pi]$ and every $\epsilon > 0$, there is a continuous function $f_\epsilon$ such that $|\{x \in [0,2\pi]; f_\epsilon(x) \neq f(x)\}| < \epsilon$ and the Fourier series of the function $f_\epsilon$ converges uniformly in $[0,2\pi]$.*

In 1988 we were able to show that the trigonometric system possesses the $L$-strong property for integrable functions: for each $\epsilon > 0$ there exists a (measurable) set $E \subset [0,2\pi]$ of measure $|E| > 2\pi - \epsilon$ such that for each function $f(x) \in L_{[0,2\pi]}$ there exists a function $g(x) \in L^1_{[0,2\pi]}$ equal to $f(x)$ on $E$ and with Fourier series convergent to $g(x)$ in $L^1_{[0,2\pi]}$- norm (see [25]).

After Men'shov's proof of the $C$-strong property, many "correction" type theorems were proved for different systems. We are not going to give a complete survey of all the research done in this area. For details we refer to [22]-[27].

**Remark.** In the D. E. Men'shov's above theorem, the "singular" set $e$, where $f(x)$ is changed, depends on $f(x)$.

Whereas in the theorems 1 and 3 of this paper the "singular" set does not depend on $f(x)$.

For $q > 0$, we put

$$||f||_{G(\ln^q)} = \sum_{k=1}^{\infty} |c_k(f)|^2 \ln^q k$$

$$G(\ln^q) = \{f(x) \in L^2[0,1]; \ \ with \ \ \sum_{k=1}^{\infty} |c_k(f)|^2 \ln^q k < \infty\}.$$

and

$$G^{\searrow}(\ln^q) = \{f(x) \in L^2[0,1]; \ with \ \sum_{k=1}^{\infty} |c_{\sigma(k)}(f)|^2 \ln^q k < \infty\},$$

where $\{\sigma(k)\}_{k=1}^{\infty}$ the permutation of natural numbers such that $|c_{\sigma(k)}(f)| \geq |c_{\sigma(k+1)}(f)|$, $\forall k \geq 1$ and

$$||f||_{G^{\searrow}(\ln^q)} = \sum_{k=1}^{\infty} |c_{\sigma(k)}(f)|^2 \ln^q k.$$

In [28 ] it is proved that if $\sum_{k=1}^{\infty} |c_k(f)|^p < \infty$, p<2 then hold Jachson inequality:

$$R_k(f) \leq \frac{||f||_{B_p}}{\frac{2}{p}-1} \frac{1}{k^{\frac{2}{p}}-1}, \ (where \ ||f||_{B_p} = \left(\sum_{k=1}^{\infty} |c_k(f)|^p\right)^{\frac{1}{p}}$$

and $R_k(f) = o\left(\frac{1}{k^{\frac{2}{p}-1}}\right)$.

Conversely, if $R_k(f) = O(\frac{1}{k^{\frac{2}{p}-1}})$ then $\sum_{k=1}^{\infty} |c_k(f)|^q < \infty$, for all $p < q$.

In this paper we prove

**Theorem 5.** If a function $f(x) \in G^{\searrow}(\ln^q)$, $q > 1$ then

$$R_k(f) \leq \frac{||f||_{G^{\searrow}}}{q-1} \frac{1}{(\ln k - \ln 2)^{q-1}}$$

and

$$R_k(f) = o\left(\frac{1}{\ln^{q-1} k}\right)$$

Conversely, if $R_k(f) = O\left(\frac{1}{\ln^q k}\right)$ then $f(x) \in G(\ln^q)$ for any $p < q-1$.

## II. Proof of the Theorems

In the proof of Theorem 3 we will use the following

**Lemma 1.** *For any $\epsilon > 0$, any $f(x) \in L[0,1]$ with $\int_0^1 |f(x)|dx > 0$ and any $N_0 > 1$, there exists a measurable set $E \subset [0,1]$, a function $g(x)$, as well as a polynomial $Q(x)$*

$$Q(x) = \sum_{|k|=N_0}^{N} a_k e^{i2\pi kx}$$

*and a rearrangement $\{\sigma(k)\}_{k=N_0}^{N}$ of natural numbers $N_0, ..., N$, which satisfy the conditions:*

1) $|E| > 1 - \epsilon$,
2) $g(x) = f(x), \quad x \in E$
3) $\frac{1}{2}\int_0^1 |f(x)|dx \leq \int_0^1 |g(x)|dx \leq 3 \cdot \int_0^1 |f(x)|dx$,
4) $\left[\int_0^1 |Q(x) - g(x)|^2 dx\right]^{\frac{1}{2}} < \epsilon$,
5) $\sum_{|k|=N_0}^{N} |a_k|^{2+\epsilon} < \epsilon$,
6) $|a_{\sigma(k)}| > |a_{\sigma(k+1)}| > 0, \ \forall k \in (N_0, N)$,
7) $\max_{N_0 \leq m \leq N} \int_0^1 \left|\sum_{|k|=N_0}^{m} a_k e^{i2\pi kx}\right| dx < 3\int_0^1 |f(x)|dx$.
8) $\max_{N_0 \leq m \leq N} \int_0^1 \left|\sum_{|k|=N_0}^{m} a_{\sigma(k)} e^{i2\pi \sigma(k)x}\right| dx < 3\int_0^1 |f(x)|dx$
9) $\sigma(-k) = -\sigma(k)$

**Proof.** This lemma is proved analogously to lemma 2 of [29].

### A. Proof of Theorems 3 and 4

Let $0 < \epsilon < 1$. An application of lemma 1 with regard to the sequence of trigonometric polynomials with rational coefficients that we denote by

$$\{f_k(x)\}_{k=1}^{\infty}, \tag{1}$$

leads to some sequences of functions $\{\bar{g}_k(x)\}_{k=1}^{\infty}$ sets $\{E_k\}$ and polynomials

$$\sum_{|k|=m_{n-1}}^{m_n-1} a_k^{(n)} e^{i2\pi kx} = \bar{Q}_n(x) =$$

$$= \sum_{|k|=m_{n-1}}^{m_n-1} a_{\sigma_n(k)}^{(n)} e^{i2\pi\sigma_n(k)x} \ , \ m_0 = 1 \ , \ a_{-k}^{(n)} = \bar{a}_k^{(n)}; \tag{2}$$

where $\{\sigma_n(k)\}_{k=m_{n-1}}^{m_n-1}$ $(\sigma_n(-k) = -\sigma_n(k))$ is some rearrangement of natural numbers $m_{n-1}, m_{n-1}+1, ..., m_n-1$ (for any fixed $n$). Besides, the following conditions are satisfied:

$$\overline{g}_n(x) = f_n(x), \quad x \in E_n, \tag{3}$$

$$|E_n| > 1 - \epsilon 4^{-8(n+2)}, \tag{4}$$

$$\frac{1}{2}\int_0^1 |f_n(x)|dx < \int_0^1 |\overline{g}_n(x)|dx < 3 \cdot \int_0^1 |f_n(x)|dx, \tag{5}$$

$$\left(\int_0^1 \left|\overline{Q}_n(x) - \overline{g}_n(x)\right|^2 dx\right)^{1/2} < 4^{-8(n+2)}, \tag{6}$$

$$\max_{m_{n-1} \leq N < m_n} \int_0^1 \left|\sum_{k=m_{n-1}}^{N} a_{\sigma_n(k)}^{(n)} e^{i2\pi\sigma_n(k)x}\right| \leq 3\int_0^1 |f_n(x)|dx, \tag{7}$$

$$\max_{m_{n-1} \leq N < m_n} \int_0^1 \sum_{k=m_{n-1}}^{N} a_k^{(n)} e^{i2\pi kx} dx \leq 3\int_0^1 |f_n(x)|dx, \tag{8}$$

$$|a_{\sigma_n(k)}^{(n)}| > |a_{\sigma_n(k+1)}^{(n)}| > |a_{\sigma_{n+1}(m_n)}^{(n+1)}| > 0, \quad \forall k \in [m_{n-1}, m_n - 1], \quad \forall n \geq 1. \tag{9}$$

$$\sum_{|k|=m_{n-1}}^{m_n} \left|a_k^{(n)}\right|^{2+4^{-8(n+2)}} < 4^{-8(n+2)}. \tag{10}$$

Taking

$$E = \bigcap_{n=1}^{\infty} E_n, \tag{11}$$

we have $|E| > 1 - \epsilon$. (see (4)).

Let $f(x) \in L^1[0,1]$. Then by (1) one can easily choose a subsequence $\{f_{k_n}(x)\}_{n=1}^{\infty}$ such that

$$\lim_{N\to\infty}\int_0^1\left|\sum_{n=1}^{N} f_{k_n}(x)-f(x)\right|dx=0, \tag{12}$$

$$\int_0^1 |f_{k_n}(x)|\,dx\le\bar\epsilon\cdot 4^{-8(n+2)},\quad n\ge 2. \tag{13}$$

where $\bar\epsilon=\min\{\frac{\epsilon}{2};||f||\}$ and $f_{k_1}(x)$ is of the form

$$\sum_{|k|=0}^{m_{\nu_1}-1} b_k e^{i2\pi kx}=f_{k_1}(x)=\sum_{|k|=0}^{m_{\nu_1}-1} b_{\overline{\sigma}(k)}e^{i2\pi\overline{\sigma}(k)x};$$

$$|b_k|>|b_{k+1}|>0,\quad\forall|k|\in[1,m_{\nu_1}], \tag{14}$$

and $\overline{\sigma}(|k|)$- is some rearrangement of natural numbers $1,2,...,m_{\nu_1}-1$ $(\overline{\sigma}(-k)=-\overline{\sigma}(k))$.

We evidently have

$$\int_0^1|f(x)-f_{k_1}(x)|\,dx<\frac{\overline{\epsilon}}{2}. \tag{15}$$

Now set

$$a_k=\begin{cases}b_k, & k\in[1,m_{\nu_1});\\ a_k^{(n)}, & k\in[m_{n-1},m_n),\quad n\ge\nu_1+1.\end{cases} \tag{16}$$

$$\sigma(k)=\begin{cases}\overline{\sigma}(k), & k\in[1,m_{\nu_1});\\ \sigma_n(k), & k\in[m_{n-1},m_n),\quad\forall n\ge\nu_1+1.\end{cases} \tag{17}$$

$$g_1(x)\equiv Q_1(x)=f_{k_1}(x)=\sum_{|k|=0}^{m_{\nu_1}-1}a_{\sigma(k)}e^{i2\pi\sigma(k)x}. \tag{18}$$

Suppose we already have determined the numbers $\nu_1<...<\nu_{q-1}$, $m_{\nu_1}-1=l(1)<l(2)<...<l(q-1)$, $\{b_{l(k)}\}_{k=1}^{q-1}$, the functions $g_n(x)$, $f_{\nu_n}(x)$, $1\le n\le q-1$ and the polynomials

$$\sum_{|k|=M_n}^{\overline{M}_n}a_ke^{i2\pi kx}=Q_n(x)=\sum_{|k|=M_n}^{\overline{M}_n}a_{\sigma(k)}e^{i2\pi\sigma(k)x},$$

$$M_n=m_{\nu_{n-1}},\quad\overline{M}_n=m_{\nu_n}-1,\quad M_1>N_1$$

satisfying the conditions:

$$g_n(x)=f_{k_n}(x),\quad x\in E_{\nu_n},\quad 1\le n\le q-1,$$

$$\int_0^1|g_n(x)|dx<4^{-3n}\cdot\bar\epsilon\quad;\quad 1\le n\le q-1$$

$$\int_0^1\left|\sum_{k=2}^{n}\left[\left(Q_k(x)+b_{l(k)}e^{i2\pi\sigma(l(k))x}\right)-g_k(x)\right]\right|dx<$$

$$<4^{-8(n+1)},\quad 1\le n\le q-1, \tag{19}$$

$$l(n)=\min\{k\in N:\quad k\notin[1,m_{\nu_1}]\cup$$

$$\cup\left(\bigcup_{j=2}^{n-1}[M_j,\overline{M}_j]\right)\cup\{l(s)\}_{s=1}^{n-1}\}\},$$

$$\max_{M_n\le N<\overline{M}_n}\int_0^1\left|\sum_{k=M_n}^{N}a_{\sigma(k)}e^{i2\pi\sigma(k)x}\right|dx<4^{-3n},\ \ 1\le n\le q-1.$$

$$\max_{M_n\le N<\overline{M}_n}\int_0^1\left|\sum_{k=M_n}^{N}a_ke^{i2\pi kx}\right|dx<4^{-3n},\quad 1\le n\le q-1.$$

$$|a_{\overline{M}_n}|>|b_{l(n)}|>|a_{M_{n+1}}|$$

We choose a natural number $\nu_q$ and a function $f_{\nu_q}(x)$ from the sequence (1) such that

$$\int_0^1\left|f_{\nu_q}(x)-\left(f_{k_q}(x)-\sum_{n=2}^{q-1}\left[\left(Q_n(x)+b_{l(n)}e^{i2\pi\sigma(l(n))x}\right)-\right.\right.\right.$$

$$\left.\left.\left.-g_n(x)\right]\right)\right|dx<4^{-8(q+2)}. \tag{20}$$

$$|a_{(M_q)}|<|b_{l(q-1)}|,\quad where\ M_q=m_{\nu_q-1} \tag{21}$$

(see (10) and (16)). Then by (13) and (19) we have

$$\int_0^1\left|f_{k_q}(x)-\sum_{n=2}^{q-1}\left[\left(Q_n(x)+b_{l(n)}e^{i2\pi\sigma(l(n))x}\right)-\right.\right.$$

$$\left.\left.-g_n(x)\right]\right|dx<4^{-8q-1}.$$

Therefore by (20) we have

$$\int_0^1\left|f_{\nu_q}(x)\right|dx<4^{-8q}. \tag{22}$$

We define

$$Q_q(x)=\overline{Q}_{\nu_q}(x)=\sum_{k=M_q}^{\overline{M}_q}a_ke^{i2\pi kx},$$

$$\overline{M}_q=m_{\nu_q}-1,\quad M_q=m_{\nu_{q-1}}, \tag{23}$$

$$g_q(x)=f_{k_q}(x)+[\overline{g}_{\nu_q}(x)-f_{\nu_q}(x)], \tag{24}$$

$$l(q)=\min\{k\in N:\quad k\notin[1,m_{\nu_1}]\cup$$

$$\cup\left(\bigcup_{n=2}^{q-1}[M_n,\overline{M}_n]\right)\cup\{l(s)\}_{s=1}^{q-1}\}\}, \tag{25}$$

$$b_{l(q)}=\min\left(4^{-8(q+3)};\quad\frac{1}{2}|a_{\overline{M}_q}|\right). \tag{26}$$

Then in view of (3)-(7), (16), (19)-(26) we get

$$g_q(x)=f_{k_q}(x),\quad x\in E_{\nu_q}, \tag{27}$$

$$\int_0^1|g_q(x)|dx\le\int_0^1\left|f_{\nu_q}(x)-\left(f_{k_q}(x)-\sum_{j=2}^{q-1}\left[\left(Q_j(x)+\right.\right.\right.\right.$$

$$+b_{l(j)}e^{i2\pi\sigma(l(j))x}\Big)-g_j(x)\Big]\Big)\Big|dx+\int_0^1|\overline{g}_{\nu_q}(x)|dx+$$

$$+\int_0^1\left|\sum_{j=2}^{q-1}\Big[\Big(Q_j(x)+b_{l(j)}e^{i2\pi\sigma(l(j))x}\Big)-g_j(x)\Big]\right|dx<4^{-3q}. \quad (28)$$

$$\int_0^1\left|\sum_{j=2}^{q-1}\Big[\Big(Q_j(x)+b_{l(j)}e^{i2\pi\sigma(l(j)x}\Big)-g_j(x)\Big]\right|dx\le$$

$$\le\int_0^1\Big|f_{\nu_q}(x)-\Big(f_{k_q}(x)-\sum_{j=2}^{q}\Big[\Big(Q_j(x)+b_{l(j)}\cdot e^{i2\pi\sigma(l(j))x}\Big)$$

$$-g_j(x)\Big]\Big)\Big|dx++|b_{l(q)}|+\int_0^1|\overline{Q}_{\nu_q}(x)-\overline{g}_{\nu_q}(x)|dx<4^{-8(q+1)}, \quad (29)$$

$$\max_{M_q\le N<\overline{M}_q}\int_0^1\left|\sum_{|k|=M_q}^{N}a_{\sigma(k)}e^{i2\pi\sigma(k)x}\right|dx\le$$

$$\le 3\cdot\int_0^1|f_{\nu_q}(x)|dx<4^{-3q}, \quad (30)$$

$$\max_{M_q\le N<\overline{M}_q}\int_0^1\left|\sum_{|k|=M_q}^{N}a_ke^{i2\pi kx}\right|dx\le 3\cdot\int_0^1|f_{\nu_q}(x)|dx<4^{-3q}, \quad (31)$$

$$|a_{\sigma(M_q}|>...>|a_{\sigma(k)}|>...>|a_{\overline{M}_q}|>$$

$$>|b_{l(q)}|>|a_{M_{q+1}}|,\quad \forall q\ge 1. \quad (32)$$

Clearly, we can use induction to determine a sequence $\{g_q(x)\}$ of functions, numbers $\{l(q)\}_{q=2}^{\infty}$, $\{b_{l(q)}\}_{q=2}^{\infty}$ and a sequence $\{Q_q(x)\}$ of polynomials satisfying the conditions (25)- (32) for all $q\ge 1$.

Taking into account the choice of $\{\sigma(k)\}_{k=1}^{\infty}$, $\{[M_q,\overline{M}_q]\}_{q=2}^{\infty}$ and $\{l(q)\}_{q=2}^{\infty}$ (see (17), (23), (25)) we obtain, that the sequence of natural numbers

$$\sigma(1)...\sigma(m_{\nu_1}-1);\quad l(1),\sigma(M_2)...\sigma(\overline{M}_2);$$

$$;\ l(2),...,l(n-1),\sigma(M_n)...\sigma(k)...\sigma(\overline{M}_n);\quad l(n).... \quad (33)$$

is some rearrangement of sequence $1,2,...,n,....$

We may write the sequence (33) in the form

$$\sigma_f^{\circ}(1),\quad \sigma_f^{\circ}(2),...,\sigma_f^{\circ}(k),...$$

We define function $g(x)$ and series $\sum_{k=1}^{\infty}d_{\sigma_f^{\circ}(k)}e^{i2\pi\sigma_f^{\circ}(k)x}$ in the following form

$$g(x)=\sum_{k=1}^{\infty}g_k(x);\quad g_1(x)=Q_1(x)=f_{k_1}(x)=$$

$$=\sum_{k=1}^{m_{\nu_1}-1}a_ke^{i2\pi\sigma(k)x}, \quad (34)$$

$$\sum_{k=1}^{\infty}d_{\sigma_f^{\circ}(k)}e^{i2\pi\sigma_f^{\circ}(k)x}=\sum_{k=1}^{m_{\nu_1}-1}a_{\sigma(k)}e^{i2\pi\sigma(k)x}+$$

$$+\sum_{n=2}^{\infty}\left[\sum_{|k|=M_n}^{\overline{M}_n}a_{\sigma(k)}e^{i2\pi\sigma(k)x}+b_{l(n)}e^{i2\pi\sigma(l(n))x}\right], \quad (35)$$

where $\{d_{\sigma_f^{\circ}(k)}\}_{k=1}^{\infty}$-is a sequence

$$a_{\sigma(1)}...a_{\sigma(m_{\nu_1}-1)},b_{l(1)},a_{\sigma(M_2}...a_{\overline{\sigma(M_2)}};b_{l(2)},...,b_{l(n-1)},$$

$$,a_{\sigma(M_n)}.a_{\sigma(k)}..a_{\overline{\sigma(M_n}};b_{l(n)};a_{\sigma(M_{n+1})}...$$

From this and from (11), (12), (21), (26), (27), (32)-(35) follows that

$$|d_{\sigma_f^{\circ}(k)}|>|d_{\sigma_f^{\circ}(k+1)}|,\quad \forall k\ge 1,$$

$$\sum_{k=1}^{\infty}|d_k|^r<\infty,\quad \forall r>2,$$

$$g(x)\in L^1_{[0,1]},\quad g(x)=f(x),\quad x\in E.$$

Let $N>M_1$ be an arbitrary natural number. Then for some natural $q$ we have

$$N_q\le N<N_{q+1},$$

where

$$N_q=M_1+1+\sum_{k=2}^{q}[\overline{M}_k-M_k+2]\quad \forall q\ge 2.$$

The relations (26),(28)-(35) imply that

$$\int_0^1\left|\sum_{k=1}^{N}d_{\sigma_f^{\circ}(k)}e^{i2\pi\sigma_f^{\circ}(k)x}-g(x)\right|dx\le$$

$$\le\int_0^1\left|\sum_{\gamma=2}^{q-1}\Big[\Big(Q_j(x)+b_{l(j)}e^{i2\pi\sigma(l(j))x}\Big)-g_j(x)\Big]\right|dx+$$

$$+\sum_{s=q}^{\infty}\int_0^1|g_s(x)|dx+\max_{M_q\le m\le\overline{M}_q}\int_0^1\left|\sum_{|k|=M_q}^{m}a_{\sigma(k)}e^{i2\pi\sigma(k)x}\right|dx+$$

$$+|b_{l(q)}|<2^{-q}.$$

$$||G_N(g)||_1=\int_0^1\left|\sum_{k=1}^{N}c_{\sigma_f^{\circ}(k)}e^{i2\pi\sigma_f^{\circ}(k)x}\right|dx$$

$$\le\sum_{n=1}^{\infty}\left(\max_{M_n\le N\le\overline{M}_n}\int_0^1\left|\sum_{|k|=M_n}^{N}a_{\sigma(k)}e^{i2\pi\sigma(k)x}\right|dx\right)+$$

$$+\sum_{k=1}^{\infty}|b_{p(k)}|\le 2\int_0^1|g_1(x)|dx+\overline{\epsilon}\sum_{n=1}^{\infty}4^{-n}$$

$$\le 3\int_0^1|g(x)|dx\le 12\cdot\int_0^1|f(x)|dx\ .$$

Similarly, one can show that

$$||S_N(g)-g|| = \int_0^1 \left|\sum_{k=1}^{N} d_k e^{i2\pi kx} - g(x)\right| dx < 2^{-q}.$$

$$||S_N(g)|| \le 3\int_0^1 |g(x)|dx \le 12\int_0^1 |f(x)|dx\ .$$

Consequently

$$d_{\sigma_f^{\circ}(k)} = \int_0^1 g(x)e^{-2\pi\sigma_f^{\circ}(k)x}dx;$$

$$(d_k = \int_0^1 g(x)e^{-2\pi kx}dx = c_k(g))$$

Theorem 3 is proved.

Now we will prove that the system $\{e^{i2\pi\sigma(k)x}\}_{k=-\infty}^{+\infty}$ and set $E$ (see (11) and (17)) satisfy the conditions of theorem 4.

Repeating the arguments in the proof of theorem 3 for each $f(x)\in L^1[0,1]$ we can use induction to determine a sequence of polynomials $\{Q_n(x)\}$ from the sequence (2) of the form

$$Q_n(x) = \sum_{|k|=m_{\nu_{n-1}}}^{m_{\nu_n}-1} a_{\sigma(k)}e^{i2\pi\sigma(k)x},\ |a_{\sigma(k)}| > |a_{\sigma(k+1)}| > 0,$$

$$k\in[m_{\nu_n-1}, m_{\nu_n}),\ n\ge 1, \nu_n \nearrow$$

and a function $g(x)\in L^1[0,1]$ coinciding with $f(x)$ on $E$ satisfying the conditions

$$\int_0^1 \left|\sum_{n=1}^{j}(\sum_{|k|=m_{\nu_{n-1}}}^{m_{\nu_n}-1} a_{\sigma(k)}e^{i2\pi\sigma(k)x}) - g(x)\right| dx \le 2^{-2j}, j>1$$

$$\max_{m_{\nu_{n-1}}\le m< m_{\nu_n}} \int_0^1 \left|\sum_{|k|=m_{\nu_{n-1}}}^{m} a_{\sigma(k)}e^{i2\pi\sigma(k)x}) - g(x)\right| dx \le$$

$$\le 2^{-n},\ n>1$$

Theorem 4 is proved.

*B. Proof of Theorems 2 and 5*

We need the following elementary result:

**Lemma 2.** Let $m$ be an arbitrary natural number. Given any finite sequence $\{x_k\}_{k=1}^n$ of non negative integers and a monotonically increasing finite sequence $\{y_k\}_{k=1}^n$. Then

$$\sum_{k=1}^{m} x_{\sigma(k)}y_k \le \sum_{k=1}^{m} x_k y_k,$$

where $\{\sigma(k)\}_{k=1}^m$ is a permutation of positive integers such that $x_{\sigma(1)} \ge x_{\sigma(2)} \ge ... \ge x_{\sigma(m)}$.

**Proof** Let $m=2$ and let $x_2\ge x_1$ and $y_1<y_2$. We have

$$0 \le (x_2-x_1)(y_2-y_1) = x_2y_2 + x_1y_1 - (x_2y_1 + x_1y_2),$$

hence

$$\sum_{k=1}^{2} x_{\sigma(k)}y_k = x_{\sigma(1)}y_1 + x_{\sigma(2)}y_2 = x_2y_1 + x_1y_2 \le \sum_{k=1}^{2} x_k y_k.$$

It is not hard to see that using the mathematical induction methods we can obtain inequality a) for each natural $m$.

**Lemma 3.** Given any sequences $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$, with

$$x_k \ge 0,\quad and\quad 0<y_1<y_2<...<y_k<....$$

then

$$\sum_{k=1}^{\infty} x_{n_k}y_k \le \sum_{k=1}^{\infty} x_k y_k,$$

where $\{\sigma(k)\}_{k=1}^\infty$ is a permutation of natural numbers $1,2,...$, such that $\{x_{\sigma(k)}\}\searrow$.

**Proof.** We may assume that

$$\sum_{k=1}^{\infty} x_k y_k < \infty.$$

Let $\{\sigma(k)\}_{k=1}^\infty$ be a permutation of natural numbers $1,2,...$ such that

$$x_{\sigma(1)} \ge x_{\sigma(2)} \ge ... \ge x_{\sigma(k)} \ge ...$$

For any natural number $s$ we set

$$N_s = \max\{\sigma(k);\quad 1\le k\le s\}.$$

Using lemma 2, with $m=N_s$, for $\{x_k\}_{k=1}^{N_s}$ and $\{y_k\}_{k=1}^{N_s}$ we get

$$\sum_{k=1}^{N_s} x_{\sigma(k)}y_k \le \sum_{k=1}^{N_s} x_k y_k \le \sum_{k=1}^{\infty} x_k y_k.$$

Since $x_k\ge 0$ and $y_k>0$ we obtain

$$\sum_{k=1}^{s} x_{\sigma(k)}y_k \le \sum_{k=1}^{\infty} x_k y_k,\quad for\quad all\quad s\ge 1,$$

what completes the proof of lemma 3.

From lemma 3 we obtain the following

**Lemma 4.** $G(\ln^q) \subset G^{\searrow}(\ln^q)$ for all $q>0$, and $||f||_{G^{\searrow}} \le ||f||_G$.

**Proof .** Using lemma 3 with $x_k = |c_k(f)|^2$ and $y_k = \ln^q k,\ q>0,\ \forall k\ge 1$ we have if $f(x)\in G(\ln^q)$ then $f(x)\in G^{\searrow}(\ln^q)$ and $||f||_{G^{\searrow}} \le ||f||_G$ (see definitions $G(\ln^q)$, $G^{\searrow}(\ln^q)$).

It is not hard to see that there exists a function $f_0(x)\in G^{\searrow}(\ln^q)$ but $f_0(x)\notin G(\ln^q)$, $q>0$.

**Proof of Theorem 5.** Let $f(x)\in G^{\searrow}(\ln^q), q>0$.
From the definition of $G^{\searrow}(\ln^q)$ we get $||f||_{G^{\searrow}} =$

$\sum_{k=1}^{\infty}|c_{\sigma(k)}(f)|^2\ln^q k<\infty$, where $\{\sigma(k)\}$ is a permutation of the natural numbers $1,2,...$, which

$$|c_{\sigma(k)}(f)|\geq|c_{\sigma(k+1)}(f)|\geq....$$

We put

$$\lambda_m(f)=\sum_{k=m}^{\infty}|c_{\sigma(k)}(f)|^2\ln^q k.$$

From this we have

$$k|c_{\sigma(k)}(f)|^2\ln^q k\leq\sum_{s=k}^{2k-1}|c_{\sigma(s)}(f)|^2\ln^q s<\lambda_k(f).$$

Hence

$$|c_{\sigma(k+1)}(f)|^2\leq|c_{\sigma(k)}(f)|^2\leq\frac{\lambda_k(f)}{k\ln^q k}.$$

From an approximation's error we obtain

$$R_k^2(f)=\sum_{s=k}^{\infty}|c_{\sigma(s)}(f)|^2\leq\lambda_{[\frac{k}{2}]}(f)\sum_{s=[\frac{k}{2}]}\frac{1}{s\ln^q s}\leq$$

$$\leq\lambda_{[\frac{k}{2}]}(f)\int_{[\frac{k}{2}]}^{\infty}\frac{dx}{x\ln^q x}\leq\lambda_{[\frac{k}{2}]}(f)\frac{1}{(q-1)(\ln[\frac{k}{2}])^{q-1}}.$$

Since $\lim_{k\to\infty}\lambda_k(f)=0$ and $\lambda_k(f)\leq||f||_{G^{\searrow}(\ln q)}$.

We get

$$R_k^2(f)=o\left(\frac{1}{\ln^{q-1}k}\right),$$

$$R_k^2(f)\leq\frac{||f||_{G^{\searrow}}}{q-1}\frac{1}{(\ln k-\ln 2)^{q-1}},\forall k>2.$$

Conversely suppose that there exists $C>0$ such that

$$R_k^2(f)\leq C\frac{1}{\ln^q k},\ q>0,k>1.$$

Since

$$R_k^2(f)=\sum_{s=k+1}^{\infty}|c_{\sigma(s)}(f)|^2\geq\sum_{s=k+1}^{2k}|c_{\sigma(s)}(f)|^2\geq k|c_{\sigma(2k)}(f)|^2,$$

then

$$|c_{\sigma(2k+1)}(f)|^2\leq|c_{\sigma(2k)}(f)|^2<C\frac{1}{k(\ln k)^q}.$$

Hence, if $p<q-1$ $(q-p>1)$

$$\sum_{k=1}^{\infty}|c_{\sigma(k)}(f)|^2(\ln k)^p\leq 2C\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^{q-p}}<\infty,$$

which completes the proof of Theorem 5.

In the proof of Theorem 2 we will use the following theorem of P.L. Ul'yanov [30].

Let a $\omega(t)$ be a nonnegative function, increasing in $(0,1]$ with $\int_0^1\alpha(x)dx$=$\infty$ and let for an constant C and for all $\delta\in(0,\frac{1}{4}]$

$\frac{1}{\delta^2}\int_0^{\delta}x^2\alpha(x)dx\leq C\ \int_{\delta}^1\alpha(x)dx,$

$$\frac{1}{\delta^2}\int_0^{\delta}x^2\alpha(x)dx\leq C\ \int_{\delta}^1\alpha(x)dx,$$

Then the condition

$$\int_0^1\int_0^1[f(x+t)-f(x-t)]^2\alpha(x)dx<\infty,$$

is equivalent to

$$\sum_{k=1}^{\infty}|c_k(f)|^2\omega(k)\ <\infty$$

From this theorem we obtain that the condition

$$\int_0^1\int_0^1\frac{[f(x+t)-f(x-t)]^2}{t}(\ln\frac{1}{t})^{\delta}dx<\infty,\quad\delta>0,$$

is equivalent to

$$\sum_{k=1}^{\infty}|c_k(f)|^2(\ln k)^{1+\delta}<\infty$$

Hence and from Lemma 4 and Theorem 5 (with $q=1+\delta$) we have the proof of Theorem 2.

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**Martin Grigoryan** Date of Birth: 05.05.1954.

Education (beginning with IHE):

1971-1976 Student of Yerevan State University (YSU)

1976-1979 Post Graduate Student (YSU)

Academic degree: Doctor of Phys.-Math. Sciences

candidate : *Convergence of orthogonal series in $L^p$ metrics*, 1980, Institute of Mathematics, National Academy of Sciences of Armenia

doctoral : *Convergence of regular and dual Fourier series by complete orthogonal systems*, 1997, Faculty of Mathematics of YSU

Professional experience:

From 1980 - present, Faculty of Physics of YSU

1993-1994 - Visiting Professor UAM, Madrid, Spain

Academic courses–: Mathematical analysis, Functional analysis, The theory of functions, Linear an nonlinear approximations

Scale of professional interests: Harmonic analysis and the theory of approximations

Participation in grant projects (present and past):

1994-1996 - Grant MVR000 from international Science Foundation,

1996 - Grant from Promethues-Gitutjun, Joint- Stock Company,

1997 - Grant from Promethues-Gitutjun, Joint- Stock Company,

1998 - Grant from Promethues-Gitutjun, Joint- Stock Company,

1997 - 1999 - Grant 96-880 from the Government of Armenia,

2000 - 2003 - Grant 00-310 from the Government of Armenia

2004 - 2007 - Grant- 01-00 from the Government of Armenia

2008 - present - Grant- 45 from the Government of Armenia

Honors : The best scientific work 2009,

RA President Award 2010

Languages: Armenian, Russian, English, Spanish, German

Author of more than 100 scientific works published in well known scientific journals (*Trans. Amer. Math. Soc., J. of Funct. Anal., J. of Math. Anal. and Appl., Proc. Amer. Math. Soc., Studia Math. etc.*).