Investigation and Approximate Solution of Two Systems of Nonlinear Partial Differential Equations

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***Abstract*—**Two systems of nonlinear partial differential equations are considered. Both systems are obtained at mathematical modeling of process of electromagnetic field penetration in the substance. In the quasistationary approximation, this process, taking into account of Joule law is described by nonlinear well known system of Maxwell equations. Taking into account heat conductivity of the medium and again the Joule law, the different type nonlinear system of partial differential equations is obtained. Investigation and approximate solution of the initial-boundary value problems are studied for these type models. Linear stability of the stationary solution is studied. Blow-up is fixed. Special attention is paid to construction of discrete analogs, corresponding to one-dimensional models as well as to construction, analysis and computer realization of decomposition algorithms with respect to physical processes for the second system. Averaged additive semi-discrete models, finite difference schemes are constructed and theorems of convergence are given.

***Keywords*—**Nonlinear differential equations, blow-up, stationary solution, linear stability, Hoph bifurcation, averaged additive semi-discrete models, finite difference

I. Introduction

In mathematical modeling of many natural processes nonlinear nonstationary differential models are received very often. One such model is obtained at mathematical modeling of process of electromagnetic field penetration in the substance. In the quasistationary case the corresponding system of Maxwell equations has the form [18]:

$$\frac{\partial H}{\partial t} = -rot\left(v_m rotH\right), \quad \frac{\partial \theta}{\partial t} = v_m\left(rotH\right)^2, \tag{1}$$

where $H=\left(H_1, H_2, H_3\right)$ is a vector of the magnetic field, θ is temperature, v_m characterizes the electro-conductivity of the substance. The first vector equation of system (1) describes the process of diffusion of the magnetic field and the second equation describes the change of the temperature at the expense of Joule heating.

For a more thorough description of electromagnetic field propagation in the medium, it is desirable to take into consideration different physical effects, first of all heat conductivity of the medium has to be taken into consideration. In this case, the same process is described by the following system:

$$\begin{aligned} \frac{\partial H}{\partial t} &= -rot\left(v_m rotH\right), \\ \frac{\partial \theta}{\partial t} &= v_m\left(rotH\right)^2 + div\left(kgrad\theta\right), \end{aligned} \tag{2}$$

where k is coefficient of heat conductivity. As a rule this coefficient is function of argument θ as well.

Many other processes are described by the (1) and (2) type systems and many works are dedicated to the investigation and numerical resolution of the initial-boundary value problems for these type models (see, for example, [1]-[17], [20]-[23], and references therein).

There are still many open questions in this direction. We study some properties of solutions of different kind of initial-boundary value problems for investigated systems, as well as numerical solution of those problems. Many authors are studying convergence of semi-discrete analogs and finite-difference schemes for the models described here and for the problems similar to them.

Some generalization of one-dimensional system of nonlinear partial differential equations based on Maxwell model is considered. Initial-boundary value problem with mixed type boundary conditions is discussed. It is proved that in some cases of nonlinearity there exists critical value ψ_c of the boundary data such that for $0<\psi<\psi_c$ the steady state solution of the studied problem is linearly stable, while for $\psi>\psi_c$ is unstable. It is shown that as ψ_c passes through ψ_c then the Hopf type bifurcation may take place.

We compare theoretical results to numerical ones. Special attention is paid to construction of discrete analogs, corresponding to one-dimensional models as well as to construction, analysis and computer realization of decomposition algorithms with respect to physical processes for the second system. The above-mentioned decomposition is defined by splitting this model in two parts: in the first part the Joule heat release is taken into account and in the second - part the heat conductivity of the medium is considered. Semi-discrete averaged additive models, finite difference schemes are constructed and theorems of convergence are given.

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Investigation and approximate solution of the initial-boundary value problems posed for these systems are the actual sphere of contemporary mathematical physics and numerical analysis. Note that, system (1) can be reduced to integro-differential form. Many works were published in this direction (see, for example, [16] and references therein).

Our aim is to study some properties of solutions of the initial-boundary value problems for one-dimensional variants of (1) and (2) type systems. Blow-up solution is constructed. Linear stability of the stationary solution is studied and Hoph bifurcation phenomena is fixed. The finite difference scheme are constructed for investigated problem. The additive schemes for one-dimensional analog of system (2) with one-component magnetic field are given as well.

II. Blow-up and Stability of Solutions

In the cylinder $[0,1]\times[0,\infty)$ let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^{\alpha}\frac{\partial U}{\partial x}\right),\quad \frac{\partial V}{\partial t}=V^{\alpha}\left(\frac{\partial U}{\partial x}\right)^{2},$$
$$U(0,t)=0,\ \ U(1,t)=\psi>0, \tag{3}$$
$$U(x,0)=U_0(x),\ V(x,0)=V_0(x)=v_0>0,$$

where U_0 and V_0 are known functions defined on $[0,1]$ and ψ and v_0 are constants.

It is not difficult to verify that if $\alpha\neq 1$ and $V_0(x)=v_0$ then the following functions:

$$U(x,t)=\psi x,$$
$$V(x,t)=\left[v_0^{1-\alpha}+(1-\alpha)\psi^{2}t\right]^{\frac{1}{1-\alpha}} \tag{4}$$

are solutions of the problem (3). But if $\alpha>1$ in the finite time $t_0=\delta_0^{1-\alpha}/\psi^{2}(\alpha-1)$ the function $V(x,t)$ becomes infinity. This example shows that solution of problem (3) with smooth initial and boundary conditions can be blown up in the finite time.

Let us consider the following system:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^{\alpha}\frac{\partial U}{\partial x}\right),$$
$$\frac{\partial V}{\partial t}=V^{\alpha}\left(\frac{\partial U}{\partial x}\right)^{2}+\frac{\partial^2 U}{\partial x^2} \tag{5}$$

Many facts that obtained for (3) problem are valid for (5) too. In particular, functions $U(x,t)$ and $V(x,t)$ defined by (4) satisfy the system (5). From this one can deduce that for system (5), analogical to (3) problem, adding the following boundary conditions:

$$\left.\frac{\partial V(x,t)}{\partial t}\right|_{x=0}=\left.\frac{\partial V(x,t)}{\partial t}\right|_{x=1}=0,$$

if $\alpha>1$ the theorem of global solvability does not take place.

Main part of this paragraph deals with a nonlinear model which is obtained after adding of two terms to the second equation of Maxwell one-dimensional system (1).

In the cylinder $[0,1]\times[0,\infty)$ let us consider the following problem [14]:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^{\alpha}\frac{\partial U}{\partial x}\right),$$
$$\frac{\partial V}{\partial t}=-aV^{\beta}+bV^{\gamma}\left(\frac{\partial U}{\partial x}\right)^{2}+cV^{\gamma-\alpha}\frac{\partial U}{\partial x}, \tag{6}$$

$$U(0,t)=0,\ \ \left.V^{\alpha}\frac{\partial U}{\partial x}\right|_{x=1}=\psi, \tag{7}$$

$$U(x,0)=U_0(x),\ V(x,0)=V_0(x). \tag{8}$$

Here, likewise above t and x are time and space variables respectively, $U=U(x,t)$, $V=V(x,t)$, are unknown functions, U_0,V_0 are given functions, $a,b,c,\alpha,\beta,\gamma,\psi$ known positive parameters.

It is easy to check that the unique stationary solution of problem (6) - (8) is:

$$U_s(x)=\left(\frac{b}{a}\psi^{2}+\frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}\psi x,$$

$$V_s(x)=\left(\frac{b}{a}\psi^{2}+\frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}},$$

Introducing a designation $W=V^{\alpha}\frac{\partial U}{\partial x}$, after simple transformations, we get:

$$\frac{\partial W}{\partial t}=V^{\alpha}\frac{\partial^2 W}{\partial x^2}+$$
$$a\left(-aV^{\beta-1}+bV^{\gamma-2\alpha-1}W^{2}+cV^{\gamma-2\alpha-1}W\right)W, \tag{9}$$
$$\frac{\partial V}{\partial t}=-aV^{\beta}+bV^{\gamma-2\alpha}W^{2}+cV^{\gamma-2\alpha}W,$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=0}=0,\ \ W(1,x)=\psi, \tag{10}$$

$$W(x,0)=V_0^{\alpha}\frac{\partial U_0(x)}{\partial x},\ \ V(x,0)=V_0(x). \tag{11}$$

The unique stationary solution of problem (9) - (11) is:

$$W_s(x)=\psi x,\ \ V_s(x)=\left(\frac{b}{a}\psi^{2}+\frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}.$$

Let

$$W(x,t)=W_s(x)+W_1(x)e^{\lambda t},$$
$$V(x,t)=V_s(x)+V_1(x)e^{\lambda t}.$$

We examine the linear stability of problem (9) - (11) by linearizing (9) about the stationary solution (W_s, V_s). After some transformations we have:

$$\lambda W_1 = \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}} \frac{\partial^2 W_1}{\partial x^2} +$$

$$\alpha\left(a\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}} + b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}} \right) W_1 -$$

$$\alpha a\psi(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-2}{2\alpha+\beta-\gamma}} V_1,$$

$$\left[\lambda + a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right] V_1 =$$

$$(2b\psi + c)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha}{2\alpha+\beta-\gamma}} W_1,$$

$$\frac{d^2 W_1}{dx^2} + \eta^2 W_1 = 0, \qquad \left.\frac{dW_1}{dx}\right|_{x=0} = W(1) = 0, \tag{12}$$

where

$$\eta^2 = \alpha\left(a\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{2\alpha+\beta-\gamma}} + b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}} \right) -$$

$$\alpha a(2\alpha+\beta-\gamma)(2b\psi + c)\psi\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}}$$

$$\times\left(\lambda + a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right)^{-1} -$$

$$\lambda\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}.$$

It is not difficult to show that problem (12) has nontrivial solutions if and only if

$$\eta^2 = \eta_n^2 = \left(n + \frac{1}{2}\right)^2 \pi^2, \quad n \in Z_0.$$

For corresponding $\lambda = \lambda_n$ we have:

$$\lambda_n^2 - P_n(\psi,\alpha,\beta,\gamma,a,b,c)\lambda_n + L_n(\psi,\alpha,\beta,\gamma,a,b,c) = 0, \tag{13}$$

where

$$P_n(\psi,\alpha,\beta,\gamma,a,b,c) = \left(n + \frac{1}{2}\right)^2 \pi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}} +$$

$$a(\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}} - \alpha b\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}},$$

$$L_n(\psi,\alpha,\beta,\gamma,a,b,c) = a\left(n + \frac{1}{2}\right)^2 \pi^2 \times \tag{14}$$

$$(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}} -$$

$$a\alpha(2\alpha+\beta-\gamma)\left[a\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{2(\beta-1)-\alpha}{2\alpha+\beta-\gamma}} + b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-1}{2\alpha+\beta-\gamma}} \right].$$

Let us note that the stationary solution (W_s, V_s) of the problem (9) - (11) is linearly stabile if and only if $\mathrm{Re}(\lambda_n) < 0,$ for all n and unstable if there is an integer m such that $\mathrm{Re}(\lambda_m) > 0.$ From (13), (14) it can be deduced the following statement.

Theorem 1. *If* $2\alpha+\beta-\gamma > 0,$ *then stationary solution* (W_s, V_s) *of problem (9) - (11) is linearly stable if and only if* $P_n(\psi,\alpha,\beta,\gamma,a,b,c) < 0,$ *for all* $n,$ *i.e. if and only if*

$$a(\gamma-\alpha-\beta)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{\alpha+\beta-\gamma}} +$$

$$\alpha b\psi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}}<\frac{\pi^2}{4}.$$

We examined the stability of the steady state solution which depends on a boundary condition $\psi>0$. For a sufficiently small values of ψ the steady state solution is linearly stable. But as ψ passes through a critical value, the stability changes and a Hopf bifurcation may takes place [19].

Global exponential stabilization of solution is also proved [12] for problem (6) - (8) in case $a=b=\alpha=\beta=1$, $c=\gamma=0$.

III. Difference Schemes and Semi-Discrete Splitting with Respect Physical Processes

At the beginning, let us consider first type initial-boundary value problem for the following model system:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right),\quad \frac{\partial V}{\partial t}=\left(\frac{\partial U}{\partial x}\right)^2,$$
$$U(0,t)=U(1,t)=0, \tag{15}$$
$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x).$$

The semi-discrete and finite difference second order accuracy schemes with respect of space step is constructed and studied in [7] for this case of nonlinearity. In [9] more general finite difference schemes including second order accuracy two-level scheme and tree-level type scheme are also studied.

Let us introduce the grids:

$$\omega_{h\tau}=\overline{\omega}_h\times\omega_\tau,\quad \omega_{h\tau}^*=\omega_h^*\times\omega_\tau,$$

where:

$$\omega_\tau=\{t_j=j\tau,\ j=0,1,\dots,N\},$$
$$\overline{\omega}_h=\{x_i=ih, i=0,1,\dots,M, h=1/M\},$$
$$\omega_h=\overline{\omega}_h\setminus\{x_0,x_M\},$$
$$\omega_h^*=\{x_i^*=(i-1/2)h, i=1,2,\dots,M\}.$$

Let us introduce also scalar-products, norms and well known notations:

$$(y,z)=\sum_{i=1}^{M-1}y_iz_ih,\quad (y,z]=\sum_{i=1}^{M}y_iz_ih,$$
$$\|y\|=(y,y)^{1/2},\quad \|y]|=(y,y]^{1/2},$$
$$y_x=\frac{y_{i+1}-y_i}{h},\quad y_{\bar{x}}=\frac{y_i-y_{i-1}}{h},$$
$$y_t=\frac{y^{j+1}-y^j}{\tau},\quad y_{\bar{t}t}=\frac{y^{j+1}-2y^j+y^{j-1}}{\tau^2},$$
$$y^{(\sigma)}=\sigma y^{j+1}+(1-\sigma)y^j$$

and consider the following finite-difference scheme:

$$u_t+\mu\tau u_{\bar{t}t}=\left(v^{(\sigma)}u_{\bar{x}}^{(\sigma)}\right)_x,\quad v_t+\mu\tau v_{\bar{t}t}=\left(u_{\bar{x}}^{(\sigma)}\right)^2,$$
$$u(0,t)=u(1,t)=0,$$
$$u(x,0)=U_0(x),\quad v(x,0)=V_0(x), \tag{16}$$
$$u(x,\tau)=U_0(x)+\tau\left(VU_{\bar{x}}\right)_x\Big|_{t=0},$$
$$v(x,\tau)=V_0(x)+\tau\left(U_{\bar{x}}\right)^2\Big|_{t=0}.$$

In the (16) discrete function u is defined on $\omega_{h\tau}$ and v is defined on $\omega_{h\tau}^*$.

The following statement takes place [9].

Theorem 2. If $\sigma-0.5\ge\mu\ge0$ and problem (15) has sufficiently smooth solution, then finite difference scheme (16) converges as $\tau\to0$, $h\to0$, and the following estimate is true

$$\left\|U^j-u^j\right\|+\left\|V^j-v^j\right]\!\!\big|=O(\tau^2+h^2+(\sigma-0,5-\mu)\tau)$$

It is clear that from Theorem 2 we get following result: If $\sigma=0.5$, $\mu=0$ or $\sigma=1$, $\mu=0.5$ then convergence is the second order $O\left(\tau^2+h^2\right)$.

Investigation of splitting along the physical processes in one-dimensional case is the natural beginning of studding for (2) type systems. In this direction the first step was made in the work [1].

Now in the domain $\Omega\times[0,T]$ let us consider the following problem for (5) nonlinear one-dimensional parabolic system:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^\alpha\frac{\partial U}{\partial x}\right),$$
$$\frac{\partial V}{\partial t}=V^\alpha\left(\frac{\partial U}{\partial x}\right)^2+\frac{\partial^2V}{\partial x^2}, \tag{17}$$
$$U(x,t)=\frac{\partial V(x,t)}{\partial x}=0,(x,t)\in\partial\Omega\times(0,T),$$
$$U(x,0)=U_0(x),V(x,0)=V_0(x)\ge v_0>0,$$

where $-1/2\le\alpha\le1/2$, $\alpha\ne0$ and $\Omega=[0,1]$ with boundary $\partial\Omega$.

If we denote $V^{1/2}=W$, $2\alpha=\delta$, then problem (17) can be rewritten in the following equivalent form [1]:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(W^\delta\frac{\partial U}{\partial x}\right),$$
$$\frac{\partial W}{\partial t}=\frac{1}{2}W^{\delta-1}\left(\frac{\partial U}{\partial x}\right)^2+\frac{\partial^2W}{\partial x^2}+\frac{1}{W}\left(\frac{\partial W}{\partial x}\right)^2, \tag{18}$$
$$U(x,t)=\frac{\partial W(x,t)}{\partial x}=0,(x,t)\in\partial\Omega\times(0,T),$$
$$U(x,0)=U_0(x),W(x,0)=W_0(x)=V_0^{1/2}(x).$$

Let us use well known notations:

$$r_t = \frac{r^{j+1} - r^j}{\tau}, \quad r^j_{\alpha,t} = \frac{r^{j+1}_\alpha - r^j}{\tau}, \quad \alpha = 1,2$$

and correspond following additive averaged semi-discrete scheme to the initial-boundary value problem (18):

$$\begin{gathered} u_t = \frac{d}{dx}\left[\left(\eta_1 w_1^\delta + \eta_2 w_2^\delta\right)\frac{du}{dx}\right], \\ \eta_1 w_{1t} = \frac{1}{2} w_1^{\delta-1}\left(\frac{du}{dx}\right)^2, \\ \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\left(\frac{dw_2}{dx}\right)^2, \\ u^0 = U_0, \quad w_1^0 = w_2^0 = W_0, \end{gathered} \tag{19}$$

where

$$w = \eta_1 w_1 + \eta_2 w_2,$$

$$\eta_1 + \eta_2 = 1, \quad \eta_1 > 0, \quad \eta_2 > 0,$$

with suitable boundary conditions.

The following statement takes place.

Theorem 3. If $-1 \le \delta \le 1$ and problem (18) has a sufficiently smooth solution, then the solution of the additive averaged semi-discrete scheme (19) converges to the solution of problem (18) as $\tau \to 0$, and the following estimate is true

$$\left\| U(t_j) - u^j \right\| + \left\| W(t_j) - w^j \right\| = O\left(\tau^{1/2}\right).$$

Here $\|\cdot\|$ is an usual norm of the space $L_2(0,1)$.

Let us also correspond to the problem (18) the following semi-discrete additive model:

$$\begin{gathered} u_{1t} = \frac{d}{dx}\left(w_1^\delta \frac{du_1}{dx}\right), \\ \eta_1 w_{1t} = \frac{1}{2} w_1^{\delta-1}\left(\frac{du_1}{dx}\right)^2, \\ u_{2t} = \frac{d}{dx}\left(w_2^\delta \frac{du_2}{dx}\right), \\ \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\left(\frac{dw_2}{dx}\right)^2, \\ u_1^0 = u_2^0 = U_0, \quad w_1^0 = w_2^0 = W_0, \end{gathered} \tag{20}$$

with suitable boundary conditions, where

$$u = \eta_1 u_1 + \eta_2 u_2, \quad w = \eta_1 w_1 + \eta_2 w_2.$$

One must note that the analogous result as Theorem 3 is valid for scheme (20).

Note also that result of Theorem 3 and result analogical to it with Dirichlet boundary conditions for function V in problem (17) is obtained in the work [1].

Now let us consider the fully discrete finite difference schemes for the problem (18).

First of all let us consider difference scheme analogical to scheme studied in [1]:

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} = \left(w_{i+3/2}^{j+1}\right)^\delta \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h^2} \\ -\left(w_{i+1/2}^{j+1}\right)^\delta \frac{u_i^{j+1} - u_{i-1}^{j+1}}{h^2}, \\ \frac{w_{i+1/2}^{j+1} - w_{i+1/2}^j}{\tau} \\ = \frac{1}{2}\left(w_{i+1/2}^{j+1}\right)^{\delta-1}\left(\frac{u_i^{j+1} - u_{i-1}^{j+1}}{h}\right)^2 \\ + \frac{w_{i+3/2}^{j+1} - 2w_{i+1/2}^{j+1} + w_{i-1/2}^{j+1}}{h^2} \\ + \frac{1}{2w_{i+1/2}^{j+1}}\left[\left(\frac{w_{i+1/2}^{j+1} - w_{i-1/2}^{j+1}}{h}\right)^2 \right. \\ \left. + \left(\frac{w_{i+3/2}^{j+1} - w_{i+1/2}^{j+1}}{h}\right)^2\right] \end{gathered} \tag{21}$$

with suitable initial and boundary conditions. Here $w^j_{i\pm 1/2} = w(x_{i\pm 1/2}, t_j)$, $w^j_{i+3/2} = w(x_{i+3/2}, t_j)$.

Following statement shows how well the scheme (21) approximates the initial-boundary value problem (18).

Theorem 4. If $-1 \le \delta \le 1$ and problem (18) has a sufficiently smooth solution, then the solution of the finite difference scheme (21) converges to the solution of problem (18) as $\tau \to 0$, $h \to 0$, and the following estimate is true

$$\left\| U^j - u^j \right\| + \left\| W^j - w^j \right\| = O\left(\tau + h^2\right).$$

In [20] for problem (17) following difference scheme is considered:

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} = \left(v_{i+1/2}^j\right)^\alpha \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h^2} \\ -\left(v_{i-1/2}^j\right)^\alpha \frac{u_i^{j+1} - u_{i-1}^{j+1}}{h^2}, \\ \frac{v_i^{j+1} - v_i^j}{\tau} = \left(v_i^j\right)^\alpha \left(\frac{u_{i+1}^j - u_{i-1}^j}{2h}\right)^2 \\ + \frac{v_{i+1}^{j+1} - 2v_i^{j+1} + v_{i-1}^{j+1}}{h^2}, \\ u_1^j = u_M^j = 0, \\ \frac{v_{M+1}^j - v_{M-1}^j}{2h} = \frac{v_1^j - v_{-1}^j}{2h} = 0, \\ u_i^0 = U_0(x_i), \quad v_i^0 = V_0(x_i), \end{gathered} \tag{22}$$

where $v^j_{i\pm 1/2} = v(x_{i\pm 1/2}, t_j)$, v_{-1} and v_{M+1} are the values at the ghost points.

IV. Conclusion

Various numerical experiments using above mentioned discrete models (16), (19) - (22) are carried out. These experiments agree with theoretical investigations

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