Estimating volume tensor by Fakir probe.

Jiří Janáček and Daniel Jirák

Abstract**—**Interactive Fakir method for object volume measurements using virtual spatial grid of lines is used for basic shape analysis evaluating second moment tensor of intersections of the spatial grid with the studied object. The method is applied on developmental study of Pheasant brain.

Keywords**—**Minkowski tensor, Fakir probe, bird brain.

I. Introduction

The volume of 3D object can be estimated using implementations the virtual grids of lines, or fakir probe, on desktop computer [1,2]. The mean length of intersection of the object with the grid in random position is proportional to the grid length density (m^{-2}) and to the volume of the object (Fig. 1).

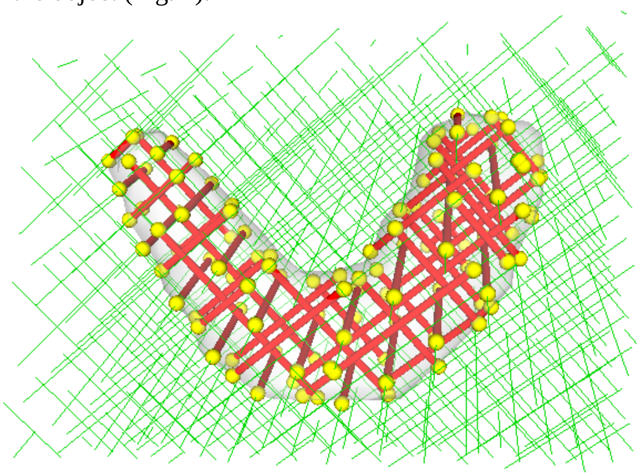
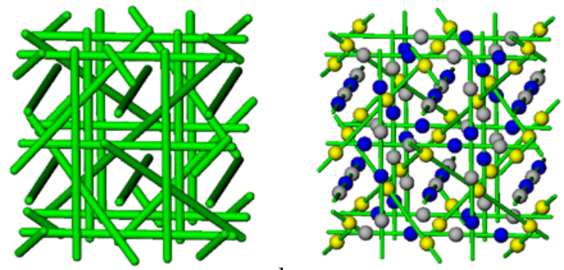


Fig 1. Threefold line grid with intersections with an 3D object.

Variance of the estimators using randomly oriented grids can be estimated from the grid density and properties of the measured objects: the variance of IUR volume estimator of objects in R^d, using spatial grid has asymptotic term proportional to the surface area of the object multiplied by u^{d+1} as the scale u of the lattice tends to zero [3]. The particularly efficient grid of lines can be found in nature, interestingly, as the atoms in the crystal of garnet are aligned along sevenfold grid of lines filling the space optimally (Fig. 2).



a b

Fig 2. Sevenfold line grid (a) with atoms in crystal of garnet (b).

II. Volume Tensor of the Line Segments

Let X be a measurable subset of d-dimensional Euclidean space R^d, the volume second moment Minkowski tensor is defined by the integral [4]:

$$\Phi_{\mathrm{d},2,0}(X) = \frac{1}{2}\int_X \mathbf{x}\otimes\mathbf{x}\,d\mathbf{x}, \tag{1}$$

where $\mathbf{x}\otimes\mathbf{x}$ is the matrix of the coordinates products $\{x_i x_j\}_{i,j=1}^d$ and the integration is done with respect to the Lebesgue measure.

Let the intersection of the set X with some regular grid of lines with intensity λ (m/m^d) consist from N line segments with endpoints $\mathbf{a}_k$ and $\mathbf{b}_k$. Let $l_k = \|\mathbf{b}_k - \mathbf{a}_k\|$ be the length of the *k*-th segment. Then natural estimate of the tensor $\Phi_{\mathrm{d},2,0}(X)$ is obtained by summing the tensors of individual lines obtained by integrating $\mathbf{x}\otimes\mathbf{x}$ with respect to 1-dimensional measure divided by intensity λ:

$$\tilde{\Phi}_{\mathrm{d},2,0}(X) = \frac{1}{12\lambda}\sum_{k=1}^{N} l_k \{2a_{k,i}a_{k,j} + a_{k,i}b_{k,j} + b_{k,i}a_{k,j} + 2b_{k,i}b_{k,j}\}_{i,j=1}^d. \tag{2}$$

The centered volume tensor is used to remove the dependence of the volume tensor on the actual position of the object in space. The centered tensor is defined as

$$\mathbf{D} = \tilde{\Phi}_{\mathrm{d},2,0}(X) - \frac{1}{\lambda L}\{C_i C_j\}_{i,j=1}^d, \tag{3}$$

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where L is total length of all line segments and C_i and C_j are coordinates of the center of mass of the union of the segments. It is convenient to visualize the symmetric tensor by an equivalent ellipsoid. Let d_i be the eigenvalues of tensor **D**. The ellipsoid with the volume tensor with eigenvalues d_i has semiaxis lengths [5]:

$$s_i = \sqrt{d_i}\left(\frac{2(d+2)}{\kappa_d}\prod_{j=1}^{d}\sqrt{d_j}\right)^{\frac{1}{d+2}}. \qquad (4)$$

We define anisotropy measure of the set as the Procrustes anisotropy of the centered tensor **D** [6]:

$$PA(X)=\sqrt{\frac{d}{d-1}\sum_{i'=1}^{d}(s_i-s)^2 \Big/ \sum_{i'=1}^{d} s_i^2}. \qquad (5)$$

The Procrustes anisotropy takes values in interval from 0 to 1.

III. Analysis of the Developing Pheasant Brain

Heads of 2 hatchlings, 4 juvenile and 6 adult ring-necked pheasants (Phasianus colchicus) fixed in formalin were scanned on a 4.7T MR spectrometer.

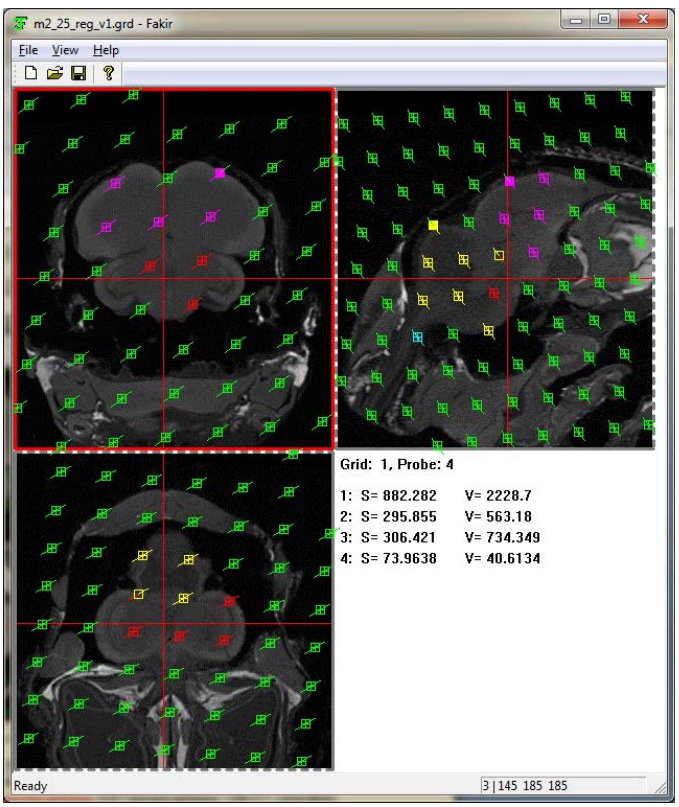

Fig 3. The pheasant brain with virtual grid of lines in Fakir program.

Volume and surface area of brain divisions were measured interactively by sevenfold Fakir probe with grid density 0.76 mm^{-2} and the results can be found in [7]. Structures of avian brain were identified in histological atlas [8]. The anisotropy characteristics of the brain compartments are in (Tab. 1).

	Forebrain	Midbrain	Hindbrain
Hatchlings (n=2)	0.30 (0.01)	0.83 (0.01)	0.39 (0.02)
Juveniles (n=4)	0.36 (0.01)	0.85 (0.01)	0.44 (0.01)
Adults (n=6)	0.43 (0.01)	0.83 (0.01)	0.35 (0.01)

Tab. 1 Procrustes anisotropy *PA* of Pheasant brain compartments calculated according to (5). Anisotropy values are presented with the standard error of the mean. The differences in forebrain and hindbrain anisotropy between age groups are highly significant (ANOVA $p<0.01$).

	s_1	s_2	s_3
Hatchlings	7.0	5.2	4.5
Juveniles	10.3	7.2	6.3
Adults	11.7	7.8	6.3

Tab. 2 Semiaxis lengths (mm) of ellipsoids corresponding (4) to forebrain volume tensor.

The semiaxis corresponding to the biggest eigenvalue is oriented dorsally in forebrain and midbrain and rostraly in hindbrain.

The change of relative width is the major change of the forebrain shape during the development (Tab. 2).

IV. Conclusion

The values of width obtained by the moment method are more robust, that direct measurements of the width, because the later depends on the selection of extremal points that may be often rather arbitrary.

The described method detected changes in Pheasant forebrain shape in development successfully and can be used in similar morphometric studies of macro- or microscopic objects and implemented in programs using Fakir grid for volume measurement.

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