Characterization of the Three-Bar Linkage System Generated Symmetric and Asymmetric Lemniscate-like Curves

Hee Seok Nam, Gaspar Porta, and Hyung Ju Nam

***Abstract*—**In this paper, we investigate the characterization of the family of curves generated by three-bar linkage systems. Depending on the location of the marker on the middle rod, symmetric or asymmetric lemniscate-like curves are constructed. An algebraic representation is presented using the distances from the foci as a generalization of the Lemniscate of Bernoulli. It turns out that the corresponding Cartesian equation is of the form of the Hippopede defined as the intersection of a torus and a plane. A geometric construction shows how we can construct a family of symmetric or asymmetric lemniscate-like curves using a circle and a fixed point. It leads to a polar representation. Finally, a parametric representation is given for the completion of the characterization.

***Keywords*—** asymmetric lemniscate-like curve, Lemniscate of Bernoulli, symmetric lemniscate-like curve, three-bar linkage system

I. Introduction

THIS paper is the result of extended observations originating during an exploration of A. B. Kempe's work "How to Draw a Straight Line; a Lecture on Linkages" [2] by the authors in [3].

Given two fixed points in the plane, Cassini Ovals can be characterized as the set of points so that the product of the distances from the two fixed points (called the foci) is a constant. When the foci are $2a$ apart, if we collect points with fixed constant b^2 for the product of distances from the foci, we obtain either two disconnected loops, the Lemniscate of Bernoulli, or a connected simple loop enclosing both foci depending on the range of b: $b < a$, $b = a$ and $b > a$. See Fig. 1 for an illustration[1]. Among many construction methods of the Lemniscate of Bernoulli, linkage systems are of great interest in this paper due to their usefulness in generalization and characterization in several ways.

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[1] http://mathworld.wolfram.com/CassiniOvals.html

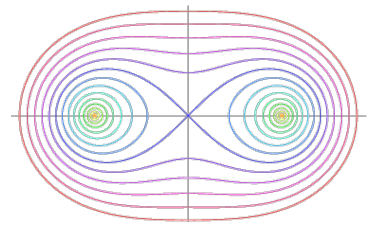

Fig. 1 Cassini Ovals

A linkage system corresponds to a set of bars connected to each other and the plane by articulations (called linkages) typically allowing a marking point on one of the bars to move describing a curve in the plane. More generally a linkage system can have the marking point describe a region as opposed to a curve--sometimes with special geometric effects. The formal terms used to describe the anatomy of a Linkage System throughout the rest of the paper follows.

1. Fixed point/Fixed Pivot: A fixed point of a linkage is a point which does not change its location during the motion of the linkage system.
2. Rod/Bar: A rod or bar is the line segment that connects two distinct points. A rod can take only rigid motions during the motion of the linkage system.
3. Hinge Point/Pivot Point: A hinge point is the "linkage point", and thus serves as the connection between two rods.
4. Mover: A mover is the end point on a rod, so called the driver, which rotates around a fixed point. It has one degrees of freedom and, in fact, we can use the angle of rotation to represent the location of the marker which draws the desired curve as a function of mover.
5. Marker: A marker, also known as tracer, is a point on a rod which will draw the whole or part of the desired curve as the mover rotates around a fixed point on a rod.

To draw the families of curves that include the Lemniscate of Bernoulli, one can use a three-bar linkage system. This system corresponds to a chain of three bars linked (hinged) to each other with the outside two bars also linked to the plane at two fixed points. The relative lengths of these bars, the distance between the fixed points, and the position of the marker (along the middle bar) determine the particular curve being drawn.

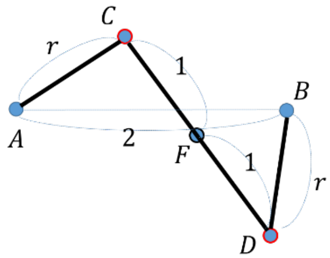

Fig. 2 Three-bar linkage system

Fig. 2 is a description of a three-bar linkage system: A and B are fixed points; AC, CD, DB are rods; C and D are hinge points; AC is the driver and C is the mover rotating around the fixed point A; F is the marker. The Lemniscate of Bernoulli is a particular case of the full set of curves that may be drawn in this way. Symmetric lemniscate-like curves, skewed or asymmetric lemniscate-like curves, and 'teardrop' curves also appear.

It turns out that the trace of the marker of three-bar linkages with specific rod lengths produces a family of symmetric lemniscate-like curves and it is natural to ask how we can characterize such a family of curves. This paper will focus on answering this question in various ways. In the following sections, we will investigate an algebraic characterization, a geometric construction, a parametric representation, and a generalization to skewed lemniscate-like curves, or asymmetric lemniscate-like curves.

II. Algebraic Characterization

In this section, we give an algebraic characterization of the family of symmetric lemniscate-like curves generated by three-bar linkage systems. See Fig. 2 for an illustration. We initially pick the distance $\text{dist}(A,B)=2$ and $|CD|=2$ for our computations and later generalize to encompass any scale.

Theorem 1 *If a three-bar linkage system is constructed by two fixed points A, B with $\text{dist}(A,B)=2$, and three bars AC, BD, CD with $|AC|=|BD|=r$, and $|CD|=2$, then the family of symmetric lemniscate-like curves generated by the marker F, the midpoint of the line segment CD, can be characterized as*

$$\left(d_1^2+1-\frac{r^2}{2}\right)\left(d_2^2+1-\frac{r^2}{2}\right)=\left(2-\frac{r^2}{2}\right)^2 \tag{1}$$

where $d_1=|AF|$ and $d_2=|BF|$.

Proof. The proof comes from the combination of Pappus's theorem and the law of cosines. In fact, the quadrilateral $ACBD$ is an anti-parallelogram and $\angle BCD=\angle ADC$. For the triangles ADC and BCD, the law of cosines gives

$$\frac{|BC|^2+2^2-r^2}{4\,|BC|}=\cos\angle BCD$$
$$=\cos\angle ADC=\frac{|AD|^2+2^2-r^2}{4\,|AD|}.$$

Solving this equation leads to the relation

$$|BC|\cdot|AD|=4-r^2. \tag{2}$$

On the other hand, we apply Pappus's theorem to those triangles and get

$$r^2+|AD|^2=2(1^2+|AF|^2)=2(1+d_1^2), \tag{3}$$
$$r^2+|BC|^2=2(1^2+|BF|^2)=2(1+d_2^2). \tag{4}$$

Combining (3), (4) and (2), we have the desired result (1). ■

Characterization (1) generalizes the Lemniscate of Bernoulli. In fact, if $r=\sqrt{2}$, then (1) reduces to

$$d_1\,d_2=1=\left(\frac{\text{dist}(A,B)}{2}\right)^2$$

which exactly coincides with the definition of the Lemniscate of Bernoulli.

Corollary 2 *For the three-bar linkage system in Theorem 1, if $A=(-1,0)$ and $B=(1,0)$, then the Cartesian equation of the marker F(x,y) is given as*

$$\left((x+1)^2+y^2+1-\frac{r^2}{2}\right)\left((x-1)^2+y^2+1-\frac{r^2}{2}\right)$$
$$=\left(2-\frac{r^2}{2}\right)^2.$$

If the two fixed points are $2a$ away and the connecting rods have lengths r, $2a$, and r, then the trace of the marker can be characterized as

$$\left(d_1^2+a^2-\frac{r^2}{2}\right)\left(d_2^2+a^2-\frac{r^2}{2}\right)=\left(2a^2-\frac{r^2}{2}\right)^2. \tag{5}$$

A proof of this relation (5) is essentially identical to that of Theorem 1. If we set the two fixed points as $(-a,0)$ and $(a,0)$, then the Cartesian equation of the graph can be simplified to

$$(x^2+y^2)^2=r^2x^2+(r^2-4a^2)\,y^2$$

which is exactly the Hippopede[2] of the form

$$(x^2+y^2)^2=cx^2+dy^2$$

where $c>0$ and $c>d$.

III. Geometric Construction and Polar Equation

A geometric characterization of symmetric lemniscate-like curves highlights a connection to three-bar linkage systems. This part is mainly motivated from the work of Akopyan [1].

Lemma 3 *Given three-bar linkage system as in Fig. 3 where* $\text{dist}(A,B)=|CD|=2$ *and $|AC|=|BD|=r$, if M, N and O are midpoints of AC, AD and AB, respectively and $ANOX$ is a parallelogram, then the followings hold.*

(a) Points M, F, X, and O are collinear.

(b) $\overrightarrow{XM}=\overrightarrow{OF}$

(c) $|AM|=|AX|=r/2$

Proof. Since $ACBD$ is an anti-parallelogram, we have $AD\parallel CB$ and $AMFN$ is also a parallelogram. Therefore M, F, X and O are collinear and $\overrightarrow{FM}=\overrightarrow{OX}$ which lead to $\overrightarrow{XM}=\overrightarrow{OF}$. Finally, $|AX|=|NO|=|BD|/2=r/2=|AM|$.

[2] https://en.wikipedia.org/wiki/Hippopede

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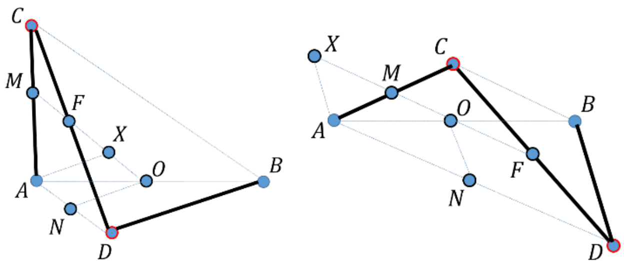

Fig. 3 Characterizing the location of F

Lemma 3 holds true for general values of r and we can obtain another characterization of symmetric lemniscate-like curves, or the Hippopede. A direct consequence of this lemma can be stated as follows.

Theorem 4 *Given three-bar linkage system as in Fig. 4 where $dis(A,B) = |CD| = 2, O$ is the midpoint of AB, $|AC| = |BD| = r$, and the circle with radius $r/2$ and center A, the set of all points F where $\overrightarrow{OF} = \overrightarrow{XM}$ draws a symmetric lemniscate-like curve generated by the three-bar linkage system.*

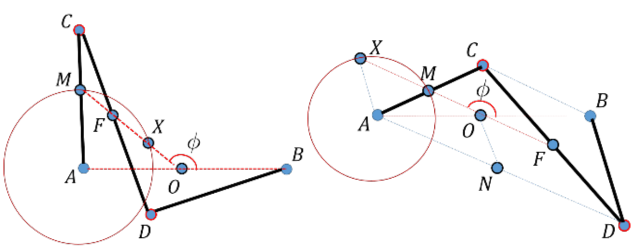

Fig. 4 Geometric construction

From the triangle AXO, the law of cosine gives

$$\left(\frac{r}{2}\right)^2 = 1 + |OX|^2 - 2\,|OX|\cos(\pi - \phi)$$

or

$$|OX| = -\cos\phi - \sqrt{\left(\frac{r}{2}\right)^2 - \sin^2\phi}\,. \qquad (6)$$

If we apply intersecting secants theorem to the secant OXM, we have

$$|OX|\cdot|OM| = |OX|\cdot(|OX| + |MX|) = \left(1 - \frac{r}{2}\right)\left(1 + \frac{r}{2}\right)$$

or

$$|MX| = -|OX| + \frac{1}{|OX|}\left(1 - \frac{r^2}{4}\right). \qquad (7)$$

Substituting (6) to (7), we obtain the following result:

Theorem 5 *For a three-bar linkage system with rod lengths r, 2, r and foci at $(-1,0)$ and $(1,0)$, the trace of the marker, or the midpoint of the middle rod can be represented by a polar equation:*

$$\rho = 2\sqrt{\left(\frac{r}{2}\right)^2 - \sin^2\phi}\,.$$

Finally, from this polar representation, we can classify the family of curves as follows:

(a) If $0 < r < 2$, then we have a figure-8 curve, or symmetric lemniscate-like curve.

(b) If $r = 2$, then we have two unit circles centered at foci.

(c) If $r > 2$, then we have a connected simple loop enclosing both foci.

IV. PARAMETRIC REPRESENTATION

If we want to construct a graph, it is very helpful to get a parametric representation. To find a parametric representation of the marker F, we assume that $A = (-1,0)$, $B = (1,0)$, $|AC| = |BD| = r$, $|CD| = 2$, and F is the midpoint of CD as in Fig. 5. We measure the angle θ counterclockwise from the ray AB to the driver AC so that $C = (-1 + r\cos\theta, r\sin\theta)$. From the key fact that the quadrilateral $ACBD$ is an anti-parallelogram, we have $\overrightarrow{CB} = (2 - r\cos\theta, -r\sin\theta)$ and $\overrightarrow{AD} = t\overrightarrow{CB}$ for some constant t. The point $D = (-1,0) + t(2 - r\cos\theta, -r\sin\theta)$ satisfies $|BD| = r$ which gives a quadratic equation in t:

$$\{t(2 - r\cos\theta) - 2\}^2 + (t\cdot r\sin\theta)^2 = r^2.$$

Since $t = 1$ is a trivial solution, it is not surprising to factor the equation and get the solution set as

$$\left\{\frac{4 - r^2}{4 - 4r\cos\theta + r^2}, 1\right\}.$$

We discard the possibility $t = 1$ to represent D as

$$D = \left(-1 + \frac{(2 - r\cos\theta)(4 - r^2)}{4 - 4r\cos\theta + r^2}, \frac{-(4 - r^2)r\sin\theta}{4 - 4r\cos\theta + r^2}\right)$$

and the marker F as

$$F = \frac{C + D}{2}$$

$$= \left(\frac{r(2 - r\cos\theta)(2\cos\theta - r)}{4 - 4r\cos\theta + r^2}, \frac{r^2\sin\theta(r - 2\cos\theta)}{4 - 4r\cos\theta + r^2}\right).$$

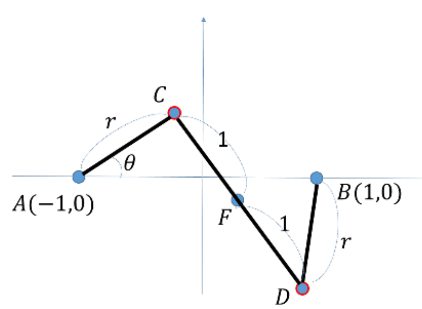

Fig. 5 Three-bar linkage system: Midpoint marker

V. GENERALIZATION TO SKEWED OR ASYMMETRIC LEMNISCATE-LIKE CURVES

In this section, we generalize the location of the marker so that it can be any fixed point on the middle rod. See Fig. 6 where $m + n = 2$. We observe a family of skewed or asymmetric lemniscate-like curves if $m \neq n$.

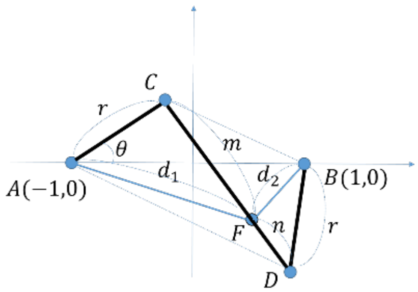

Fig. 6 Three-bar linkage system: Generalized marker

For an algebraic characterization, we need a generalized version of Pappus's theorem.

Lemma 6 *For Fig. 7, we have the relation:*

$$c_1c_2 + d^2 = \frac{c_2}{c_1+c_2}a^2 + \frac{c_1}{c_1+c_2}b^2.$$

In particular, if $c_1 = c_2 = c$, then the relation reduces to

$$c^2 + d^2 = \frac{1}{2}(a^2+b^2).$$

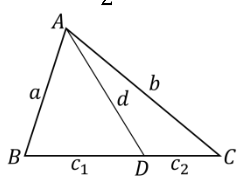

Fig. 7 Generalized Pappus's Theorem

Proof. We use vectors and the dot product to prove the theorem. In fact,

$$\overrightarrow{AD} = \frac{c_2}{c_1+c_2}\overrightarrow{AB} + \frac{c_1}{c_1+c_2}\overrightarrow{AC}$$

which leads to

$$(c_1+c_2)^2\left|\overrightarrow{AD}\right|^2 = c_2^2\left|\overrightarrow{AB}\right|^2 + c_1^2\left|\overrightarrow{AC}\right|^2 + 2c_1c_2\overrightarrow{AB}\cdot\overrightarrow{AC}.$$

Since $|\overrightarrow{AC} - \overrightarrow{AB}| = |\overrightarrow{BC}| = c_1 + c_2$, we have

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = \frac{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 - (c_1+c_2)^2}{2}.$$

We plug this dot product into (8) and simplify to get the desired result. ■

Now we are ready to give a geometric characterization of the family of curves generated by a generalized marker of three-bar linkage systems. See Fig. 6.

Theorem 7 *For a three-bar linkage system with two fixed points A and B, and three bars AC, BD, CD such that $dist(A,B) = |CD| = 2$ and $|AC| = |BD| = r$, the family of skewed or asymmetric lemniscate-like curves generated by the marker F with $|CF| = m$ and $|FD| = n = 2 - m$ can be characterized as*

$$\left(d_1^2 + mn - \frac{nr^2}{2}\right)\left(d_2^2 + mn - \frac{mr^2}{2}\right) = mn\left(2 - \frac{r^2}{2}\right)^2 \quad (9)$$

where $d_1 = |AF|$ and $d_2 = |BF|$.

The proof is similar to that of Theorem 1 and left to the reader. Note that the relation (2) is still valid while we need a generalized version of Pappus's theorem (Lemma 6) to replace (3) and (4).

Under the same assumptions of Theorem 7, we can construct skewed or asymmetric lemniscate-like curves geometrically. When the marker is located at F such that $|CF| = m < n = |FD|$, we construct the following points in this order (see Fig. 8):

- M is the intersection of AC and the circle Y with center at A and radius $nr/2$,
- O' is on AB such that $|AO'| = n = |FD|$,
- N is on AD such that $|AN| = m|AD|/2$,
- N' is on AD such that $|AN'| = n|AD|/2$,
- X is the point such that the quadrilateral $AN'O'X$ is a parallelogram.

Elementary geometry shows that

$$MF \parallel AD,$$

$$|MF| = |AN| = \tfrac{m}{2}|AD|, \quad (10)$$

$$|AX| = |N'O'| = \tfrac{n}{2}|DB| = \tfrac{nr}{2} = |AM|. \quad (11)$$

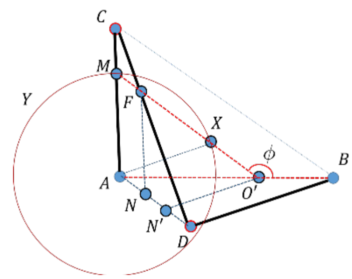

Fig. 8 Geometric construction

Let's consider the case that the angle $\phi = \angle BO'X \in \left[\pi - \sin^{-1}\left(\frac{r}{2}\right), \pi + \sin^{-1}\left(\frac{r}{2}\right)\right]$. The four points O', X, F and M are collinear in this order. From the relation (2) and (10), we have a system of equations:

$$|MX| = |MO'| - |XO'| = \frac{n}{2}\cdot\frac{4-r^2}{|AD|} - \frac{n}{2}|AD|,$$

$$|FO'| = |MO'| - |MF| = \frac{n}{2}\cdot\frac{4-r^2}{|AD|} - \frac{m}{2}|AD|.$$

Solving for $|FO'|$ in terms of $|MX|$ leads to:

$$|FO'| = \frac{|MX|}{n} + \sqrt{\frac{1-mn}{n^2}|MX|^2 + (m-n)^2\left(1 - \frac{r^2}{4}\right)}.$$

Since M and X are on the same circle Y by (11), we can determine the location of F as described in the next theorem. Note that if $\phi \in \left[-\left(\pi - \sin^{-1}\left(\frac{r}{2}\right)\right), \pi - \sin^{-1}\left(\frac{r}{2}\right)\right]$, then X, M, O', and F are collinear in this order, but the following geometric construction still remains accurate.

Theorem 8 *Let A and B be two fixed points with $dist(A,B) = 2$, and O' is on the line segment AB such that $dist(A,O') = n$, $dist(O',B) = m = 2 - n$. For any point M on the circle Y with center at A and radius $nr/2$, let X and F be determined as*

(a) X is the other intersection of the circle Y and the line passing through O' and M,
(b) $\overrightarrow{O'F} = k\,\overrightarrow{XM}$ where

$$k = \frac{1}{n} + \sqrt{\frac{1-mn}{n^2} + \frac{(m-n)^2}{|XM|^2}\left(1 - \frac{r^2}{4}\right)}.$$

Then the locus of F is an asymmetric lemniscate-like curve generated by the three-bar linkage system with two fixed points A and B, and three bars AC, CD, DB such that $\overrightarrow{AC} = \frac{2}{n}\overrightarrow{AM}$ and $\overrightarrow{CD} = \frac{2}{m}\overrightarrow{CF}$. Moreover, dist$(A,B) = |CD| = 2$, $|AC| = |BD| = r$, $|CF| = m$ and $|FD| = n = 2 - m$.

From the triangle AXO' (see Fig. 8), similar to the derivation of (6) and (7), we have

$$|O'X| = -n\cos\phi - n\sqrt{\frac{r^2}{4} - \sin^2\phi}\,,$$

$$|MX| = -|O'X| + \frac{n^2}{|O'X|}\left(1 - \frac{r^2}{4}\right) = 2n\sqrt{\frac{r^2}{4} - \sin^2\phi}\,,$$

$$|O'M| = |O'X| + |XM| = -n\cos\phi + n\sqrt{\frac{r^2}{4} - \sin^2\phi}\,.$$

Since $|O'M| = \frac{n}{2}\cdot\frac{4-r^2}{|AD|}$, we have

$$|AD| = -2\left(\cos\phi + \sqrt{\frac{r^2}{4} - \sin^2\phi}\right).$$

Finally,

$$|O'F| = |O'M| - \frac{m}{2}|AD| = (m-n)\cos\phi + 2\sqrt{\frac{r^2}{4} - \sin^2\phi}\,.$$

Theorem 9 *For a three-bar linkage system with two fixed points $A = (-n, 0)$ and $B = (m, 0)$, and three bars AC, BD, CD such that dist$(A,B) = m + n = |CD| = 2$ and $|AC| = |BD| = r$, then the family of skewed or asymmetric lemniscate-like curves generated by the marker F with $|CF| = m$ and $|FD| = n = 2 - m$ can be represented by a polar equation:*

$$\rho = (m-n)\cos\phi + 2\sqrt{\frac{r^2}{4} - \sin^2\phi}.$$

See Fig. 9 for an illustration generated using Maple.

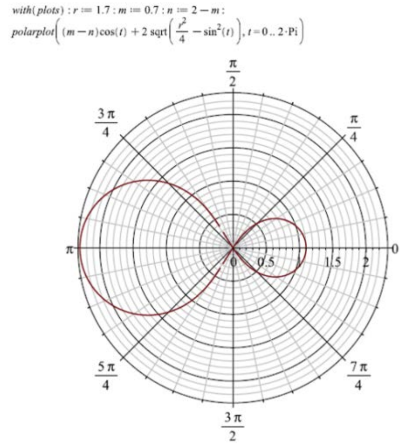

Fig. 9 Asymmetric lemniscate-like curve: $r = 1.7$, $m = 0.7$, $n = 1.3$

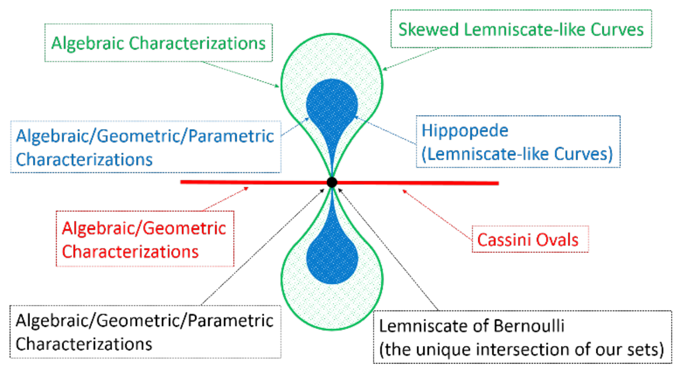

Fig. 10 Relationship between sets of curves

VI. Conclusion

Fig. 10 indicates the relationship between the sets made up of the families of curves we have connected. The geometric relationship that characterizes Cassini Ovals leads to an algebraic expression that, in the case of the Lemniscate of Bernoulli, can be connected with a construction using a three-bar linkage system. We verify this connection algebraically for a three-bar linkage system including the scale-independent case. From the three-bar linkage construction, we extend the algebraic relation to symmetric lemniscate-like curves, or the Hippopede, and represent these parametrically.

With this we incorporate another geometric characterization (the geometric representation described in Fig. 4) for which we construct a polar representation. This allows us to broaden the set of curves for which we have an algebraic characterization connected with a geometric characterization to include all our symmetric lemniscate-like curves. We then further expand our algebraic characterization from the three-bar linkage system to include skewed or asymmetric lemniscate-like curves where the marker is located on any fixed point on the middle rod.

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