Interiors and Closures of Sets and Applications

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***Abstract*—We prove $(U \cup V)^\circ = U^\circ \cup V^\circ$, $\overline{U \cap V} = \overline{U} \cap \overline{V}$, $U^\circ \setminus \overline{V} = (U \setminus V)^\circ$, and $\overline{U} \setminus V^\circ = \overline{U \setminus V}$ under some conditions, where U and V are subsets of a topological space. More precisely, we give an equivalent condition to each of equalities $(U \cup V)^\circ = U^\circ \cup V^\circ$, $\overline{U \cap V} = \overline{U} \cap \overline{V}$, and $\overline{U} \setminus V^\circ = \overline{U \setminus V}$. Furthermore, some sufficient conditions are given for the intersection of subsets to be open.**

***Index Terms*—interior, closure, subset, union, intersection.**

I. Introduction

Throughout this paper, let X be a topological space. For any subset U of a topological space X, we denote by $\overline{U}$, U°, and ∂U the closure, the interior, and the boundary of U in X, respectively.

It is well known that

$$\begin{aligned} (U \cup V)^\circ &\supseteq U^\circ \cup V^\circ, &(1)\\ (U \cap V)^\circ &= U^\circ \cap V^\circ, \\ \overline{U \cup V} &= \overline{U} \cup \overline{V}, \\ \overline{U \cap V} &\subseteq \overline{U} \cap \overline{V} &(2) \end{aligned}$$

for any subsets U and V of a topological space X (see [1], [2]), where $U \subseteq V$ or $V \supseteq U$ means that U is a subset of V or V is a superset of U, while $U \subset V$ or $V \supset U$ means that U is a proper subset of V or V is a proper superset of U.

For example, let $X = \mathbf{R}$ be a (Euclidean) topological space, $U = \mathbf{Q}$, and let $V = \mathbf{R} \setminus \mathbf{Q}$. Then $(U \cup V)^\circ = \mathbf{R}^\circ = \mathbf{R}$, $U^\circ = \mathbf{Q}^\circ = \emptyset$ and $V^\circ = (\mathbf{R} \setminus \mathbf{Q})^\circ = \emptyset$. Hence, we see that $(U \cup V)^\circ \supset U^\circ \cup V^\circ$. Analogously, we see that $\overline{U \cap V} = \overline{\mathbf{Q} \cap (\mathbf{R} \setminus \mathbf{Q})} = \overline{\emptyset} = \emptyset$, $\overline{U} = \overline{\mathbf{Q}} = \mathbf{R}$ and $\overline{V} = \overline{\mathbf{R} \setminus \mathbf{Q}} = \mathbf{R}$. Hence, we see that $\overline{U \cap V} \subset \overline{U} \cap \overline{V}$. This example shows that we cannot expect the equalities in (1) and (2) in general case.

In many practical applications, however, it would be very useful for us to know under what conditions we can expect the equalities in the relations (1) and (2). This is a motivation of this paper, and we investigate some necessary and sufficient conditions.

The following theorem is well known and it is essential for proving Corollaries 2.2 and 3.2 and Theorem 4.1.

Theorem 1.1 *If U is a subset of a topological space X, then it holds that*
(i) $X \setminus U^\circ = \overline{X \setminus U}$*; and*
(ii) $X \setminus \overline{U} = (X \setminus U)^\circ$.

In Section 4, we generalize Theorem 1.1 by proving the relations $U^\circ \setminus \overline{V} = (U \setminus V)^\circ$ and $\overline{U} \setminus V^\circ = \overline{U \setminus V}$ under some conditions. Finally, by using Theorems 1.1 and 4.1, we give some sufficient conditions for the intersection of subsets to be open.

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II. Interior of union of subsets

In general, it holds that $(U \cup V)^\circ \supseteq U^\circ \cup V^\circ$ for any subsets U and V of a topological space X. The following theorem deals with a necessary and sufficient condition under which the interior of the union of two subsets equals the union of their interiors.

We can easily see that all 'solid' subsets of $\mathbf{R}^n$ satisfy the condition (b) of Theorem 2.1.

Theorem 2.1 *For arbitrary subsets U and V of a topological space X, the following two conditions are equivalent:*
(a) $(U \cup V)^\circ = U^\circ \cup V^\circ$;
(b) $\partial(U \cup V) = \big(\partial U \setminus V^\circ\big) \cup \big(\partial V \setminus U^\circ\big)$.

Proof. First, we assume that U and V satisfy the condition (b). For the boundary $\partial(U \cup V)$ of $U \cup V$, it follows from the assumption (b) that

$$\begin{aligned} & X \setminus \partial(U \cup V) \\ &= X \setminus \big((\partial U \setminus V^\circ) \cup (\partial V \setminus U^\circ)\big) \\ &= \big(X \setminus (\partial U \setminus V^\circ)\big) \cap \big(X \setminus (\partial V \setminus U^\circ)\big) \\ &= \big(X \setminus (\partial U \cap X \setminus V^\circ)\big) \cap \big(X \setminus (\partial V \cap X \setminus U^\circ)\big) \\ &= \big((X \setminus \partial U) \cup V^\circ\big) \cap \big((X \setminus \partial V) \cup U^\circ\big) &(3)\\ &= \big((X \setminus \partial U) \cap (X \setminus \partial V)\big) \cup \big((X \setminus \partial U) \cap U^\circ\big) \\ &\quad \cup \big(V^\circ \cap (X \setminus \partial V)\big) \cup \big(V^\circ \cap U^\circ\big) \\ &= \big(X \setminus (\partial U \cup \partial V)\big) \cup U^\circ \cup V^\circ \end{aligned}$$

and by (3), we further obtain

$$\begin{aligned} & \overline{U} \cap X \setminus \partial(U \cup V) \\ &= \overline{U} \cap \big(\big(X \setminus (\partial U \cup \partial V)\big) \cup U^\circ \cup V^\circ\big) &(4)\\ &= \overline{U} \setminus (\partial U \cup \partial V) \cup U^\circ \cup (\overline{U} \cap V^\circ) \end{aligned}$$

and

$$\begin{aligned} & \overline{V} \cap X \setminus \partial(U \cup V) \\ &= \overline{V} \cap \big(\big(X \setminus (\partial U \cup \partial V)\big) \cup U^\circ \cup V^\circ\big) &(5)\\ &= \overline{V} \setminus (\partial U \cup \partial V) \cup (U^\circ \cap \overline{V}) \cup V^\circ. \end{aligned}$$

We now have

$$\begin{aligned} & (U \cup V)^\circ \\ &= \overline{U \cup V} \setminus \partial(U \cup V) \\ &= \overline{U \cup V} \cap \big(X \setminus \partial(U \cup V)\big) \\ &= (\overline{U} \cup \overline{V}) \cap \big(X \setminus \partial(U \cup V)\big) \\ &= \big(\overline{U} \cap X \setminus \partial(U \cup V)\big) \cup \big(\overline{V} \cap X \setminus \partial(U \cup V)\big). \end{aligned}$$

Moreover, it follows from (4), (5) and the last relation that

$$
\begin{aligned}
&(U\cup V)^\circ\\
&\quad= \overline{U}\backslash(\partial U\cup\partial V)\cup\overline{V}\backslash(\partial U\cup\partial V)\\
&\qquad\cup(\overline{U}\cap V^\circ)\cup(U^\circ\cap\overline{V})\cup(U^\circ\cup V^\circ)\\
&\quad= \big(U^\circ\cap(\overline{U}\backslash\partial V)\big)\cup\big((\overline{V}\backslash\partial U)\cap V^\circ\big)\cup(U^\circ\cup V^\circ)\\
&\quad= U^\circ\cup V^\circ.
\end{aligned}
$$

If U and V satisfy (a), then by Theorem 1.1 (i) and (a), we have

$$
\begin{aligned}
&\big(\partial U\backslash V^\circ\big)\cup\big(\partial V\backslash U^\circ\big)\\
&\quad= \big(\partial U\cap(X\backslash V^\circ)\big)\cup\big(\partial V\cap(X\backslash U^\circ)\big)\\
&\quad= (\partial U\cup\partial V)\cap\big(\partial U\cup(X\backslash U^\circ)\big)\\
&\qquad\cap\big((X\backslash V^\circ)\cup\partial V\big)\cap\big((X\backslash V^\circ)\cup(X\backslash U^\circ)\big)\\
&\quad= (\partial U\cup\partial V)\cap(X\backslash U^\circ)\cap(X\backslash V^\circ)\cap\big(X\backslash(U^\circ\cap V^\circ)\big)\\
&\quad= (\partial U\cup\partial V)\cap(X\backslash U^\circ)\cap(X\backslash V^\circ)\cap\big(X\backslash(U\cap V)^\circ\big) \qquad (6)\\
&\quad= (\partial U\cup\partial V)\cap(X\backslash U^\circ)\cap(X\backslash V^\circ)\cap\overline{X\backslash(U\cap V)}\\
&\quad= (\partial U\cup\partial V)\cap\big(X\backslash(U^\circ\cup V^\circ)\big)\cap\overline{X\backslash(U\cap V)}\\
&\quad= (\partial U\cup\partial V)\cap\big(X\backslash(U\cup V)^\circ\big)\cap\overline{X\backslash(U\cap V)}\\
&\quad= (\partial U\cup\partial V)\cap\overline{X\backslash(U\cup V)}\cap\overline{X\backslash(U\cap V)}\\
&\quad= (\partial U\cup\partial V)\cap\overline{X\backslash(U\cup V)}.
\end{aligned}
$$

Since $\partial U\subseteq\overline{U}$ and $\partial V\subseteq\overline{V}$, it follows from (6) that

$$
\begin{aligned}
\big(\partial U\backslash V^\circ\big)\cup\big(\partial V\backslash U^\circ\big) &\subseteq \big(\overline{U}\cup\overline{V}\big)\cap\overline{X\backslash(U\cup V)}\\
&= \overline{U\cup V}\cap\overline{X\backslash(U\cup V)}\\
&= \partial(U\cup V).
\end{aligned}
$$

On the other hand, by (1), we have

$$
\begin{aligned}
&\partial(U\cup V)\\
&\quad= \overline{U\cup V}\cap\overline{X\backslash(U\cup V)}\\
&\quad= \big(\overline{U}\cup\overline{V}\big)\cap\big(X\backslash(U\cup V)^\circ\big)\\
&\quad\subseteq \big(\overline{U}\cup\overline{V}\big)\cap\big(X\backslash(U^\circ\cup V^\circ)\big)\\
&\quad= \big(\overline{U}\cup\overline{V}\big)\cap\big((X\backslash U^\circ)\cap(X\backslash V^\circ)\big)\\
&\quad= \big(\overline{U}\cap(X\backslash U^\circ)\cap(X\backslash V^\circ)\big)\\
&\qquad\cup\big(\overline{V}\cap(X\backslash U^\circ)\cap(X\backslash V^\circ)\big)\\
&\quad= \big((\overline{U}\backslash U^\circ)\cap(X\backslash V^\circ)\big)\cup\big((\overline{V}\backslash V^\circ)\cap(X\backslash U^\circ)\big)\\
&\quad= \big(\partial U\backslash V^\circ\big)\cup\big(\partial V\backslash U^\circ\big),
\end{aligned}
$$

which completes the proof.

The following two conditions are sufficient for subsets U and V to satisfy the condition (b) in Theorem 2.1, and hence, to satisfy the relation $(U\cup V)^\circ=U^\circ\cup V^\circ$.

Corollary 2.2 *Let U and V be arbitrary subsets of a topological space X. If U and V satisfy the following both conditions*
(a) $\partial U\backslash V^\circ=\overline{U}\backslash(U\cup V)^\circ$*; and*
(b) $\partial V\backslash U^\circ=\overline{V}\backslash(U\cup V)^\circ$,
then it holds that $(U\cup V)^\circ=U^\circ\cup V^\circ$.

Proof. For the boundary $\partial(U\cup V)$ of $U\cup V$, it follows from Theorem 1.1 (i) and the conditions (a) and (b) that

$$
\begin{aligned}
&\partial(U\cup V)\\
&\quad= \overline{U\cup V}\cap\overline{X\backslash(U\cup V)}\\
&\quad= \big(\overline{U}\cup\overline{V}\big)\cap\big(X\backslash(U\cup V)^\circ\big)\\
&\quad= \big(\overline{U}\cap\big(X\backslash(U\cup V)^\circ\big)\big)\cup\big(\overline{V}\cap\big(X\backslash(U\cup V)^\circ\big)\big)\\
&\quad= \big(\overline{U}\backslash(U\cup V)^\circ\big)\cup\big(\overline{V}\backslash(U\cup V)^\circ\big)\\
&\quad= \big(\partial U\backslash V^\circ\big)\cup\big(\partial V\backslash U^\circ\big),
\end{aligned}
$$

and the assertion of this corollary immediately follows from Theorem 2.1.

III. Closure of intersection of subsets

It is well known that $\overline{U\cap V}\subseteq\overline{U}\cap\overline{V}$ for any subsets U and V of a topological space X. In the following theorem, we study a necessary and sufficient condition under which the closure of the intersection of two subsets equals the intersection of their closures. Roughly speaking, this theorem is a counterpart of Theorem 2.1 for the relation $\overline{U\cap V}=\overline{U}\cap\overline{V}$.

We can also see that all 'solid' subsets of $\mathbf{R}^n$ satisfy the condition (b) of Theorem 3.1.

Theorem 3.1 *For arbitrary subsets U and V of a topological space X, the following two conditions are equivalent:*
(a) $\overline{U\cap V}=\overline{U}\cap\overline{V}$*;*
(b) $\partial(U\cap V)=(\partial U\cup\partial V)\cap\big(\overline{U}\cap\overline{V}\big)$.

Proof. It follows from the assumption (b) that

$$
\begin{aligned}
&\overline{U\cap V}\\
&\quad= (U\cap V)^\circ\cup\partial(U\cap V)\\
&\quad= (U^\circ\cap V^\circ)\cup\big((\partial U\cup\partial V)\cap\big(\overline{U}\cap\overline{V}\big)\big)\\
&\quad= \big((U^\circ\cap V^\circ)\cup(\partial U\cup\partial V)\big)\cap\big((U^\circ\cap V^\circ)\cup\big(\overline{U}\cap\overline{V}\big)\big)\\
&\quad= \big((U^\circ\cap V^\circ)\cup(\partial U\cup\partial V)\big)\cap\big(\overline{U}\cap\overline{V}\big)\\
&\quad= \big((U^\circ\cup\partial U\cup\partial V)\cap(V^\circ\cup\partial U\cup\partial V)\big)\cap\big(\overline{U}\cap\overline{V}\big)\\
&\quad= \big(\big(\overline{U}\cup\partial V\big)\cap\big(\overline{V}\cup\partial U\big)\big)\cap\big(\overline{U}\cap\overline{V}\big)\\
&\quad= \big(\big(\overline{U}\cap\overline{V}\big)\cup\big(\overline{U}\cap\partial U\big)\cup\big(\partial V\cap\overline{V}\big)\cup\big(\partial V\cap\partial U\big)\big)\\
&\qquad\cap\big(\overline{U}\cap\overline{V}\big)\\
&\quad= \big(\big(\overline{U}\cap\overline{V}\big)\cup\partial U\cup\partial V\cup\big(\partial U\cap\partial V\big)\big)\cap\big(\overline{U}\cap\overline{V}\big)\\
&\quad= \overline{U}\cap\overline{V}.
\end{aligned}
$$

On the other hand, if we assume that (a) is true, then by using Theorem 1.1 (i), we get

$$
\begin{aligned}
&\partial(U\cap V)\\
&\quad= \overline{U\cap V}\cap\overline{X\backslash(U\cap V)}\\
&\quad= \big(\overline{U}\cap\overline{V}\big)\cap X\backslash(U\cap V)^\circ\\
&\quad= \big(\overline{U}\cap\overline{V}\big)\cap\big(X\backslash(U^\circ\cap V^\circ)\big)\\
&\quad= \big(\overline{U}\cap\overline{V}\big)\cap\big((X\backslash U^\circ)\cup(X\backslash V^\circ)\big)\\
&\quad= \big(\big(\overline{U}\cap\overline{V}\big)\cap(X\backslash U^\circ)\big)\cup\big(\big(\overline{U}\cap\overline{V}\big)\cap(X\backslash V^\circ)\big)\\
&\quad= \big(\big(\overline{U}\backslash U^\circ\big)\cap\overline{V}\big)\cup\big(\overline{U}\cap\big(\overline{V}\backslash V^\circ\big)\big)\\
&\quad= \big(\partial U\cap\overline{V}\big)\cup\big(\overline{U}\cap\partial V\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \big(\partial U \cup \overline{U}\big) \cap \big(\partial U \cup \partial V\big) \cap \big(\overline{U} \cup \overline{V}\big) \cap \big(\overline{V} \cup \partial V\big) \\
&= \overline{U} \cap \big(\partial U \cup \partial V\big) \cap \big(\overline{U} \cup \overline{V}\big) \cap \overline{V} \\
&= (\partial U \cup \partial V) \cap \big(\overline{U} \cap \overline{V}\big),
\end{aligned}
$$

which completes the proof.

In the following corollary, we introduce some conditions by which the condition (b) of Theorem 3.1 is satisfied. Under those conditions, the subsets U and V of a topological space satisfy the relation $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

Corollary 3.2 *Let U and V be arbitrary subsets of a topological space X. If U and V satisfy the following both conditions*

(a) $\overline{U} \cap \partial V = \overline{U \cap V} \setminus U^\circ$; *and*
(b) $\overline{V} \cap \partial U = \overline{U \cap V} \setminus V^\circ$,

then it holds that $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

Proof. For the boundary $\partial(U \cap V)$ of $U \cap V$, it follows from Theorem 1.1 (i) and the conditions (a) and (b) that

$$
\begin{aligned}
&\partial(U \cap V) \\
&= \overline{U \cap V} \cap \overline{X \setminus (U \cap V)} \\
&= \overline{U \cap V} \cap \big(X \setminus (U \cap V)^\circ\big) \\
&= \overline{U \cap V} \cap \big(X \setminus (U^\circ \cap V^\circ)\big) \\
&= \overline{U \cap V} \cap \big((X \setminus U^\circ) \cup (X \setminus V^\circ)\big) \\
&= \big(\overline{U \cap V} \setminus U^\circ\big) \cup \big(\overline{U \cap V} \setminus V^\circ\big) \\
&= \big(\overline{U} \cap \partial V\big) \cup \big(\overline{V} \cap \partial U\big) \\
&= \big(\overline{U} \cup \overline{V}\big) \cap \big(\overline{U} \cup \partial U\big) \cap \big(\partial V \cup \overline{V}\big) \cap (\partial U \cup \partial V) \\
&= \big(\overline{U} \cup \overline{V}\big) \cap \big(\overline{U} \cap \overline{V}\big) \cap (\partial U \cup \partial V) \\
&= (\partial U \cup \partial V) \cap \big(\overline{U} \cap \overline{V}\big),
\end{aligned}
$$

and the assertion of this corollary immediately follows from Theorem 3.1.

IV. Generalization of Theorem 1.1

We are now in the position to prove a variation of Theorem 1.1 for subsets of a topological space. Indeed, we prove that $U^\circ \setminus \overline{V} = (U \setminus V)^\circ$ and $\overline{U} \setminus V^\circ = \overline{U \setminus V}$ under some additional assumption.

Theorem 4.1 *If U and V are subsets of a topological space X, then $U^\circ \setminus \overline{V} = (U \setminus V)^\circ$. In addition, the following two conditions are equivalent:*

(a) $\overline{U} \setminus V^\circ = \overline{U \setminus V}$;
(b) $\big((X \setminus U) \cup V\big)^\circ = (X \setminus U)^\circ \cup V^\circ$.

Proof. By Theorem 1.1 (ii), we have

$$
\begin{aligned}
U^\circ \setminus \overline{V} &= U^\circ \cap \big(X \setminus \overline{V}\big) = U^\circ \cap (X \setminus V)^\circ \\
&= (U \cap X \setminus V)^\circ = (U \setminus V)^\circ.
\end{aligned}
$$

Moreover, if U and V satisfy the condition (b), then it follows from Theorem 1.1 (ii) that

$$
\begin{aligned}
X \setminus \overline{U \setminus V} &= X \setminus \overline{U \cap X \setminus V} = \big(X \setminus (U \cap X \setminus V)\big)^\circ \\
&= \big((X \setminus U) \cup V\big)^\circ = (X \setminus U)^\circ \cup V^\circ \\
&= (X \setminus \overline{U}) \cup V^\circ = (X \setminus \overline{U}) \cup \big(X \setminus (X \setminus V^\circ)\big) \\
&= X \setminus \big(\overline{U} \cap (X \setminus V^\circ)\big) = X \setminus \big(\overline{U} \setminus V^\circ\big),
\end{aligned}
$$

which implies that $\overline{U} \setminus V^\circ = \overline{U \setminus V}$. On the other hand, if we assume that $\overline{U} \setminus V^\circ = \overline{U \setminus V}$, then we have

$$X \setminus \big(\overline{U} \setminus V^\circ\big) = X \setminus \overline{U \setminus V}.$$

Further, it follows from Theorem 1.1 (ii) that

$$
\begin{aligned}
X \setminus \big(\overline{U} \setminus V^\circ\big) &= X \setminus \big(\overline{U} \cap (X \setminus V^\circ)\big) = \big(X \setminus \overline{U}\big) \cup V^\circ \\
&= (X \setminus U)^\circ \cup V^\circ
\end{aligned}
$$

and

$$
\begin{aligned}
X \setminus \overline{U \setminus V} &= \big(X \setminus (U \setminus V)\big)^\circ = \big(X \setminus (U \cap (X \setminus V))\big)^\circ \\
&= \big((X \setminus U) \cup V\big)^\circ.
\end{aligned}
$$

By using the last three equalities, we conclude that (b) is true.

If $U = V$ and $\partial U \neq \emptyset$, then U and V do not satisfy the condition (b) of Theorem 4.1. In this case, we see that $\overline{U} \setminus V^\circ = \partial U \neq \emptyset = \overline{U \setminus V}$.

We remark that the condition (b) of Theorem 4.1 is equivalent to

$$\overline{U \cap (X \setminus V)} = \overline{U} \cap \overline{X \setminus V}.$$

We now introduce some specific conditions by which the condition (b) of Theorem 4.1 is satisfied. Under those conditions, the subsets U and V of a topological space satisfy the relation $\overline{U} \setminus V^\circ = \overline{U \setminus V}$.

Corollary 4.2 *Let U and V be arbitrary subsets of a topological space X. If U and V satisfy the following conditions*

(a) $\partial U \setminus V^\circ = \overline{X \setminus U} \setminus \big((X \setminus U) \cup V\big)^\circ$; *and*
(b) $\overline{U} \cap \partial V = \overline{V} \setminus \big((X \setminus U) \cup V\big)^\circ$,

then it holds that $\overline{U} \setminus V^\circ = \overline{U \setminus V}$.

Proof. We know that $\partial U = \partial(X \setminus U)$ and, by Theorem 1.1 (ii), we have

$$X \setminus (\overline{U} \cap \partial V) = (X \setminus \overline{U}) \cup (X \setminus \partial V) = (X \setminus U)^\circ \cup (X \setminus \partial V).$$

Hence, we get

$$
\begin{aligned}
&\overline{U} \cap \partial V \\
&= X \setminus \big(X \setminus (\overline{U} \cap \partial V)\big) = X \setminus \big((X \setminus U)^\circ \cup (X \setminus \partial V)\big) \\
&= X \setminus (X \setminus U)^\circ \cap \partial V = \partial V \setminus (X \setminus U)^\circ.
\end{aligned}
$$

Thus, it follows from the conditions (a), (b) and the last equality that

$$
\begin{aligned}
&\partial(X \setminus U) \setminus V^\circ \; (= \partial U \setminus V^\circ) = \overline{X \setminus U} \setminus \big((X \setminus U) \cup V\big)^\circ, \\
&\partial V \setminus (X \setminus U)^\circ \; (= \overline{U} \cap \partial V) = \overline{V} \setminus \big((X \setminus U) \cup V\big)^\circ.
\end{aligned}
\tag{7}
$$

Let us substitute $X\backslash U$ for U in Corollary 2.2. Then, in view of (7), it follows from Corollary 2.2 that

$$\big((X\backslash U)\cup V\big)^\circ = (X\backslash U)^\circ \cup V^\circ.$$

By using Theorem 4.1, we conclude that our assertion is true.

V. Application of Theorem 4.1

First, we will introduce some sufficient conditions under which the intersection of two subsets of a topological space is open.

Theorem 5.1 *Let U and V be arbitrary subsets of a topological space X containing at least 2 points. If U and V satisfy the conditions*
(a) $U \cap V \cap \partial V = \emptyset$; and
(b) U is open,
then $U \cap V$ is open.

Proof. The assertion is obviously true for the case $U \cap V = \emptyset$ or $U \cap V = X$. We now assume that $U \cap V \neq \emptyset$, $U \cap V \neq X$, and $x \in U \cap V$. Then the assumption (a) implies that $x \notin \partial V$, and since $x \in V$ it follows that $x \in V^\circ$, which means that $U \cap V \subseteq U \cap V^\circ$ or $U \cap V = U \cap V^\circ$. Finally, by (b), $U \cap V = U \cap V^\circ$ is open.

In the following theorem, by using Theorems 1.1 and 4.1, we will introduce some sufficient conditions for the intersection of subsets to be open. The first part of this theorem is analogous to the first part of the proof of preceding theorem.

Theorem 5.2 *Let U and V be arbitrary subsets of a topological space X containing at least 2 points. If U and V satisfy the following properties*
(a) $\big((X\backslash U)\cup V\big)^\circ = (X\backslash U)^\circ \cup V^\circ$;
(b) U is open; and
(c) $\overline{U}\backslash V$ is closed,
then $U \cap V$ is open.

Proof. Obviously, the assertion holds for the case $U \cap V = \emptyset$ or $U \cap V = X$. Hence, we assume that $U \cap V \neq \emptyset$, $U \cap V \neq X$, and that $U \cap V$ were not open. Then, $U \cap V$ would include at least one of its boundary points. Let x be the point with the property

$$x \in \partial(U \cap V) \quad \text{and} \quad x \in U \cap V. \tag{8}$$

If $x \in \partial U$, then $x \notin U$ because of (b), which is contrary to (8). Hence, we notice that $x \notin \partial U$. Since $x \in \partial(U \cap V) \subseteq \partial U \cup \partial V$, it should be

$$x \in \partial V. \tag{9}$$

By Theorem 1.1 (i), we have

$$\begin{aligned} U \cap \partial V &= U \cap \big(\overline{V} \cap \overline{X\backslash V}\big) \\ &= U \cap \big(\overline{V} \cap X\backslash V^\circ\big) = \overline{V} \cap \big(U\backslash V^\circ\big) \\ &\subseteq U\backslash V^\circ. \end{aligned} \tag{10}$$

Furthermore, by (8), (9), and (10), we get

$$x \in U \cap \partial V \subseteq U\backslash V^\circ \subseteq \overline{U}\backslash V^\circ. \tag{11}$$

By Theorem 4.1, (a), and (11), we get

$$x \in \overline{U}\backslash V^\circ = \overline{U\backslash V}.$$

Since $\overline{U\backslash V}$ is the 'smallest' closed set including $U\backslash V$ and, by (c), $\overline{U}\backslash V$ is a closed set including $U\backslash V$, we have

$$x \in \overline{U\backslash V} \subseteq \overline{U}\backslash V. \tag{12}$$

However, in view of (8), we see that $x \in V$, and hence, $x \notin \overline{U}\backslash V$, which is contrary to (12). Therefore, $U \cap V$ should be open.

VI. Conclusions

The relations (1) and (2) are true with the 'strict inclusion' for some specific subsets U and V of a topological space X; for example, let $U = \mathbf{Q}$ and $V = \mathbf{R}\backslash\mathbf{Q}$ be subsets of the (Euclidean) topological space $X = \mathbf{R}$. In many practical applications, it seems very important to investigate conditions under which the equality holds in relations (1) and (2). Unfortunately, the author could not find any literature concerning this subject. One of the main purposes of this paper is to resolve this problem by introducing some necessary and sufficient conditions. Moreover, the author prove the relations $U^\circ \setminus \overline{V} = (U \setminus V)^\circ$ and $\overline{U} \setminus V^\circ = \overline{U\setminus V}$ for subsets U and V of a topological space under an additional condition in Theorem 4.1. These equalities are of great importance in practical uses for investigating subjects of general topology or set-theoretic topology; for example, they are used to show the openness of the intersection of two subsets under some additional conditions (see Theorem 5.2).

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