Correctness of the Initial-Boundary Value Problem and Discrete Analogs for One Nonlinear Parabolic Integro-Differential Equation

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***Abstract*—**First type initial-boundary value problem for one nonlinear parabolic integro-differential equation is considered. This model is based on Maxwell system describing the process of the penetration of a magnetic field into a substance. Semi-discrete and finite difference schemes are studied. Attention is paid to the investigation not only power type that already were studied but more wide cases of nonlinearities. Existence, uniqueness and long-time behavior of solutions are fixed too.

***Keywords*—**Nonlinear parabolic integro-differential equation, existence and uniqueness of solutions, long-time behavior, semi-discrete and finite difference schemes.

I. Introduction

IN mathematical modeling of many processes nonlinear integro-differential models are received very often (see, for example, [6], [14], [15], [19], [20] and references therein).

One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. In the quasistationary approximation the corresponding system of Maxwell equations has the form [16]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rotH), \tag{1}$$

$$\frac{\partial \theta}{\partial t} = \nu_m (rotH)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the electro-conductivity of the substance. System (1) describes the process of diffusion of the magnetic field and equation (2) - change of the temperature at the expense of Joule heating. If ν_m depends on temperature θ, i.e., $\nu_m = \nu_m(\theta)$, then the system (1), (2) can be rewritten in the following form [5]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2 d\tau \right) rotH \right], \tag{3}$$

where function $a = a(S)$ is defined for $S \in [0,\infty)$.

Note that partial integro-differential models of (3) type are complex and still yields to the investigation only for special cases (see, for example, [2]-[5], [14], [17], [19], [21] and references therein).

Study of the models of type (3) have begun in the work [5]. In this work, in particular, are proved the theorems of existence of solution of the initial-boundary value problem with first kind boundary conditions for scalar and one-dimensional space case while $a(S) = 1 + S$ and uniqueness for more general cases. One-dimensional scalar variant for the case $a(S) = (1+S)^p$, $0 < p \le 1$ is studied in [4]. Investigations for multi-dimensional space cases are discussed in the following works [2], [3], [9], [17].

Asymptotic behavior as $t \to \infty$ of solutions of initial-boundary value problems for (1), (2) and (3) type models are studied in the works [1], [8], [9], [12], [14] and in a number of other works as well. In these works main attentions, are paid to one-dimensional analogs.

One must note that for the cylindrical conductors to the study of modeling of physical process of penetrating of the electromagnetic field some amounts of works were also devoted. In this case above-mentioned type models, written in cylindrical coordinates, are studied in many works (see, for example, [14] and references therein). To the investigation of periodic problem for one-dimensional (3) type model in cylindrical coordinates the work [21] is also devoted.

Interest to above-mentioned differential and integro-differential models is more and more arising and initial-boundary value problems with different kinds of boundary and

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initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance. Finite element analogues and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (3) type and similar one-dimensional integro-differential equations are studied in [7], [10], [11], [13], [14], [18], [21] and in the other works as well.

Our aim is to study of semi-discrete and finite difference schemes for numerical solution of initial-boundary value problem for the one-dimensional (3) type equation. Attention is paid to the investigation more wide cases of nonlinearity than already were studied. The work is organized as follows. Existence, uniqueness and long-time behavior of solution are fixed in Section 2. Semi-discrete scheme is constructed and studied in Section 3. Convergence statement of finite difference scheme is done in Section 4. Some conclusions are given in Section 5.

II. Existence, Uniqueness and Long-time Behavior of Solution

Let us consider the cylinder $[0,1]\times[0,\infty)$. If the magnetic field has the form $H=(0,0,U)$, $U=U(x,t)$, then from (3) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right]=0, \tag{4}$$

where
$$\frac{\partial \theta}{\partial t}=\nu_m(rotH)^2, \tag{5}$$

Let us consider the following boundary and initial conditions:

$$U(0,t)=U(1,t)=0, \tag{6}$$

$$U(x,0)=U_0(x), \tag{7}$$

where U_0 is a given function.

The following statement of existence and uniqueness of the solution takes place.

Theorem 1 If $a(S)\ge a_0=Const>0$, $a'(S)\ge 0$, $a''(S)\le 0$ and $U_0\in H^2(0,1)\cap \overset{o}{H}{}^1(0,1)$, then where exists unique solution U of the problem (4) - (7) such that: $U\in L_2\left(0,\infty;H^2(0,1)\right)$, $U_{xt}\in L_2\left(0,\infty;L_2(0,1)\right)$.

We use usual $L_2(0,1)$ and Sobolev spaces $H^k(0,1)$, $\overset{o}{H}{}^1(0,1)$ and the corresponding norms.

The existence part of the Theorem 1 is proved using Galerkin modified method and compactness arguments as in [20], [23] for nonlinear parabolic equations.

The study long-time behavior of solution of the problem (7) - (10) is also very important.

Theorem 2 If $a(S)\ge a_0=Const>0$, $a'(S)\ge 0$, $a''(S)\le 0$ and $U_0\in H^3(0,1)\cap \overset{o}{H}{}^1(0,1)$, then for the solution of problem (4) - (7) the following estimate holds as $t\to\infty$

$$\left\|\frac{\partial U(x,t)}{\partial x}\right\|_{L_2(0,1)}\le C\exp\left(-\frac{a_0 t}{2}\right),$$

uniformly in x on $[0,1]$.

Symbol C denotes positive constant independent of t.

Results of Theorem 2 show that asymptotic behavior of the solution has an exponential character.

III. Semi-discrete Scheme

Let us consider the following problem:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[a\left(\int_0^t\left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right]=0, \tag{8}$$

$$U(0,t)=U(1,t)=0, \tag{9}$$

$$U(x,0)=U_0(x), \tag{10}$$

where $a(S)\ge a_0=Const>0$, $a'(S)\ge 0$, $a''(S)\le 0$ and U_0 is a given function.

On [0,1] let us introduce a net with mesh points denoted by $x_i=ih$, $i=0,1,\ldots,M$, with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. In this section the semi-discrete approximation at (x_i,t) is designed by $u_i=u_i(t)$. The exact solution to the problem at (x_i,t) is denoted by $U_i=U_i(t)$. At points $i=1,2,\ldots,M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences. We will use the following known notations [22]:

$$r_{x,i}(t)=\frac{r_{i+1}(t)-r_i(t)}{h},\quad r_{\bar{x},i}(t)=\frac{r_i(t)-r_{i-1}(t)}{h}.$$

Let us correspond to problem (8) - (10) the following semi-discrete scheme:

$$\frac{du_i}{dt}-\left\{a\left(\int_0^t\left(u_{\bar{x},i}\right)^2 d\tau\right)u_{\bar{x},i}\right\}_x=0 \tag{11}$$

$$i=1,2,\ldots,M-1,$$

$$u_0(t)=u_M(t)=0, \tag{12}$$

$$u_i(0)=U_{0,i},\quad i=0,1,\ldots,M. \tag{13}$$

So, we obtained Cauchy problem (11) - (13) for nonlinear system of ordinary integro-differential equations.

Introduce usual inner products and norms:

$$(r,g)_h = h\sum_{i=1}^{M-1} r_i g_i, \quad (r,g]_h = h\sum_{i=1}^{M} r_i g_i,$$
$$\|r\|_h = (r,r)_h^{1/2}, \quad \|r]|_h = (r,r]_h^{1/2}.$$

Multiplying equations (11) scalarly by, $u(t)=(u_1(t),u_2(t),\dots,u_{M-1}(t))$ after simple transformations we get

$$\frac{d}{dt}\|u(t)\|_h^2 + h\sum_{i=1}^{M} a\left(\int_0^t (u_{\bar{x},i})^2 d\tau\right)(u_{\bar{x},i})^2 = 0.$$

From this we obtain the inequality

$$\|u(t)\|_h^2 + \int_0^t \|u_{\bar{x}}]|_h^2 d\tau \le C, \tag{14}$$

where, here and below in this section, C denotes a positive constant which does not depend on h.

The a priori estimate (14) guarantee the global solvability of the problem (8) - (10).

The principal aim of the present section is the proof of the following statement.

Theorem 3 *If problem (8) - (10) has a sufficiently smooth solution* $U=U(x,t)$ *, then when* $a(S)\ge a_0 = Const > 0$, $a'(S)\ge 0$, $a''(S)\le 0$ *the solution* $u=u(t)=(u_1(t),u_2(t),\dots,u_{M-1}(t))$ *of problem (8) - (10) tends to* $U=U(t)=(U_1(t),U_2(t),\dots,U_{M-1}(t))$ *as* $h\to 0$ *and the following estimate is true*

$$\|u(t)-U(t)\|_h \le Ch. \tag{15}$$

Proof. For $U=U(x,t)$ we have:

$$\frac{dU_i}{dt} - \left\{ a\left(\int_0^t (U_{\bar{x},i})^2 d\tau\right) U_{\bar{x},i}\right\}_x = \psi_i(t), \tag{16}$$

$$i = 1,2,\dots,M-1,$$

$$U_0(t) = U_M(t) = 0, \tag{17}$$

$$U_i(0) = U_{0,i}, \ i = 0,1,\dots,M, \tag{18}$$

where

$$\psi_i(t) = O(h).$$

Let $z_i(t) = u_i(t) - U_i(t)$. From (8) - (10) and (16) - (18) we have:

$$\frac{dz_i}{dt} - \left\{ a\left(\int_0^t (u_{\bar{x},i})^2 d\tau\right) u_{\bar{x},i} - a\left(\int_0^t (U_{\bar{x},i})^2 d\tau\right) U_{\bar{x},i}\right\}_x = -\psi_i(t), \tag{19}$$

$$z_0(t) = z_M(t) = 0,$$

$$z_i(0) = 0.$$

Multiplying equation (19) scalarly by $z(t)=(z_1(t),z_2(t),\dots,z_{M-1}(t))$, using the discrete analogue of the formula of integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + h\sum_{i=1}^{M}\left\{ a\left(\int_0^t (u_{\bar{x},i})^2 d\tau\right)u_{\bar{x},i} - a\left(\int_0^t (U_{\bar{x},i})^2 d\tau\right)U_{\bar{x},i}\right\}(u_{\bar{x},i} - U_{\bar{x},i}) = -h\sum_{i=1}^{M-1}\psi_i z_i. \tag{20}$$

Note that,

$$\left[a\left(\int_0^t (u_{\bar{x},i})^2 d\tau\right)u_{\bar{x},i} - a\left(\int_0^t (U_{\bar{x},i})^2 d\tau\right)U_{\bar{x},i}\right](u_{\bar{x},i} - U_{\bar{x},i})$$
$$= \int_0^1 \frac{d}{d\xi} a\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})]^2 d\tau\right) \times [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})] d\xi (u_{\bar{x},i} - U_{\bar{x},i})$$
$$= \int_0^1 a'\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})]^2 d\tau\right) \times \int_0^t 2[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})](u_{\bar{x},i} - U_{\bar{x},i}) d\tau \times [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})] d\xi (u_{\bar{x},i} - U_{\bar{x},i})$$
$$+ \int_0^1 a\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})]^2 d\tau\right) \times (u_{\bar{x},i} - U_{\bar{x},i}) d\xi (u_{\bar{x},i} - U_{\bar{x},i})$$
$$= \int_0^1 a'\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})]^2 d\tau\right) \times \frac{d}{dt}\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})](u_{\bar{x},i} - U_{\bar{x},i}) d\tau\right)^2 d\xi$$
$$+ \int_0^1 a\left(\int_0^t [U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})]^2 d\tau\right) d\xi (u_{\bar{x},i} - U_{\bar{x},i})^2.$$

Using this relation, after integrating on $(0,t)$ and applying formula of integrating by parts we get from (20)

$$\|z\|^2 + 2h\sum_{i=1}^{M}\int_0^t\int_0^1 a\left(\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau'\right)$$
$$\times\left(u_{\bar{x},i}-U_{\bar{x},i}\right)^2 d\xi d\tau$$
$$+h\sum_{i=1}^{M}\int_0^1 a'\left(\int_0^t\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau\right)$$
$$\times\left(\int_0^t\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau\right)^2 d\xi$$
$$-h\sum_{i=1}^{M}\int_0^1\int_0^t a''\left(\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau'\right)$$
$$\times\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2$$
$$\times\left(\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau'\right)^2 d\xi d\tau$$
$$\le -2h\sum_{i=1}^{M-1}\psi_i z_i.$$

Taking into account restrictions on function $a = a(S)$, we have from last equality

$$\|z(t)\|^2 \le \int_0^t \|z(\tau)\|^2 d\tau + \int_0^t \|\psi_i\|^2 d\tau. \qquad (21)$$

IV. Finite Difference Scheme

Now, consider problem (8) - (10) in the cylinder $[0,1]\times[0,T]$, where T is given positive constant.

On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih, j\tau)$, where $i=0,1,...,M;\ j=0,1,...,N$ with $h=1/M$, $\tau = T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by u_i^j and the exact solution to the problem (8) - (10) by U_i^j. We will use the following known notations [22]:

$$r_{t,i}^j=\frac{r_i^{j+1}-r_i^j}{\tau},\quad r_{\bar{t},i}^j=r_{t,i}^{j-1}=\frac{r_i^j-r_i^{j-1}}{\tau}.$$

For problem (8) - (10) let us consider the finite difference scheme:

$$u_{t,i}^j-\left\{a\left(\tau\sum_{k=1}^{j+1}\left(u_{\bar{x},i}^k\right)^2\right)u_{\bar{x},i}^{j+1}\right\}_x=0,$$
$$i=1,2,...,M-1;\ \ j=0,1,...,N-1, \qquad (22)$$
$$u_0^j=u_M^j=0,\ \ j=0,1,...,N,$$
$$u_i^0=U_{0,i},\ \ i=0,1,...,M.$$

Theorem 4 *If problem (8) - (10) has sufficiently smooth solution* $U=U(x,t)$, $a(S)\ge a_0=Const>0$, $a'(S)\ge 0$, $a''(S)\le 0$, *then solution* $u^j=(u_1^j,u_2^j,\ldots,u_M^j)$, $j=1,2,\ldots,N$ *of the difference scheme (22) tends to the solution of continuous problem* $U^j=(U_1^j,U_2^j,\ldots,U_M^j)$, $j=1,2,\ldots,N$ *as* $\tau\to 0$, $h\to 0$ *and the following estimate is true*

$$\|u^j-U^j\|\le C(\tau+h).$$

Here C is a positive constant independent of h and τ.

For solving the difference schemes (22) Newton iterative process is used.

V. Conclusion

Nonlinear parabolic integro-differential equation associated with the penetration of a magnetic field in a substance is considered. Existence, uniqueness and long-time behavior of solution of initial-boundary value problem are fixed. The semi-discrete and finite difference scheme are investigated for this model as well.

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