Remark on small analytic solutions to the Schrödinger equation with cubic convolution

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***Abstract*—We consider the Cauchy problem for the Schrödinger equation with cubic convolution in space dimension $d \geq 3$. We assume that the interaction potential V belongs to the weak $L^{d/\sigma}$ space with $2 \leq \sigma < d$. We prove that if the initial data ϕ is sufficiently small in the sense of the Sobolev space $H^{\sigma/2-1}$ and either ϕ or its Fourier transform $\mathcal{F}\phi$ satisfies a real-analytic condition, then the solution $u(t)$ is also real-analytic for any $t \neq 0$. We also prove that if ϕ and V satisfy some strong condition, then $u(t)$ can be extended to an entire function on $\mathbb{C}^d$ for any $t \neq 0$. We remark that no $H^{\sigma/2-1}$ smallness condition is imposed on first and higher order partial derivatives of ϕ and $\mathcal{F}\phi$.**

***Index Terms*—Nonlinear Schrödinger equation, Analytic solution, Hartree term,**

I. Introduction

In this paper, we establish an extended result of main theorems in [23]. We consider the Cauchy problem for the nonlinear Schrödinger equation of the form

$$\begin{cases} iu_t + \Delta u &= F(u), \\ u(0,x) &= \phi(x). \end{cases} \quad (1)$$

Here, u is a complex-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $d \geq 3$, $i = \sqrt{-1}$, Δ is the Laplacian in $\mathbb{R}^d$, $F(u)$ denotes $(V * |u|^2)u$, which is called the Hartree term or cubic convolution, and $*$ is the convolution in $\mathbb{R}^d$. Throughout this paper, we assume that $2 \leq \sigma < d$ and the interaction potential V is a given function on $\mathbb{R}^d$ and belongs to the weak $L^{d/\sigma}$ space. In other words, we assume that

$$\sup_{\lambda > 0} \lambda \, \mu\Big(\big\{ x \in \mathbb{R}^d; |V(x)| > \lambda \big\} \Big)^{\sigma/d} < \infty, \quad (2)$$

where μ is the Lebesgue measure on $\mathbb{R}^d$. There is a large literature on the Cauchy problem for nonlinear Schrödinger equations (see, e.g., [2], [13], [25] and references therein).

To state a global existence theorem for (1), we set some notation. For $q \in [1, \infty]$, we denote the Lebesgue space $L^q(\mathbb{R}^d)$ and the L^q-norm by L^q and $\|\cdot\|_q$, respectively, and we set $\|\cdot\| = \|\cdot\|_2$. Furthermore, for $s \in \mathbb{R}$, H^s_q denotes the inhomogeneous Sobolev space $H^s_q(\mathbb{R}^d)$, we abbreviate H^s_2 to H^s, the H^s-norm denotes $\|\cdot\|_{(s)}$. For $t \in \mathbb{R}$, $U(t)$ denotes the propagator $e^{it\Delta}$ for the free Schrödinger equation $iu_t + \Delta u = 0$. Define $s(\sigma) = \sigma/2 - 1$ for $2 \leq \sigma < d$. Mochizuki [14] has proved that if the condition

$$\text{either} \quad |V(x)| \leq C|x|^{-\sigma} \quad \text{or} \quad V \in L^{d/\sigma},$$

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which is stronger than (2), holds and $\|\phi\|_{(s(\sigma))}$ is sufficiently small, then there exists a time-global solution u to the integral equation of the form

$$u(t) = U(t)\phi - i \int_0^t U(t-t')F(u(t'))dt', \quad t \in \mathbb{R}, \quad (3)$$

which is equivalent to (1), such that $u(t)$ behaves like a free solution $U(t)\phi_+$ as $t \to \infty$ in $H^{s(\sigma)}$. In particular, the inverse wave operator $\mathbf{V}_+ : \phi \mapsto \phi_+$ is well-defined on a neighborhood of 0 in $H^{s(\sigma)}$. Remark that in the above existence theorem, no $H^{s(\sigma)}$ smallness condition is imposed on first and higher order partial derivatives of ϕ and its Fourier transform $\mathcal{F}\phi$.

A. Main results

In this paper, assuming that either ϕ or $\mathcal{F}\phi$ satisfies a real-analytic condition, we study the analyticity of the small solution $u(t)$ to (3), the final data $\mathbf{V}_+(\phi)$ and $\mathcal{F}\mathbf{V}_+(\phi)$. Remark that we do not impose any $H^{s(\sigma)}$ smallness condition on any partial derivative of ϕ and $\mathcal{F}\phi$. We now briefly state a part of our main result. We show that

(I) There exists some $\eta > 0$ such that if $0 < \|\phi\|_{(s(\sigma))} < \eta$ and

$$\limsup_{|\alpha| \to \infty} \left(\frac{\|x^\alpha \phi\|_{(s(\sigma))}}{\alpha!} \right)^{1/|\alpha|} < \infty, \quad (4)$$

then $u(t)$ is real-analytic for any $t \neq 0$. More precisely, the map $x \mapsto M(-t)u(t,x)$ can be extended to a function holomorphic on $\{z \in \mathbb{C}^d; |\mathrm{Im}\, z| < |2t|/C(\phi, d, V)\}$. Here, we have defined for $t \in \mathbb{R} \setminus \{0\}$

$$M(t) : \mathcal{S}'(\mathbb{R}^d_x) \ni f \mapsto \exp\left(\frac{i|x|^2}{4t} \right) f \in \mathcal{S}'(\mathbb{R}^d_x)$$

and

$$C(\phi, d, V) = \sup_{|\alpha| > 0} \left(\frac{(1+|\alpha|)^{d+1} \|x^\alpha \phi\|_{(s(\sigma))}}{\alpha! \, \|\phi\|_{(s(\sigma))}} \right)^{1/|\alpha|}.$$

As a corollary, the following property is proved:

(II) If ϕ and V satisfy some strong condition, then the mapping $x \mapsto M(-t)u(t,x)$ can be extended an entire function on $\mathbb{C}^d$ for any $t \neq 0$.

In [23], Properties (I) and (II) with $\sigma = 2$ were shown. Therefore, in this paper we establish an extended result of Theorems in [23]. For the detailed statement of (I)–(II), see Theorem I.3(3) and Corollary I.5.

Remark I.1. *There are many papers on the analytic Cauchy problem for nonlinear Schrödinger equations and related* *equations (see, e.g., [1], [3]–[12], [15], [18]–[22], [24], [27]). In particular, we can use methods in [9], [10], [19] to show the analyticity of the solution $u(t)$ of (3) and more detailed properties. However, it does not seem that we can prove (I)-(II) by using the methods directly.*

To state our main results precisely, we list some notation. Let $2 \le \sigma < d$. For $\lambda > 0$, $B_\lambda H^{s(\sigma)}$ denotes the closed ball of $H^{s(\sigma)}$ with radius λ centered at 0. Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\alpha \in \mathbb{N}_0^d$, we denote $1 + |\alpha|$ by $\langle\alpha\rangle$. Furthermore, for $t \in \mathbb{R}$, we define $J^\alpha = U(t)x^\alpha U(-t)$. Remark that the following identity holds:

$$J^\alpha = M(t)(2it\partial)^\alpha M(-t), \quad \alpha \in \mathbb{N}_0^d,\ t \neq 0. \tag{5}$$

We put $r = 6d/(3d-4)$. For an interval I in $\mathbb{R}$, we denote $L^3(I; H_r^{s(\sigma)})$ and $(C \cap L^\infty)(I; H^{s(\sigma)}) \cap L^3(I; H_r^{s(\sigma)})$ by $Y(\sigma, I)$ and $Z(\sigma, I)$, respectively. We define $Y(\sigma) = Y(\sigma, \mathbb{R})$, $Z(\sigma) = Z(\sigma, \mathbb{R})$,

$$Z(\sigma)^\infty = \left\{v \in Z(\sigma); \partial^\alpha v \in Z(\sigma),\ \alpha \in \mathbb{N}_0^d\right\},$$
$$Z(\sigma)_\infty = \left\{v \in Z(\sigma); J^\alpha v \in Z(\sigma),\ \alpha \in \mathbb{N}_0^d\right\}$$

and

$$H(\sigma)^\infty = \left\{\psi \in H^{s(\sigma)}; \partial^\alpha \psi \in H^{s(\sigma)},\ \alpha \in \mathbb{N}_0^d\right\},$$
$$H(\sigma)_\infty = \left\{\psi \in H^{s(\sigma)}; x^\alpha \psi \in H^{s(\sigma)},\ \alpha \in \mathbb{N}_0^d\right\}.$$

We set for a Banach space $X \subset \mathcal{S}'(\mathbb{R}^d)$,

$$\mathfrak{L}(\partial, \psi, X) = \limsup_{|\alpha|\to\infty} \left(\frac{\|\partial^\alpha \psi\|_X}{\alpha!}\right)^{1/|\alpha|}$$

if $\|\partial^\alpha \psi\|_X < \infty$ for any $\alpha \in \mathbb{N}_0^d$. Similarly, we set

$$\mathfrak{L}(x, \psi, H^{s(\sigma)}) = \limsup_{|\alpha|\to\infty} \left(\frac{\|x^\alpha \psi\|_{(s(\sigma))}}{\alpha!}\right)^{1/|\alpha|}, \ \psi \in H(\sigma)_\infty,$$
$$\mathfrak{L}(\partial, v, Z(\sigma)) = \limsup_{|\alpha|\to\infty} \left(\frac{\|\partial^\alpha v\|_{Z(\sigma)}}{\alpha!}\right)^{1/|\alpha|}, \ v \in Z(\sigma)^\infty,$$
$$\mathfrak{L}(J, v, Z(\sigma)) = \limsup_{|\alpha|\to\infty} \left(\frac{\|J^\alpha v\|_{Z(\sigma)}}{\alpha!}\right)^{1/|\alpha|}, \ v \in Z(\sigma)_\infty.$$

Remark I.2. *The Sobolev embedding theorem implies that $\mathfrak{L}(\partial, \psi, L^\infty) \le \mathfrak{L}(\partial, \psi, L^q)$ for any $q \in [1, \infty]$. Therefore, if $\mathfrak{L}(\partial, \psi, L^q) < \infty$ for some $q \in [1, \infty]$, then ψ can be extended to a holomorphic function on the domain $\{z \in \mathbb{C}^d; |\mathrm{Im}\, z| < 1/\mathfrak{L}(\partial, \psi, L^q)\}$.*

We are ready to state our main results.

Theorem I.3. *Let $2 \le \sigma < d$. Assume (2). Then a positive number η satisfies the following properties:*

1) *For any $\phi \in B_\eta H^{s(\sigma)}$, there exist a unique time-global solution $u \in Z(\sigma)$ to (3) and a function ϕ_+ such that $U(-t)u(t) \to \phi_+$ as $t \to +\infty$ in $H^{s(\sigma)}$. Hence the inverse wave operator $\mathbf{V}_+ : B_\eta H^{s(\sigma)} \ni \phi \mapsto \phi_+ \in H^{s(\sigma)}$ is well-defined.*
2) *If*

$$\phi \in B_\eta H^{s(\sigma)} \cap H(\sigma)^\infty \quad \textit{and} \quad \mathfrak{L}(\partial, \phi, H^{s(\sigma)}) < \infty, \tag{6}$$

then $u \in Z(\sigma)^\infty$ and $\mathbf{V}_+(\phi) \in H(\sigma)^\infty$. Furthermore, if $\phi \neq 0$, then we have

$$\mathfrak{L}(\partial, u, Z(\sigma)),\ \mathfrak{L}(\partial, \mathbf{V}_+(\phi), H^{s(\sigma)}) \le \sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^{d+1} \|\partial^\alpha \phi\|_{(s(\sigma))}}{\alpha!\, \|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}. \tag{7}$$

3) *If*

$$\phi \in B_\eta H^{s(\sigma)} \cap H(\sigma)_\infty \quad \textit{and} \quad \mathfrak{L}(x, \phi, H^{s(\sigma)}) < \infty, \tag{8}$$

then $u \in Z(\sigma)_\infty$ and $\mathbf{V}_+(\phi) \in H(\sigma)_\infty$. Furthermore, if $\phi \neq 0$, then we have

$$\mathfrak{L}(J, u, Z(\sigma)),\ \mathfrak{L}(x, \mathbf{V}_+(\phi), H^{s(\sigma)}) \le \sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^{d+1} \|x^\alpha \phi\|_{(s(\sigma))}}{\alpha!\, \|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}. \tag{9}$$

We give a remark on the above theorem.

Remark I.4. *(1) If (6) holds, then it follows that*

$$\sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^{q} \|\partial^\alpha \phi\|_{(s(\sigma))}}{\alpha!\, \|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|} < \infty, \quad q > 0$$

and hence that $\mathfrak{L}(\partial, \mathbf{V}_+(\phi), H^{s(\sigma)}), \mathfrak{L}(\partial, u(t), H^{s(\sigma)}) < \infty$ for any $t \in \mathbb{R}$. We see from Remark I.2 that $u(t)$ and $\mathbf{V}_+(\phi)$ are real-analytic.

(2) Assume (8) and that $\phi \neq 0$. Since

$$\begin{aligned}&\mathfrak{L}(\partial, \mathcal{F}\mathbf{V}_+(\phi), L^2) = \mathfrak{L}(x, \mathcal{F}\mathbf{V}_+(\phi), L^2)\\ &\le \mathfrak{L}(x, \mathbf{V}_+(\phi), H^{s(\sigma)})\\ &\le \sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^{d+1} \|x^\alpha \phi\|_{(s(\sigma))}}{\alpha!\, \|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|} =: C(\phi, d, V) < \infty,\end{aligned}$$

it follows from Remark I.2 that $\mathcal{F}\mathbf{V}_+(\phi)$ is real-analytic. For any $t \neq 0$, we see from the identity (5) that $|2t|\, \mathfrak{L}(\partial, M(-t)u(t), L^2) \le \mathfrak{L}(J, u, Z(\sigma)) \le C(\phi, d, V)$, and hence that the mapping $x \mapsto M(-t)u(t,x)$ can be extended to a function holomorphic on the domain $\{z \in \mathbb{C}^d; |\mathrm{Im}\, z| < |2t|/C(\phi, d, V)\}$.

It is a natural question to ask whether the solution $u(t)$ can be extended to an entire function on $\mathbb{C}^d$ if ϕ satisfies some strong condition. The following result is a partial answer:

Corollary I.5. *Let $2 \le \sigma < d$. Assume (2) and that $\phi \in B_\eta H^{s(\sigma)}$, where η is the positive number mentioned in Theorem I.3. Let u be the solution to (3).*

(1) If

$$\phi \in H(\sigma)^\infty, \quad \mathfrak{L}(\partial, \phi, H^{s(\sigma)}) = 0 \tag{10}$$

and

$$\partial^\alpha V \in L^{d/\sigma}\ (\alpha \in \mathbb{N}_0^d), \quad \mathfrak{L}(\partial, V, L^{d/\sigma}) = 0, \tag{11}$$

then

$$\mathfrak{L}(\partial, u, Z(\sigma)),\ \mathfrak{L}(\partial, \mathbf{V}_+(\phi), H^{s(\sigma)}) = 0 \tag{12}$$

and

$$\lim_{t\to+\infty}\sup_{\alpha\in\mathbb{N}_0^d}\frac{\|\partial^\alpha(U(-t)u(t)-\mathbf{V}_+(\phi))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}=0$$

for any $\varepsilon>0$.

(2) If

$$\phi\in H(\sigma)_\infty,\quad \mathfrak{L}(x,\phi,H^{s(\sigma)})=0 \tag{13}$$

and

$$\partial^\alpha V\in L^\infty\ (\alpha\in\mathbb{N}_0^d),\quad \mathfrak{L}(\partial,V,L^\infty)=0, \tag{14}$$

then

$$\mathfrak{L}(\partial,M(-t)u(t),L^2)=0\quad \text{for any } t\neq 0. \tag{15}$$

Remark I.6. *Assume (12). We see from Remark I.2 that* $u(t)$ *and* $\mathbf{V}_+(\phi)$ *are extended to entire functions*

$$\widetilde{u}(t,z):=\sum_{\alpha\in\mathbb{N}_0^d}\frac{\partial^\alpha u(t,0)}{\alpha!}z^\alpha$$

and

$$\sum_{\alpha\in\mathbb{N}_0^d}\frac{\partial^\alpha\mathbf{V}_+(\phi)|_{x=0}}{\alpha!}z^\alpha,\quad z\in\mathbb{C}^d,$$

respectively. Similarly, if $t\neq 0$*, then* $M(-t)u(t)$ *satisfying (15) is extended to*

$$\sum_{\alpha\in\mathbb{N}_0^d}\frac{\partial^\alpha M(-t)u(t)|_{x=0}}{\alpha!}z^\alpha,\quad z\in\mathbb{C}^d.$$

Since the function $\exp(i|x|^2/(4t))$ *can be extended to an entire function, the solution* $u(t)$ $(=M(t)M(-t)u(t))$ *is extended to the entire function* $\widetilde{u}(t,z)$.

The rest of this paper is organized as follows. In Section 2, we list some useful inequalities. In order to show Theorem I.3(2) (resp. (3)), in Section 3 we define the function space $Z(\sigma)^\phi$ (resp. $Z(\sigma)_\phi$), which is included in the set of all functions v satisfying that $\mathfrak{L}(\partial,v,Z(\sigma))<\infty$ (resp. $\mathfrak{L}(J,v,Z(\sigma))<\infty$). In Section 4, we establish Theorem I.3. In Sections 5, we establish Corollary I.5.

II. Preliminaries

In this section, we list some inequalities to prove our main results. Using Strichartz type estimates for linear Schrödinger equations (see, e.g., [2], [13], [25], [26], [28]), we obtain the following time-space estimates:

Proposition II.1. *Let* I *be an interval in* $\mathbb{R}_t$ *and fix* $t_0\in\overline{I}$*. Then for any* $f\in H^{s(\sigma)}$ *and* $G\in L^1(I;H^{s(\sigma)})$*, we have*

$$U(t)f,\ \int_{t_0}^t U(t-t')G(t')dt'\in Z(\sigma,I).$$

Furthermore, there exists a positive number C *independent of* I *and* t_0 *such that*

$$\|U(t)f\|_{Z(\sigma,I)}\le C\|f\|_{(s(\sigma))},$$
$$\left\|\int_{t_0}^t U(t-t')G(t')dt'\right\|_{Z(\sigma,I)}\le C\|G\|_{L^1(I;H^{s(\sigma)})}.$$

We use the following lemma in the proof of Proposition II.3 below:

Lemma II.2. *Let* $1<q_1,q_2<\infty$*. Assume (2) and that*

$$1+\frac{1}{q_1}=\frac{\sigma}{d}+\frac{1}{q_2}.$$

Then

$$\|V*f\|_{q_1}\le C\|f\|_{q_2},\quad f\in L^{q_2}.$$

Proof. Since V belongs to the Lorentz space $L(d/\sigma,\infty)$, the desired inequality follows from Theorem 2.6 in O'Neil [17]. □

In order to estimate the nonlinearity F, we use the following two propositions:

Proposition II.3. *(1) Assume (2). Then*

$$\|(V*f_1f_2)f_3\|_{(s(\sigma))}\le C\prod_{k=1}^3\|f_k\|_{H_r^{s(\sigma)}}.$$

(2) If $h\in L^\infty$*, then*

$$\|(h*f_1f_2)f_3\|_{(s(\sigma))}\le C\|h\|_\infty\prod_{k=1}^3\|f_k\|_{H^{s(\sigma)}}.$$

(3) If $h\in L^{d/\sigma}$*, then*

$$\|(h*f_1f_2)f_3\|_{(s(\sigma))}\le C\|h\|_{d/\sigma}\prod_{k=1}^3\|f_k\|_{H_r^{s(\sigma)}}.$$

Proof. We use the generalized Hölder inequality

$$\|fg\|_{H_q^s}\le C\|f\|_{H_{q_1}^s}\|g\|_{q_2}+C\|f\|_{q_3}\|g\|_{H_{q_4}^s},$$

where $s>0$, $1<q,q_1,q_2,q_3,q_4<\infty$ and $1/q=1/q_1+1/q_2=1/q_3+1/q_4$. For the detail, see, e.g., Nakanishi–Ozawa [16]. Put

$$\begin{aligned} r(\sigma)&=\left(\frac12-\frac{2}{3d}-\frac{s(\sigma)}{d}\right)^{-1},\\ q_1&=\left(-1+\frac{\sigma}{d}+\frac1r+\frac{1}{r(\sigma)}\right)^{-1},\\ q_2&=\left(-1+\frac{\sigma}{d}+\frac{2}{r(\sigma)}\right)^{-1},\ q_3=\left(\frac1r+\frac{1}{r(\sigma)}\right)^{-1}.\end{aligned}$$

By Lemma II.2, we have

$$\begin{aligned}&\|(V*f_1f_2)f_3\|_{(s(\sigma))}\\ &\lesssim\|V*f_1f_2\|_{H_{q_1}^s}\|f_3\|_{r(\sigma)}+\|V*f_1f_2\|_{q_2}\|f_3\|_{H_r^s}\\ &\lesssim\|f_1f_2\|_{H_{q_3}^s}\|f_3\|_{r(\sigma)}+\|f_1f_2\|_{r(\sigma)/2}\|f_3\|_{H_r^s}\\ &\lesssim\|f_1\|_{H_r^s}\|f_2\|_{r(\sigma)}\|f_3\|_{r(\sigma)}+\|f_3\|_{H_r^s}\|f_1\|_{r(\sigma)}\|f_2\|_{r(\sigma)}\\ &\quad+\|f_2\|_{H_r^s}\|f_3\|_{r(\sigma)}\|f_1\|_{r(\sigma)}.\end{aligned}$$

We see from the embedding $H_r^s\hookrightarrow L^{r(\sigma)}$ that (1) holds true. Similarly, we can obtain (2) and (3). □

Proposition II.4 (Proposition 2.4 in [23]). *If $p > d$, then*

$$\sup_{\alpha\in\mathbb{N}_0^d}\sum_{\beta+\gamma=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle}\right)^p<\infty,$$
$$\sup_{\alpha\in\mathbb{N}_0^d}\sum_{\beta+\gamma+\delta=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle\langle\delta\rangle}\right)^p<\infty.$$

III. Function spaces

In this section, we introduce some function spaces to prove our main results. Fix $2 \le \sigma < d$. By $\mathbf{0}$, we denote the zero multi-index in d-dimensions. For $\phi \in H(\sigma)^\infty \setminus \{0\}$, we define $g^\phi(\mathbf{0}) = 1$,

$$g^\phi(\alpha)=\left(\max_{\substack{\beta\le\alpha\\|\beta|>0}}\left(\frac{\langle\beta\rangle^{d+1}\left\|\partial^\beta\phi\right\|_{(s(\sigma))}}{\beta!\left\|\phi\right\|_{(s(\sigma))}}\right)^{1/|\beta|}\right)^{|\alpha|}$$

if $\alpha \in \mathbb{N}_0^d \setminus \{0\}$, and

$$Z(\sigma)^\phi = \left\{v\in Z(\sigma)^\infty; \|v\|_{Z(\sigma)^\phi} := \sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\left\|\partial^\alpha v\right\|_{Z(\sigma)}}{\alpha! g^\phi(\alpha)}<\infty\right\}.$$

For $\phi \in H(\sigma)_\infty \setminus \{0\}$, we define $g_\phi(\mathbf{0}) = 1$,

$$g_\phi(\alpha)=\left(\max_{\substack{\beta\le\alpha\\|\beta|>0}}\left(\frac{\langle\beta\rangle^{d+1}\left\|x^\beta\phi\right\|_{(s(\sigma))}}{\beta!\left\|\phi\right\|_{(s(\sigma))}}\right)^{1/|\beta|}\right)^{|\alpha|}$$

if $\alpha \in \mathbb{N}_0^d \setminus \{0\}$, and

$$Z(\sigma)_\phi = \left\{v\in Z(\sigma)_\infty; \|v\|_{Z(\sigma)_\phi} = \sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\left\|J^\alpha v\right\|_{Z(\sigma)}}{\alpha! g_\phi(\alpha)}<\infty\right\}.$$

IV. Proof of Theorem I.3

Using the method in [23], we give a proof of Theorem I.3. It suffices to consider the case where $\phi \in H^{s(\sigma)} \setminus \{0\}$ and $V \ne 0$. Set $a \vee b = \max\{a, b\}$ ($a, b \in \mathbb{R}$). Let C be some constant independent of α.

We first prove (1). We see from Proposition II.3(1) that

$$\|F(v)\|_{L^1(\mathbb{R};H^{s(\sigma)})}\le C\|v\|_{Y(\sigma)}^3\le C\|v\|_{Z(\sigma)}^3,\ v\in Z(\sigma).$$

It follows from Proposition II.1 that the mapping

$$Z(\sigma)\ni v\mapsto\widetilde{v}:=U(t)\phi-i\int_0^t U(t-t')F(v(t'))dt'\in Z(\sigma)$$

is well-defined and that

$$\|\widetilde{v}\|_{Z(\sigma)}\le C\|\phi\|_{H^{s(\sigma)}}+C\|v\|_{Z(\sigma)}^3,\quad v\in Z(\sigma). \tag{16}$$

Similarly, we obtain

$$\|\widetilde{v_1}-\widetilde{v_2}\|_{Z(\sigma)}\le C\left(\|v_1\|_{Z(\sigma)}\vee\|v_2\|_{Z(\sigma)}\right)^2\|v_1-v_2\|_{Z(\sigma)}$$

for any $v_1, v_2 \in Z(\sigma)$. Therefore, we see that for any $\phi \in B_\eta H^{s(\sigma)}$, the mapping $v \mapsto \widehat{v}$ becomes a contraction provided that η is sufficiently small. Moreover, we see that the fixed point u becomes a time-global solution to (3) and unique in the sense of $Z(\sigma)$. Define

$$\phi_+=\phi-i\int_0^\infty U(t-t')F(u(t'))dt'. \tag{17}$$

Then it follows from the proof of (16) that $\phi_+ \in H^{s(\sigma)}$ and

$$\begin{aligned}\|U(-t)u(t)-\phi_+\|_{H^{s(\sigma)}}&\le C\|u\|_{Y(\sigma,[t,\infty))}^3\\&\to 0\quad\text{as } t\to\infty.\end{aligned}$$

Hence we have $\phi_+ = \mathbf{V}_+(\phi)$.

We next show (2). Let η be the positive number appearing in the proof of (1). Put $\phi \in H(\sigma)^\infty$, $v \in Z(\sigma)^\phi$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| > 0$. We see from the Leibniz rule and Proposition II.3(1) that

$$\begin{aligned}&\frac{\|\partial^\alpha F(v)\|_{L^1(\mathbb{R};H^{s(\sigma)})}}{\alpha!}\\&\le\sum_{\beta+\gamma+\delta=\alpha}\left\|\left(V*\frac{\partial^\beta v}{\beta!}\frac{\overline{\partial^\gamma v}}{\gamma!}\right)\frac{\partial^\delta v}{\delta!}\right\|_{L^1(\mathbb{R};H^{s(\sigma)})}\\&\le C\sum_{\beta+\gamma+\delta=\alpha}\frac{\left\|\partial^\beta v\right\|_{Z(\sigma)}}{\beta!}\frac{\left\|\partial^\gamma v\right\|_{Z(\sigma)}}{\gamma!}\frac{\left\|\partial^\delta v\right\|_{Z(\sigma)}}{\delta!}.\end{aligned}$$

By Proposition II.4, we obtain

$$\begin{aligned}&\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(v)\|_{L^1(\mathbb{R};H^{s(\sigma)})}}{\alpha! g^\phi(\alpha)}\\&\le C\sum_{\beta+\gamma+\delta=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle\langle\delta\rangle}\right)^{d+1}\left\{\frac{\langle\beta\rangle^{d+1}\left\|\partial^\beta v\right\|_{Z(\sigma)}}{\beta! g^\phi(\beta)}\right.\\&\quad\left.\times\frac{\langle\gamma\rangle^{d+1}\left\|\partial^\gamma v\right\|_{Z(\sigma)}}{\gamma! g^\phi(\gamma)}\frac{\langle\delta\rangle^{d+1}\left\|\partial^\delta v\right\|_{Z(\sigma)}}{\delta! g^\phi(\delta)}\right\}\\&\le C\|v\|_{Z(\sigma)^\phi}^3.\end{aligned}$$

Here, we have used the estimate

$$g^\phi(\alpha)\ge g^\phi(\beta)g^\phi(\gamma)g^\phi(\delta),\quad \beta+\gamma+\delta=\alpha$$

in the first inequality. Using Proposition II.1, we have $\widetilde{v} \in Z^\phi$ and

$$\begin{aligned}\|\widetilde{v}\|_{Z(\sigma)^\phi}&\le C\frac{\langle\alpha\rangle^{d+1}\left\|\partial^\alpha\phi\right\|_{(s(\sigma))}}{\alpha! g^\phi(\alpha)}+C\|v\|_{Z(\sigma)^\phi}^3\\&\le C\|\phi\|_{(s(\sigma))}+C\|v\|_{Z(\sigma)^\phi}^3.\end{aligned} \tag{18}$$

Similarly, we obtain

$$\begin{aligned}&\|\widetilde{v_1}-\widetilde{v_2}\|_{Z(\sigma)^\phi}\\&\le C\left(\|v_1\|_{Z(\sigma)^\phi}\vee\|v_2\|_{Z(\sigma)^\phi}\right)^2\|v_1-v_2\|_{Z(\sigma)^\phi}\end{aligned}$$

for any $v_1, v_2 \in Z(\sigma)^\phi$. Therefore, we see that if η' is sufficiently small and $\eta' \le \eta$, then for any $\phi \in B_{\eta'} H^{s(\sigma)}$, the mapping $v \mapsto \widehat{v}$ becomes a contraction and $\|v\|_{Z(\sigma)^\phi} \le$

$C\|\phi\|_{(s(\sigma))}$. Moreover, we see that the fixed point coincides with the unique solution u to (3). We also obtain

$$\limsup_{|\alpha|\to\infty}\left(\frac{\|\partial^\alpha u\|_{Z(\sigma)}}{\alpha!}\right)^{1/|\alpha|} \le \limsup_{|\alpha|\to\infty}\left(\frac{\|u\|_{Z(\sigma)^\phi}\, g^\phi(\alpha)}{\langle\alpha\rangle^{d+1}}\right)^{1/|\alpha|}$$
$$\le \limsup_{|\alpha|\to\infty}\left\{\left(\frac{C\|\phi\|_{(s(\sigma))}}{\langle\alpha\rangle^{d+1}}\right)^{1/|\alpha|} \times \max_{\substack{\beta\le\alpha\\|\beta|>0}}\left(\frac{\langle\beta\rangle^{d+1}\|\partial^\beta\phi\|_{(s(\sigma))}}{\beta!\,\|\phi\|_{(s(\sigma))}}\right)^{1/|\beta|}\right\}$$
$$\le \sup_{|\alpha|>0}\left(\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\,\|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}.$$

By (17), the proofs of (18) and the above estimate, we have $\mathbf{V}_+(\phi)\in H^\infty$,

$$\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\mathbf{V}_+(\phi)\|_{(s(\sigma))}}{\alpha! g^\phi(\alpha)} \le C\|\phi\|_{(s(\sigma))}$$

and

$$\limsup_{|\alpha|\to\infty}\left(\frac{\|\partial^\alpha\mathbf{V}_+(\phi)\|_{(s(\sigma))}}{\alpha!}\right)^{1/|\alpha|} \le \sup_{|\alpha|>0}\left(\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\,\|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}.$$

Hence (2) holds.

We next show (3). Let η be the positive number appearing in the proof of (1). Put $\phi\in H(\sigma)_\infty$, $v\in Z(\sigma)_\phi$ and $\alpha\in\mathbb{N}_0^d$ with $|\alpha|>0$. We see from (5), the Leibniz rule and Proposition II.3(1) that

$$\frac{\|J^\alpha F(v)\|_{L^1(\mathbb{R};H^{s(\sigma)})}}{\alpha!} = \frac{|2t|^{|\alpha|}\|M(t)\partial^\alpha M(-t)F(v)\|_{L^1(\mathbb{R};H^{s(\sigma)})}}{\alpha!}$$
$$\le \sum_{\beta+\gamma+\delta=\alpha}\left\|\left(V*\frac{J^\beta v}{\beta!}\frac{\overline{J^\gamma v}}{\gamma!}\right)\frac{J^\delta v}{\delta!}\right\|_{L^1(\mathbb{R};H^{s(\sigma)})}$$
$$\le C\sum_{\beta+\gamma+\delta=\alpha}\frac{\|J^\beta v\|_{Z(\sigma)}}{\beta!}\frac{\|J^\gamma v\|_{Z(\sigma)}}{\gamma!}\frac{\|J^\delta v\|_{Z(\sigma)}}{\delta!}.$$

Therefore, as in the proof of (2), we see that (3) holds.

V. Proof of Corollary I.5

Using the method in [23], we give a proof of Corollary I.5. It suffices to consider the case when $\phi\in B_\eta H^{s(\sigma)}\setminus\{0\}$ and $V\neq 0$, where η is the positive number mentioned in Theorem I.3. We fix $\varepsilon\in(0,1)$.

We first prove (1). Suppose (10) and (11). Then it follows from Theorem I.3 that the time-global solution u to (3) and the final data $\mathbf{V}_+(\phi)$ satisfy that $u\in Z(\sigma)^\infty$ and $\mathbf{V}_+(\phi)\in H(\sigma)^\infty$. We see from (10), (11) and the embedding $H^3_{d/2}\hookrightarrow L^\infty$ that

$$\lim_{|\alpha|\to\infty}\left(\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}\|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}=0$$

and

$$\lim_{|\alpha|\to\infty}\left(\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_q}{\alpha!\varepsilon^{|\alpha|}\|V\|_q}\right)^{1/|\alpha|}=0 \quad (q=d/\sigma,\infty).$$

In particular, there exists some $L\in\mathbb{N}$ such that

$$\sup_{|\alpha|>L}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\le\|\phi\|_{(s(\sigma))}$$

and

$$\sup_{|\alpha|>L}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_q}{\alpha!\varepsilon^{|\alpha|}}\le\|V\|_q \quad (q=d/\sigma,\infty).$$

We define

$$K^\phi=\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!}$$

and

$$K_{V,q}=\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_q}{\alpha!} \quad (q=d/\sigma,\infty).$$

Then we have

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\le\varepsilon^{-L}K^\phi$$

and

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_q}{\alpha!\varepsilon^{|\alpha|}}\le\varepsilon^{-L}K_{V,q} \quad (q=d/\sigma,\infty).$$

Since u solves (3), we see from the Leibniz rule, Propositions II.3(2) and II.4 that for any $\alpha\in\mathbb{N}_0^d$ and $t\ge0$

$$\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u(t)\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}} \le \frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}} + \int_0^t\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(u(t'))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'$$
$$\le \varepsilon^{-L}K^\phi + \sum_{\beta+\gamma=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle}\right)^{d+1}\int_0^t\left\|\frac{\langle\gamma\rangle^{d+1}\partial^\gamma u(t')}{\gamma!\varepsilon^{|\gamma|}} \times\left(\frac{\langle\beta\rangle^{d+1}\partial^\beta V}{\beta!\varepsilon^{|\beta|}}*|u(t')|^2\right)\right\|_{(s(\sigma))}dt'$$
$$\le \varepsilon^{-L}K^\phi + \|u\|^2_{L^\infty(\mathbb{R};H^{(s(\sigma))})}C\left(\max_{\beta\le\alpha}\frac{\langle\beta\rangle^{d+1}\|\partial^\beta V\|_\infty}{\beta!\varepsilon^{|\beta|}}\right) \times\int_0^t\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|\partial^\gamma u(t')\|_{(s(\sigma))}}{\gamma!\varepsilon^{|\gamma|}}dt'$$
$$\le \varepsilon^{-L}K^\phi + \varepsilon^{-L}CK_{V,\infty}\int_0^t\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|\partial^\gamma u(t')\|_{(s(\sigma))}}{\gamma!\varepsilon^{|\gamma|}}dt'. \quad (19)$$

Using the Gronwall inequality, we obtain

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u(t)\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\le\varepsilon^{-L}K^\phi\exp\left(\varepsilon^{-L}CK_{V,\infty}t\right)$$

for any $t\ge 0$. Furthermore, by Propositions II.1 and II.3(3), we have for any $\alpha\in\mathbb{N}_0^d$ and $T>0$

$$\begin{aligned}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u\|_{Z(\sigma,[0,T])}}{\alpha!\varepsilon^{|\alpha|}}&\le C\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}} \qquad (20)\\&+C\int_0^T\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(u(t'))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'\\&\le\varepsilon^{-L}CK^\phi\\&+\varepsilon^{-L}CK_{V,\infty}\int_0^T\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u(t')\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'\\&\le\varepsilon^{-L}CK^\phi\exp\left(\varepsilon^{-L}CK_{V,\infty}T\right) \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u\|_{Z(\sigma,(T,\infty))}}{\alpha!\varepsilon^{|\alpha|}}&\le C\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u(T)\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\\&+C\int_T^\infty\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(u(t'))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'\\&\le\varepsilon^{-L}CK^\phi\exp\left(\varepsilon^{-L}CK_{V,\infty}T\right)\\&+C\|u\|_{Y(\sigma,(T,\infty))}^2\left\{\sum_{\beta+\gamma=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle}\right)^{d+1}\right.\\&\left.\times\frac{\langle\beta\rangle^{d+1}\|\partial^\beta V\|_{d/\sigma}}{\beta!\varepsilon^{|\beta|}}\frac{\langle\gamma\rangle^{d+1}\|\partial^\gamma u\|_{Y(\sigma,(T,\infty))}}{\gamma!\varepsilon^{|\gamma|}}\right\}\\&\le\varepsilon^{-L}CK^\phi\exp\left(\varepsilon^{-L}CK_{V,\infty}T\right)\\&+\varepsilon^{-L}\|u\|_{Y(\sigma,(T,\infty))}^2\\&\times CK_{V,d/\sigma}\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|\partial^\gamma u\|_{Z(\sigma,(T,\infty))}}{\gamma!\varepsilon^{|\gamma|}}. \qquad (22)\end{aligned}$$

Choose $T(\varepsilon)>0$ so that

$$\varepsilon^{-L}\left(\|u\|_{Y(\sigma,(-\infty,T(\varepsilon)))}\vee\|u\|_{Y(\sigma,(T(\varepsilon),\infty))}\right)^2CK_{V,d/\sigma}\le\frac{1}{2} \qquad (23)$$

and put

$$C_\varepsilon=\varepsilon^{-L}CK^\phi\exp\left(\varepsilon^{-L}T(\varepsilon)CK_{V,\infty}\right).$$

Then we obtain

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u\|_{Z(\sigma,(T(\varepsilon),\infty))}}{\alpha!\varepsilon^{|\alpha|}}\le 2C_\varepsilon.$$

It follows from (21) that

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u\|_{Z(\sigma,[0,\infty))}}{\alpha!\varepsilon^{|\alpha|}}\le 3C_\varepsilon.$$

Similarly, we see that

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha u\|_{Z(\sigma,(-\infty,0])}}{\alpha!\varepsilon^{|\alpha|}}\le 3C_\varepsilon$$

and hence that $\mathfrak{L}(\partial,u,Z(\sigma))\le\varepsilon$. Using (17) and the proof of (22), we have

$$\begin{aligned}&\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha\mathbf{V}_+(\phi)\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\\&\le\varepsilon^{-L}K^\phi+\int_{\mathbb{R}}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(u(t'))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'\\&\le\varepsilon^{-L}K^\phi+\varepsilon^{-L}\|u\|_{Y(\sigma)}^2CK_{V,d/\sigma}\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|\partial^\gamma u\|_{Z(\sigma)}}{\gamma!\varepsilon^{|\gamma|}}\\&\le\varepsilon^{-L}K^\phi+6\varepsilon^{-L}\|u\|_{Y(\sigma)}^2CC_\varepsilon K_{V,d/\sigma},\quad\alpha\in\mathbb{N}_0^d,\end{aligned}$$

which implies that $\mathfrak{L}(\partial,\mathbf{V}_+(\phi),H^{s(\sigma)})\le\varepsilon$, and we obtain

$$\begin{aligned}&\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha(U(-t)u(t)-\mathbf{V}_+(\phi))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\\&\le\int_t^\infty\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha F(u(t'))\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}dt'\\&\le 6\varepsilon^{-L}\|u\|_{Y(\sigma,(t,\infty))}^2CC_\varepsilon K_{V,d/\sigma},\quad\alpha\in\mathbb{N}_0^d.\end{aligned}$$

Since we can choose ε arbitrarily, (1) holds.

We next prove (2). Suppose (13) and (14). Then it follows from Theorem I.3 that the time-global solution u to (3) and the final data $\mathbf{V}_+(\phi)$ satisfy that $u\in Z(\sigma)_\infty$ and $\mathbf{V}_+(\phi)\in H(\sigma)_\infty$. We see from (13) and (14) that

$$\lim_{|\alpha|\to\infty}\left(\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}\|\phi\|_{(s(\sigma))}}\right)^{1/|\alpha|}=0$$

and

$$\lim_{|\alpha|\to\infty}\left(\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_\infty}{\alpha!\varepsilon^{|\alpha|}\|V\|_\infty}\right)^{1/|\alpha|}=0.$$

In particular, there exists some $N\in\mathbb{N}$ such that for any $\alpha\in\mathbb{N}_0^d$ with $|\alpha|>N$

$$\sup_{|\alpha|>N}\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\le\|\phi\|_{(s(\sigma))}$$

and

$$\sup_{|\alpha|>N}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_\infty}{\alpha!\varepsilon^{|\alpha|}}\le\|V\|_\infty.$$

We define

$$K_\phi=\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|_{(s(\sigma))}}{\alpha!}.$$

Then we have

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|_{(s(\sigma))}}{\alpha!\varepsilon^{|\alpha|}}\le\varepsilon^{-N}K_\phi$$

and

$$\sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^{d+1}\|\partial^\alpha V\|_\infty}{\alpha!\varepsilon^{|\alpha|}}\le\varepsilon^{-N}K_{V,\infty},$$

where $K_{V,\infty}$ is the positive constant defined in the proof of (1).

Set $\langle \tau \rangle = \sqrt{1+\tau^2}$ ($\tau \in \mathbb{R}$) and fix $t > 0$. Since u solves (3), we see from the Leibniz rule, (5), Propositions II.3(2) and II.4 that for any $\alpha \in \mathbb{N}_0^d$

$$\frac{\langle \alpha \rangle^{d+1} \|J^\alpha u(t)\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|} \langle 2t \rangle^{|\alpha|}} \leq \frac{\langle \alpha \rangle^{d+1} \|x^\alpha \phi\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|}} + \int_0^t \frac{\langle \alpha \rangle^{d+1} \|M(t')(2it'\partial)^\alpha F(M(-t')u(t'))\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|} \langle 2t \rangle^{|\alpha|}} dt'$$

and

$$\begin{aligned} &\frac{\langle \alpha \rangle^{d+1} \|M(t')(2it'\partial)^\alpha F(M(-t')u(t'))\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|} \langle 2t \rangle^{|\alpha|}} \\ &\leq \sum_{\beta+\gamma=\alpha} \left(\frac{\langle \alpha \rangle}{\langle \beta \rangle \langle \gamma \rangle} \right)^{d+1} \Bigg\{ \frac{|2t'|^{|\beta|} \langle 2t' \rangle^{|\gamma|}}{\langle 2t \rangle^{|\alpha|}} \\ &\quad \times \Bigg\| \left(\frac{\langle \beta \rangle^{d+1} \partial^\beta V}{\beta! \varepsilon^{|\beta|}} * |u(t')|^2 \right) \\ &\quad \times \frac{\langle \gamma \rangle^{d+1} M(t')(2it'\partial)^\gamma M(-t')u(t')}{\gamma! \varepsilon^{|\gamma|} \langle 2t' \rangle^{|\gamma|}} \Bigg\|_{(s(\sigma))} \Bigg\} \\ &\leq \varepsilon^{-N} \|u\|^2_{L^\infty(\mathbb{R}; H^{s(\sigma)})} C \left(\max_{\beta \leq \alpha} \frac{\langle \beta \rangle^{d+1} \|\partial^\beta V\|_\infty}{\beta! \varepsilon^{|\beta|}} \right) \\ &\quad \times \left(\max_{\gamma \leq \alpha} \frac{\langle \gamma \rangle^{d+1} \|J^\gamma u(t')\|_{(s(\sigma))}}{\gamma! \varepsilon^{|\gamma|} \langle 2t' \rangle^{|\gamma|}} \right) \\ &\leq \varepsilon^{-N} C K_{V,\infty} \max_{\gamma \leq \alpha} \frac{\langle \gamma \rangle^{d+1} \|J^\gamma u(t')\|_{(s(\sigma))}}{\gamma! \varepsilon^{|\gamma|} \langle 2t' \rangle^{|\gamma|}} \end{aligned}$$

for any $t' \in (0, t)$. Therefore, we have

$$\frac{\langle \alpha \rangle^{d+1} \|J^\alpha u(t)\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|} \langle 2t \rangle^{|\alpha|}} \leq \varepsilon^{-N} K_\phi + \varepsilon^{-N} C K_{V,\infty} \times \int_0^t \max_{\gamma \leq \alpha} \frac{\langle \gamma \rangle^{d+1} \|J^\gamma u(t')\|_{(s(\sigma))}}{\gamma! \varepsilon^{|\gamma|} \langle 2t' \rangle^{|\gamma|}} dt'$$

for any $\alpha \in \mathbb{N}_0^d$. Using the Gronwall inequality, we obtain

$$\sup_{\alpha \in \mathbb{N}_0^d} \frac{\langle \alpha \rangle^{d+1} \|J^\alpha u(t)\|_{(s(\sigma))}}{\alpha! \varepsilon^{|\alpha|} \langle 2t \rangle^{|\alpha|}} \leq \varepsilon^{-N} K_\phi \exp\left(\varepsilon^{-N} C K_{V,\infty} t \right).$$

It follows from (5) that

$$\begin{aligned} \frac{\|\partial^\alpha M(-t)u(t)\|_2}{\alpha!} &= |2t|^{-|\alpha|} \frac{\|J^\alpha u(t)\|_2}{\alpha!} \\ &\leq \left(\frac{\varepsilon \langle 2t \rangle}{|2t|} \right)^{|\alpha|} \frac{\varepsilon^{-N} K_\phi}{\langle \alpha \rangle^{d+1}} \exp\left(\varepsilon^{-N} C K_{V,\infty} t \right), \end{aligned}$$

and hence that

$$\limsup_{|\alpha| \to \infty} \left(\frac{\|\partial^\alpha M(-t)u(t)\|_2}{\alpha!} \right)^{1/|\alpha|} \leq \frac{\varepsilon \langle 2t \rangle}{|2t|}.$$

Since the above estimate holds even if $t < 0$, we obtain (15).

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