Prime geodesic theorem for compact even-dimensional locally symmetric spaces of real rank one

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***Abstract*—We improve the error term in DeGeorge's prime geodesic theorem for compact, even-dimensional, locally symmetric Riemannian manifolds of strictly negative sectional curvature.**

***Index Terms*—locally symmetric spaces, prime geodesic theorem, Selberg zeta and Ruelle zeta function.**

I. Introduction

LET $Y = \Gamma \backslash G/K = \Gamma \backslash X$ be a compact, $n-$dimensional (n even), locally symmetric Riemannian manifold with strictly negative sectional curvature, where G is a connected semi-simple Lie group of real rank one, K is a maximal compact subgroup of G and Γ is a discrete, co-compact, torsion-free subgroup of G.

We assume that the Riemannian metric over Y induced from the Killing form is normalized so that the sectional curvature of Y varies between -4 and -1.

As well known, a prime geodesic C_γ over Y corresponds to a conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$.

Let $\pi_\Gamma(x)$ be the number of prime geodesics C_γ of length $l(\gamma)$ whose norm $N(\gamma) = e^{l(\gamma)}$ is not larger than x (see Section 3).

DeGeorge [7] derived the following form of the prime geodesic theorem with an error term

$$\pi_\Gamma(x) = \int_1^{\log x} \frac{e^{\alpha u}}{u} du + O(x^\eta) \tag{1}$$

as $x \to +\infty$, where η is a constant such that $\left(1 - \frac{1}{2n}\right)\alpha \leq \eta < \alpha$ and $\alpha = n + q - 1$, with $q = 0,\ 1,\ 3,\ 7$ depending on whether X is a real, a complex or a quaternionic hyperbolic space or the hyperbolic Cayley plane, respectively (see, sections I and V of [7]).

Integrating (1) by parts, one easily deduces a weaker form of the prime geodesic theorem

$$\pi_\Gamma(x) \sim \frac{x^\alpha}{\alpha \log x}, \tag{2}$$

as $x \to +\infty$, where $f(x) \sim g(x)$ means $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$.

Note that (2) was also proved by Gangolli [12] and by Gangolli-Warner [14] when Y has a finite volume.

By adapting Hejhal's techniques [16], [17], Park [19] refined the corresponding result of Gangolli-Warner [14] for real hyperbolic manifolds with cusps. Inspired by Randol's approach [20], we [1] further improved Park's result [19] to the form

$$\begin{aligned}\pi_\Gamma(x) = &\sum_{\frac{3}{2}d_0 < s_j(k) \leq 2d_0} (-1)^k \operatorname{li}\left(x^{s_j(k)}\right) \\ &+ O\left(x^{\frac{3}{2}d_0} (\log x)^{-1}\right)\end{aligned} \tag{3}$$

as $x \to +\infty$,
where $d_0 = \frac{1}{2}(n-1)$, $(s_j(k) - k)\ (2d_0 - k - s_j(k))$ is a small eigenvalue in $\left[0, \frac{3}{4}d_0^2\right]$ of Δ_k on $\pi_{\sigma_k, \lambda_j(k)}$ with $s_j(k)$ $= d_0 + \mathrm{i}\,\lambda_j(k)$ or $s_j(k) = d_0 - \mathrm{i}\,\lambda_j(k)$ in $\left(\frac{3}{2}d_0, 2d_0\right]$, Δ_k is the Laplacian acting on the space of $k-$forms over Y and $\pi_{\sigma_k, \lambda_j(k)}$ is the principal series representation.

Note that the error term in (3) is in accordance with the best known estimate in the case of compact Riemann surfaces (see, e.g., [20], [4]).

The main purpose of this paper is to improve the error term in the prime geodesic theorem (1) of DeGeorge [7] for compact, even-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature so to correspond to (3).

We shall use the zeta functions of Selberg and Ruelle described by Bunke and Olbrich [6]. In particular, we utilize the fact that for even n these functions are meromorphic functions of order not larger than n (see, [2], [3]).

II. Preliminaries

In the sequel, we follow the notation of [6].

Assume that G is a linear group.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and M the centralizer of $\mathfrak{a}$ in K with the Lie algebra $\mathfrak{m}$.

Let $\Phi(\mathfrak{g}, \mathfrak{a})$ be the root system and $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$ a system of positive roots. Let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{n}_\alpha$$

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be the sum of the root spaces. Then, the Iwasawa decomposition $G = KAN$ corresponds to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a})} \dim(\mathfrak{n}_\alpha)\, \alpha.$$

Let $\mathfrak{a}^+$ be the half line in $\mathfrak{a}$ on which the positive roots take positive values. Put $A^+ = \exp(\mathfrak{a}^+) \subset A$.

Let $\sigma \in \hat{M}$.

By [6, p. 27], there is an element $\gamma \in R(K)$ such that $i^*(\gamma) = \sigma$ (see also [6, p. 23, Prop. 1.2]). Here, $i^* : R(K) \to R(M)$ is the restriction map induced by the embedding $i : M \hookrightarrow K$, where $R(K)$ and $R(M)$ are the representation rings over $\mathbb{Z}$ of K and M, respectively.

In [6, p. 28], the authors introduced the operators $A_d(\gamma,\sigma)$ and $A_{Y,\chi}(\gamma,\sigma)$. These operators correspond to spaces X_d and Y, respectively. Here, χ is a finite-dimensional unitary representation of Γ and X_d denotes a compact dual space of the symmetric space X.

Let $E_A(.)$ be the family of spectral projections of a normal operator A. Put

$$m_\chi(s,\gamma,\sigma) = \operatorname{Tr} E_{A_{Y,\chi}(\gamma,\sigma)}(\{s\}),$$

and

$$m_d(s,\gamma,\sigma) = \operatorname{Tr} E_{A_d(\gamma,\sigma)}(\{s\}),$$

for $s \in \mathbb{C}$.

Definition 1. *[6, p. 49, Def. 1.17] Let $\sigma \in \hat{M}$. Then, $\gamma \in R(K)$ is called $\sigma-$admissible if $i^*(\gamma) = \sigma$ and $m_d(s,\gamma,\sigma) = P_\sigma(s)$ for all $0 \le s \in L(\sigma)$.*

Here, $P_\sigma(s)$ resp. $L(\sigma)$ denote the polynomial resp. the lattice given by [6, Definition 1.13, p. 47; see also p. 40]. In particular, $L(\sigma) = T(\epsilon_\sigma + \mathbb{Z})$, where T and $\epsilon_\sigma \in \left\{0, \frac{1}{2}\right\}$ are given by the same definition.

By [6, p. 49, Lemma 1.18], there exists a $\sigma-$admissible $\gamma \in R(K)$ for every $\sigma \in \hat{M}$.

III. Zeta functions

Since $\Gamma \subset G$ is co-compact and torsion-free, there are only two types of conjugacy classes: the class of the identity $e \in \Gamma$ and classes of hyperbolic elements.

Let Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ denote the set of the $\Gamma-$conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

It is well known that every hyperbolic element $g \in G$ is conjugated to some element $a_g m_g \in A^+M$ (see, e.g., [12]–[14]). Following [6, p. 59], we put $l(g) = |\log(a_g)|$.

For $s \in \mathbb{C}$, $\operatorname{Re}(s) > \rho$, the Selberg zeta function is defined by the infinite product (see, [6, p. 97])

$$\begin{aligned} &Z_{S,\chi}(s,\sigma) \\ &= \prod_{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \prod_{k=0}^{+\infty} \det\left(1 - \left(\sigma(m_{\gamma_0}) \otimes \chi(\gamma_0) \otimes S^k\left(\operatorname{Ad}(m_{\gamma_0} a_{\gamma_0})_{\bar{\mathfrak{n}}}\right)\right) e^{-(s+\rho)l(\gamma_0)}\right), \end{aligned}$$

where σ and χ are finite-dimensional unitary representations of M and Γ, respectively, S^k is the $k-$th symmetric power of an endomorphism, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ and θ is the Cartan involution of $\mathfrak{g}$.

For $s \in \mathbb{C}$, $\operatorname{Re}(s) > 2\rho$, the Ruelle zeta function is defined by the infinite product (see, [6, p. 96])

$$Z_{R,\chi}(s,\sigma) = \prod_{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \det\left(1 - \left(\sigma(m_{\gamma_0}) \otimes \chi(\gamma_0)\right) e^{-sl(\gamma_0)}\right)^{(-1)^{n-1}}.$$

As known, the Ruelle zeta function can be expressed in terms of Selberg zeta functions (see, e.g., [9]–[11]). By [6, pp. 99–100], there exist sets $I_p = \left\{(\tau,\lambda) \mid \tau \in \hat{M}, \lambda \in \mathbb{R}\right\}$ such that

$$\begin{aligned} &Z_{R,\chi}(s,\sigma) \\ &= \prod_{p=0}^{n-1} \prod_{(\tau,\lambda) \in I_p} Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)^{(-1)^p}. \end{aligned} \tag{4}$$

Let Λ resp. Υ denote the set of all elements λ resp τ that appear in (4).

Note that [6, p. 113, Theorem 3.15] gives precise description of the locations and the orders of the singularities of $Z_{S,\chi}(s,\sigma)$.

We have proved the following theorem.

Theorem A. *[2, p. 528, Th. 4.1] If γ is $\sigma-$admissible, then there exist entire functions $Z_1(s)$, $Z_2(s)$ of order at most n such that*

$$Z_{S,\chi}(s,\sigma) = \frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $Z_{S,\chi}(s,\sigma)$ and the zeros of $Z_2(s)$ correspond to the poles of $Z_{S,\chi}(s,\sigma)$. The orders of the zeros of $Z_1(s)$ resp. $Z_2(s)$ equal the orders of the corresponding zeros resp. poles of $Z_{S,\chi}(s,\sigma)$.

IV. Auxiliary results

Lemma 2. *If γ is $\sigma-$admissible, then*

$$P_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1},$$

where

$$p_{n-2k-1} = \frac{2T}{\left(\frac{n}{2} - k - 1\right)!} c_{-\left(\frac{n}{2}-k\right)},\ k = 0, 1, ..., \frac{n}{2} - 1,$$

$$c_{-\frac{n}{2}} = \frac{\left(\frac{n}{2} - 1\right)!}{2T}$$

and the numbers c_k are defined by the asymptotic expression

$$\operatorname{Tr} e^{-tA_d(\gamma,\sigma)^2} \overset{t\to 0}{\sim} \sum_{k=-\frac{n}{2}}^{\infty} c_k t^k.$$

Proof. By [6, pp. 47–48], $P_\sigma(0) = 0$, $P_\sigma(-w) = -P_\sigma(w)$ and $P_\sigma(w) = w \cdot Q_\sigma(w)$, where Q_σ is an even polynomial. Hence, P_σ is an odd polynomial. Moreover, P_σ is a monic polynomial of degree $n-1$ (see, e.g., [5, pp. 17–19], [22, pp. 240–243]).

Put

$$P_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1},\ p_{n-1} = 1.$$

By [6, p. 118], $Q_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} q_{n-2k-2} w^{n-2k-2}$, where $q_{2i} = \frac{2T}{i!} c_{-(i+1)}$, $i = 0, 1,..., \frac{n}{2}-1$. In other words,

$$p_{n-2k-1} = q_{n-2k-2} = \frac{2T}{\left(\frac{n}{2}-k-1\right)!} c_{-\left(\frac{n}{2}-k\right)},$$

$k = 0,1,...,\frac{n}{2}-1$. This completes the proof. $\square$

Lemma 3. *Let H be a half-plane of the form $\operatorname{Re}(s) < -(2\rho+\varepsilon)$, $\varepsilon > 0$, minus the union of a set of congruent disks about the points $-s$, $s \in T(\mathbb{N} - \epsilon_{\tau\otimes\sigma}) + \rho - \lambda$, $\lambda \in \Lambda$, $\tau \in \Upsilon$. Then there exists a constant C_R such that*

$$\left| \frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)} \right| \leq C_R |s|^{n-1}$$

for $s \in H$.

Proof. The identity (4) implies

$$\begin{aligned} &\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)} \\ &= \sum_{p=0}^{n-1} (-1)^p \sum_{(\tau,\lambda)\in I_p} \frac{Z'_{S,\chi}(s+\rho-\lambda, \tau\otimes\sigma)}{Z_{S,\chi}(s+\rho-\lambda, \tau\otimes\sigma)}. \end{aligned} \tag{5}$$

Recall [6, p. 113, Theorem 3.15]. Now, it is enough to prove that if K is a half-plane of the form $\operatorname{Re}(s) < -(\rho+\varepsilon)$, $\varepsilon > 0$, minus the union of a set of congruent disks about the points $-s$, $s \in T(\mathbb{N} - \epsilon_{\tau\otimes\sigma})$, $\tau \in \Upsilon$, then there exists a constant C_S such that

$$\left| \frac{Z'_{S,\chi}(s,\tau\otimes\sigma)}{Z_{S,\chi}(s,\tau\otimes\sigma)} \right| \leq C_S |s|^{n-1}$$

for $s \in K$ and all $\tau \in \Upsilon$.

The proof is independent of the choice of τ. We simplify our notation by omitting the latter.

By [6, p. 118, Th. 3.19], $Z_{S,\chi}(s,\sigma)$ has the representation

$$\begin{aligned} Z_{S,\chi}(s,\sigma) = {} & \det\left(A_{Y,\chi}(\gamma,\sigma)^2 + s^2\right) \det\left(A_d(\gamma,\sigma)+s\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}} \cdot \\ & \exp\left(\frac{\dim(\chi)\chi(Y)}{\chi(X_d)} \sum_{m=1}^{\frac{n}{2}} c_{-m} \frac{s^{2m}}{m!} \left(\sum_{r=1}^{m-1} \frac{1}{r} - 2\sum_{r=1}^{2m-1} \frac{1}{r}\right)\right). \end{aligned}$$

Hence, (see, [6, pp. 120–122])

$$\begin{aligned} & Z_{S,\chi}(-s,\sigma) \\ & = Z_{S,\chi}(s,\sigma) \cdot \left(\frac{\det\left(A_d(\gamma,\sigma)-s\right)}{\det\left(A_d(\gamma,\sigma)+s\right)}\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}} \\ & = Z_{S,\chi}(s,\sigma) \cdot \left(\frac{D^+(s)}{D^-(s)}\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}} \\ & = Z_{S,\chi}(s,\sigma) \cdot \\ & \quad \exp\left(-\frac{\pi}{T}\int_0^s P_\sigma(w) \left\{\begin{array}{ll} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array}\right\} dw\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}} \\ & = Z_{S,\chi}(s,\sigma) \cdot e^{K\int_0^s P_\sigma(w) \left\{\begin{array}{ll} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array}\right\} dw}. \end{aligned} \tag{6}$$

Consider the case $\epsilon_\sigma = \frac{1}{2}$. The case $\epsilon_\sigma = 0$ is discussed similarly.

The identity (6) implies

$$-\frac{Z'_{S,\chi}(-s,\sigma)}{Z_{S,\chi}(-s,\sigma)} = \frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)} + K P_\sigma(s) \tan\left(\frac{\pi s}{T}\right).$$

Since $\frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}$ is bounded on every half-plane $\operatorname{Re}(s) > \rho+\varepsilon$, $\varepsilon > 0$, we conclude that $\frac{Z'_{S,\chi}(-s,\sigma)}{Z_{S,\chi}(-s,\sigma)}$ is bounded on K. Moreover, $\tan\left(\frac{\pi s}{T}\right)$ is bounded on the complement of the union of congruent disks about the points $T\left(k+\frac{1}{2}\right) = T(k+\epsilon_\sigma)$, $k \in \mathbb{Z}$. This completes the proof. $\square$

Lemma 4. *Let $c, d \in \mathbb{R}$, $c < d$. If γ is $\sigma-$admissible then there exists a sequence $\{y_j\}$, $y_j \to +\infty$ as $j \to +\infty$, such that*

$$\frac{Z'_{R,\chi}(x+\mathrm{i}\,y_j,\sigma)}{Z_{R,\chi}(x+\mathrm{i}\,y_j,\sigma)} = O\left(y_j^{2n}\right)$$

for $x \in (c,d)$.

Proof. Consider the identity (5).

It is enough to prove that there exists a sequence $\{y_j\}$, $y_j \to +\infty$ as $j \to +\infty$, such that

$$\frac{Z'_{S,\chi}(x+\mathrm{i}\,y_j,\tau\otimes\sigma)}{Z_{S,\chi}(x+\mathrm{i}\,y_j,\tau\otimes\sigma)} = O\left(y_j^{2n}\right)$$

for $x \in (a,b)$ and all $\tau \in \Upsilon$, where $a = c-\rho$, $b = d+\rho$.

We consider the interval I_1 given by $\mathrm{i}\,t$, $t_0 - 1 < t \leq t_0 + 1$, where $t_0 > 2\rho$ is fixed.

It suffices to prove that there exists $y \in (t_0 - 1, t_0 + 1]$ such that

$$\frac{Z'_{S,\chi}(x + \mathrm{i}\, y, \tau \otimes \sigma)}{Z_{S,\chi}(x + \mathrm{i}\, y, \tau \otimes \sigma)} = O\left(y^{2n}\right) \tag{7}$$

for $x \in (a, b)$ and all $\tau \in \Upsilon$.

Let S_R be the set of all singularities of all zeta functions $Z_{S,\chi}(s, \tau \otimes \sigma)$, $\tau \in \Upsilon$. Let $N_R(t)$ be the number of elements in S_R on the interval $\mathrm{i}\, x$, $0 < x \leq t$.

Let $N(t)$ be the number of singularities of $Z_{S,\chi}(s, \sigma)$ on the same interval. By [6, Th. 3.15], these singularities are given in terms of eigenvalues of $A_{Y,\chi}(\gamma_\sigma, \sigma)$ for some $\sigma-$admissible $\gamma_\sigma \in R(K)$. Hence, according to [8, p. 89, Th. 9.1.], $N(t) = D_1 t^n + O\left(t^{n-1}(\log t)^{-1}\right)$ for some explicitly known constant D_1. However, the $O-$term does not improve our result. For the sake of simplicity, we take $N(t) = O(t^n)$. Consequently, $N_R(t) = O(t^n)$.

It follows immediately that the number of singularities of $Z_{S,\chi}(s, \sigma)$ on I_1 is $O(t_0^n)$.

Similarly, the number of elements in S_R on I_1 is $O(t_0^n)$, i.e., it is at most $\lfloor C_1 t_0^n \rfloor$ for some constant C_1.

Denote by I_2 the interval $\mathrm{i}\, t$, $t_0 - \frac{3}{4} < t \leq t_0 + \frac{3}{4}$.

Since $I_2 \subset I_1$, the number of elements in S_R on I_2 is at most $\lfloor C_1 t_0^n \rfloor$.

Let us divide the interval I_2 into $1 + \lfloor C_1 t_0^n \rfloor$ equal intervals. By the Dirichlet principle, one of them does not contain any element from S_R. Let $\mathrm{i}\, y$ be the midpoint of such an interval. We shall prove that y satisfies (7) for $x \in (a, b)$ and all $\tau \in \Upsilon$. The proof does not depend on the choice of $\tau \in \Upsilon$. We simplify our notation by omitting it, i.e., we prove that

$$\frac{Z'_{S,\chi}(x + \mathrm{i}\, y, \sigma)}{Z_{S,\chi}(x + \mathrm{i}\, y, \sigma)} = O\left(y^{2n}\right)$$

for $x \in (a, b)$.

By Theorem A, $Z_1(s)$ and $Z_2(s)$ are entire functions of order at most n. Hence, there are canonical product expressions for $Z_1(s)$ and $Z_2(s)$ of the form (see, e.g., [9, p. 509])

$$Z_i(s) = s^{n_i} e^{g_i(s)} \prod_{\alpha \in R_i \setminus \{0\}} \left(1 - \tfrac{s}{\alpha}\right) \exp\left(\tfrac{s}{\alpha} + \tfrac{s^2}{2\alpha^2} + ... + \tfrac{s^n}{n\alpha^n}\right),$$

$i = 1, 2$, where R_i is the set of zeros of $Z_i(s)$, n_i is the order of the zero of $Z_i(s)$ at $s = 0$, $g_i(s)$ is a polynomial of degree at most n.

Therefore,

$$\frac{Z'_{S,\chi}(s, \sigma)}{Z_{S,\chi}(s, \sigma)} = \frac{1}{s}(n_1 - n_2) + g'_1(s) - g'_2(s) + \sum_{i=1,2} (-1)^{i-1} \sum_{\alpha \in R_i \setminus \{0\}} \left(\frac{s}{\alpha}\right)^n \frac{1}{s - \alpha}.$$

We have

$$|\mathrm{i}\, y - \alpha| \geq \frac{1}{2} \cdot \frac{\frac{3}{2}}{1 + \lfloor C_1 t_0^n \rfloor} \geq \frac{3}{4} \cdot \frac{1}{1 + C_1 t_0^n} > \frac{3}{4} \cdot \frac{1}{1 + C_1\left(y + \frac{3}{4}\right)^n} \geq \frac{C_2}{y^n}$$

for some constant C_2 and all $\alpha \in R_i$, $i = 1, 2$.

Now, for a small fixed $\varepsilon > 0$ and the choice $s_x = x + \mathrm{i}\, y$, $x \in (a, b)$, we have

$$\frac{Z'_{S,\chi}(s_x, \sigma)}{Z_{S,\chi}(s_x, \sigma)} = \frac{1}{s_x}(n_1 - n_2) + g'_1(s_x) - g'_2(s_x) + \sum_{k=1}^{8} \sum_{\beta \in A_k} \left(\frac{s_x}{\beta}\right)^n \frac{1}{s_x - \beta},$$

where β denotes a singularity of $Z_{S,\chi}(s, \sigma)$ and

$$\begin{aligned}
A_1 &= \{\beta \,|\, \beta \in -T(\mathbb{N} - \epsilon_\sigma), |\beta| > \rho + \varepsilon\}, \\
A_2 &= \{\beta \,|\, 0 < |\beta| \leq \rho + \varepsilon\}, \\
A_3 &= \{\beta \,|\, \beta = \mathrm{i}\, t, \rho + \varepsilon < t \leq t_0 - 1\}, \\
A_4 &= \{\beta \,|\, \beta \in I_1\}, \\
A_5 &= \{\beta \,|\, \beta = \mathrm{i}\, t, t > t_0 + 1\}, \\
A_6 &= \{\beta \,|\, \beta = -\mathrm{i}\, t, \rho + \varepsilon < t \leq t_0 - 1\}, \\
A_7 &= \{\beta \,|\, -\beta \in I_1\}, \\
A_8 &= \{\beta \,|\, \beta = -\mathrm{i}\, t, t > t_0 + 1\}.
\end{aligned}$$

Since $\sum_{\beta \in A_1} \frac{1}{|\beta|^n}$ converges and $|s_x - \beta| \geq y$ for $\beta \in A_1$, we get

$$\sum_{\beta \in A_1} \left(\tfrac{s_x}{\beta}\right)^n \tfrac{1}{s_x - \beta} = O\left(y^n \sum_{\beta \in A_1} \tfrac{1}{|\beta|^n} \tfrac{1}{|s_x - \beta|}\right) = O\left(y^{n-1}\right).$$

Furthermore, A_2 is a finite set. Hence,

$$\sum_{\beta \in A_2} \left(\tfrac{s_x}{\beta}\right)^n \tfrac{1}{s_x - \beta} = O\left(y^n \sum_{\beta \in A_2} \tfrac{1}{|\beta|^n} \tfrac{1}{|s_x - \beta|}\right) = O\left(y^{n-1}\right)$$

since $|s_x - \beta| \geq y - \rho - \varepsilon > C_3 y$ for some constant C_3 and all $\beta \in A_2$.

Similarly, $|s_x - \beta| \geq y - t_0 + 1 > \frac{1}{4}$ and $|\beta| > \rho + \varepsilon$ for $\beta \in A_3$. Hence,

$$\sum_{\beta \in A_3} \left(\tfrac{s_x}{\beta}\right)^n \tfrac{1}{s_x - \beta} = O\left(y^n \sum_{\beta \in A_3} \tfrac{1}{|\beta|^n} \tfrac{1}{|s_x - \beta|}\right) = O\left(y^n \sum_{\beta \in A_3} 1\right) = O\left(y^n (t_0 - 1)^n\right) = O\left(y^{2n}\right).$$

If $\beta \in A_4$, then $|s_x - \beta| \geq |\mathrm{i}\, y - \beta| > \frac{C_2}{y^n}$ and $|\beta| > y - \frac{7}{4} > C_4 y$ for some constant C_4. Therefore,

$$\sum_{\beta \in A_4} \left(\tfrac{s_x}{\beta}\right)^n \tfrac{1}{s_x - \beta} = O\left(y^n \sum_{\beta \in A_4} \tfrac{1}{|\beta|^n} \tfrac{1}{|s_x - \beta|}\right) = O\left(y^n \sum_{\beta \in A_4} 1\right) = O\left(y^n t_0^n\right) = O\left(y^n \left(y + \tfrac{3}{4}\right)^n\right) = O\left(y^{2n}\right).$$

Similarly, $|s_x - \beta| \geq t - y > C_5 t$ for some constant C_5 and $\beta = \mathrm{i}\, t \in A_5$. One has

$$\sum_{\beta\in A_5}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}$$
$$=O\left(y^n\sum_{\beta\in A_5}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)=O\left(y^n\int\limits_{t_0+1}^{+\infty}\frac{1}{t^{n+1}}dN(t)\right)$$
$$=O\left(y^n\int\limits_{t_0+1}^{+\infty}t^{-2}dt\right)=O\left(y^n(t_0+1)^{-1}\right)=O\left(y^{n-1}\right).$$

If $\beta\in A_6$, then $|s_x-\beta|>y+\rho+\varepsilon>y$ and $|\beta|>\rho+\varepsilon$. Hence,

$$\sum_{\beta\in A_6}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_6}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^{n-1}\sum_{\beta\in A_6}1\right)=O\left(y^{2n-1}\right).$$

Similarly, $|s_x-\beta|>y+t_0-1>y$ and $|\beta|>t_0-1>y-\frac{7}{4}>C_4y$ for $\beta\in A_7$. We have

$$\sum_{\beta\in A_7}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_7}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^{-1}\sum_{\beta\in A_7}1\right)=O\left(y^{n-1}\right).$$

If $\beta\in A_8$, then $|s_x-\beta|\geq y+t>t$ for $\beta=-\mathrm{i}\,t\in A_8$. Therefore,

$$\sum_{\beta\in A_8}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_8}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^n\int\limits_{t_0+1}^{+\infty}\frac{1}{t^{n+1}}dN(t)\right)=O\left(y^{n-1}\right).$$

Finally, $\frac{1}{s_x}(n_1-n_2)=O\left(y^{-1}\right)$ and $g_1'(s_x)-g_2'(s_x)=O\left(y^{n-1}\right)$.

We obtain

$$\frac{Z_{S,\chi}'(s_x,\sigma)}{Z_{S,\chi}(s_x,\sigma)}=O\left(y^{2n}\right).$$

This completes the proof. □

V. Prime geodesic theorem

Theorem 5. *Let Y be a compact, $n-$dimensional (n even), locally symmetric Riemannian manifold with strictly negative sectional curvature. Then,*

$$\pi_\Gamma(x)=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\mathrm{li}\left(x^{s^{p,\tau,\lambda}}\right)$$
$$+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)$$

as $x\to+\infty$, where $s^{p,\tau,\lambda}$ is a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$.

Proof. We fix a $\chi\in\hat{\Gamma}$.

As already mentioned, there exists a $\sigma-$admissible γ_σ for every $\sigma\in\hat{M}$. Fix $\sigma\in\hat{M}$ and choose some $\sigma-$admissible γ_σ. We simplify our notation by omitting χ and σ in the sequel.

For $g\in\Gamma$, let $n_\Gamma(g)=\#(\Gamma_g/\langle g\rangle)$, where Γ_g is the centralizer of g in Γ and $\langle g\rangle$ is the group generated by g.

If $\gamma\in\Gamma_{\mathrm{h}}$ then $\gamma=\gamma_0^{n_\Gamma(\gamma)}$ for some $\gamma_0\in\mathrm{P}\Gamma_{\mathrm{h}}$.

For $\gamma\in\Gamma_{\mathrm{h}}$ we introduce $\Lambda_0(\gamma)=\Lambda_0\left(\gamma_0^{n_\Gamma(\gamma)}\right)=\log N(\gamma_0)$.

By [6, pp. 96–97, (3.4)],

$$\frac{Z_R'(s)}{Z_R(s)}=-\sum_{\gamma\in\Gamma_{\mathrm{h}}}\Lambda_0(\gamma)N(\gamma)^{-s},\ \mathrm{Re}(s)>2\rho. \tag{8}$$

We define

$$\psi_j(x)=\int\limits_0^x\psi_{j-1}(t)\,dt,\ \ j=1,2,..., \tag{9}$$

where

$$\psi_0(x)=\sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x}\Lambda_0(\gamma).$$

Let $k\geq 2n$ be an integer and $x>1$, $c>2\rho$.

By [18, p. 31, Th. B.] and (8)

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\frac{Z_R'(s)}{Z_R(s)}\frac{x^s}{s(s+1)...(s+k)}ds$$
$$=-\sum_{\gamma\in\Gamma_{\mathrm{h}}}\Lambda_0(\gamma)\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\left(\frac{x}{N(\gamma)}\right)^s\frac{ds}{s(s+1)...(s+k)}$$
$$=-\sum_{\gamma\in\Gamma_{\mathrm{h}},\frac{x}{N(\gamma)}\geq 1}\Lambda_0(\gamma)\frac{1}{k!}\left(1-\frac{1}{\frac{x}{N(\gamma)}}\right)^k$$
$$=-\frac{1}{k!}\sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x}\Lambda_0(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^k.$$

On the other hand, by [18, p. 18, Th. A.]

$$\psi_k(x)=\frac{1}{k!}\sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x}\Lambda_0(\gamma)(x-N(\gamma))^k.$$

Hence,

$$\psi_k(x)=\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\left(-\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)...(s+k)}\right)ds.$$

Assume that $c^{'} \ll -2\rho$ is not a pole of the integrand of $\psi_k(x)$.

By Lemma 3, $\frac{Z_R^{'}(s)}{Z_R(s)} = O\left(|s|^{n-1}\right)$ on the line $\mathrm{Re}(s) = c^{'}$. Furthermore, by Lemma 4, there exists a sequence $\{y_j\}$, $y_j \to +\infty$ as $j \to +\infty$, such that

$$\frac{Z_R^{'}(t+\mathrm{i}\,y_j)}{Z_R(t+\mathrm{i}\,y_j)} = O\left(y_j^{2n}\right)$$

for $t \in \left[c^{'}, c\right]$.

Fix some $y_j \gg 1$.

By construction of $\{y_j\}$, we know that no pole of $\frac{Z_R^{'}(s)}{Z_R(s)}$ occurs on the line $\mathrm{Im}(s) = y_j$.

Applying the Cauchy residue theorem to the integrand of $\psi_k(x)$ over the rectangle $R\left(c^{'}, y_j\right)$ given by vertices $c - \mathrm{i}\,y_j$, $c + \mathrm{i}\,y_j$, $c^{'} + \mathrm{i}\,y_j$, $c^{'} - \mathrm{i}\,y_j$, we obtain

$$\begin{aligned} &\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,y_j}^{c+\mathrm{i}\,y_j} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds \\ &= \sum_{z\in R\left(c^{'},y_j\right)} \mathrm{Res}_{s=z} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) \\ &\quad + \tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}-\mathrm{i}}^{c^{'}+\mathrm{i}} + \tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}-\mathrm{i}\,y_j}^{c^{'}-\mathrm{i}} + \tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}+\mathrm{i}}^{c^{'}+\mathrm{i}\,y_j} + \tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}+\mathrm{i}\,y_j}^{c+\mathrm{i}\,y_j} + \tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,y_j}^{c^{'}-\mathrm{i}\,y_j}. \end{aligned} \tag{10}$$

We have

$$\begin{aligned} &\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}-\mathrm{i}}^{c^{'}+\mathrm{i}} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds \\ &= O\left(x^{c^{'}+k} \int\limits_{c^{'}-\mathrm{i}}^{c^{'}+\mathrm{i}} |ds|\right) = O\left(x^{c^{'}+k} \int\limits_{-1}^{1} dv\right) = O\left(x^{c^{'}+k}\right), \end{aligned}$$

$$\begin{aligned} &\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}+\mathrm{i}}^{c^{'}+\mathrm{i}\,y_j} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds \\ &= O\left(x^{c^{'}+k} \int\limits_{c^{'}+\mathrm{i}}^{c^{'}+\mathrm{i}\,y_j} \tfrac{|ds|}{|s|^{k-n+2}}\right) = O\left(x^{c^{'}+k} \int\limits_{1}^{y_j} \tfrac{dv}{v^{k-n+2}}\right) = O\left(x^{c^{'}+k}\right), \end{aligned}$$

$$\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}+\mathrm{i}\,y_j}^{c+\mathrm{i}\,y_j} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds = O\left(\tfrac{x^{c+k}}{y_j^{k+1-2n}}\right).$$

Similarly,

$$\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c^{'}-\mathrm{i}\,y_j}^{c^{'}-\mathrm{i}} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds = O\left(x^{c^{'}+k}\right),$$

$$\tfrac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,y_j}^{c^{'}-\mathrm{i}\,y_j} \left(-\tfrac{Z_R^{'}(s)}{Z_R(s)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds = O\left(\tfrac{x^{c+k}}{y_j^{k+1-2n}}\right).$$

Hence, by (10) and (5)

$$\begin{aligned} &\frac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,y_j}^{c+\mathrm{i}\,y_j} \left(-\frac{Z_R^{'}(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in R\left(c^{'},y_j\right)} c_z(p,\tau,\lambda,k) \\ &\quad + O\left(x^{c^{'}+k}\right) + O\left(\frac{x^{c+k}}{y_j^{k+1-2n}}\right), \end{aligned} \tag{11}$$

where

$$c_z(p,\tau,\lambda,k) = \mathrm{Res}_{s=z}\left(\tfrac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \tfrac{x^{s+k}}{s(s+1)\dots(s+k)}\right).$$

Letting $j \to +\infty$, $c^{'} \to -\infty$ in (11), we get

$$\begin{aligned} &\frac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty} \left(-\frac{Z_R^{'}(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\dots(s+k)}\right) ds \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in A_k^{p,\tau,\lambda}} c_z(p,\tau,\lambda,k), \end{aligned}$$

i.e.,

$$\psi_k(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in A_k^{p,\tau,\lambda}} c_z(p,\tau,\lambda,k), \tag{12}$$

where $A_k^{p,\tau,\lambda}$ denotes the set of poles of $\frac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+k}}{s(s+1)\dots(s+k)}$.

Take $k = 2n$. By (12),

$$\begin{aligned} \psi_{2n}(x) &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in A_{2n}^{p,\tau,\lambda}} c_z(p,\tau,\lambda,2n) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in A^{p,\tau,\lambda}} c_z(p,\tau,\lambda), \end{aligned} \tag{13}$$

where, for the sake of simplicity, we denote by $A^{p,\tau,\lambda}$ the set of poles of $\frac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+2n}}{s(s+1)\dots(s+2n)}$ and by $c_z(p,\tau,\lambda)$ the residue at $s = z$.

$\frac{Z_S(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+2n}}{s(s+1)\dots(s+2n)}$ corresponds to some $(\tau,\lambda) \in I_p$ for some $p \in \{0, 1, ..., n-1\}$.

By [6, p. 113, Theorem 3.15], the singularities of $Z_S(s+\rho-\lambda,\tau)$ are: at $\pm\,\mathrm{i}\,s - \rho + \lambda$ of order $m(s,\gamma_\tau,\tau)$ if $s \neq 0$ is an eigenvalue of $A_Y(\gamma_\tau,\tau)$, at $-\rho+\lambda$ of order $2m(0,\gamma_\tau,\tau)$ if 0 is an eigenvalue of $A_Y(\gamma_\tau,\tau)$, at $-s-\rho+\lambda$, $s \in T(\mathbb{N}-\epsilon_\tau)$ of order $-2(-1)^{\frac{n}{2}} \frac{\mathrm{vol}(Y)}{\mathrm{vol}(X_d)} m_d(s,\gamma_\tau,\tau)$ (in this case $s > 0$ is an eigenvalue of $A_d(\gamma_\tau,\tau)$). Here, γ_τ is some $\tau-$admissible element in $R(K)$.

Note that the singularities of $Z_S(s+\rho-\lambda,\tau)$ at $-s-\rho+\lambda$, $s \in T(\mathbb{N}-\epsilon_\tau)$ are all less than $-\rho+\lambda$. Furthermore, the singularities of $Z_S(s+\rho-\lambda,\tau)$ that correspond to $A_Y(\gamma_\tau,\tau)$ are contained in the union of the interval

$[-2\rho+\lambda,\lambda]$ with the line $-\rho+\lambda+\mathrm{i}\,\mathbb{R}$. An overlap between these two kinds of singularities may occur inside $[-2\rho+\lambda,-\rho+\lambda)$ (see, [6, pp. 114–115]).

The integers 0, -1, ... , $-2n$ are simple poles of $\frac{x^{s+2n}}{s(s+1)\dots(s+2n)}$. These integers may also appear as simple poles of $\frac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}$, i.e., as singularities of $Z_S\left(s+\rho-\lambda,\tau\right)$. Denote by $I_{p,\tau,\lambda}$ the set of such integers. Put $I_{p,\tau,\lambda}^{'}$ to be the difference $\{0,-1,...,-2n\}\setminus I_{p,\tau,\lambda}$. The set of the remaining singularities $s^{p,\tau,\lambda}$ of $Z_S\left(s+\rho-\lambda,\tau\right)$ will be denoted by $S^{p,\tau,\lambda}$.

Reasoning as in [16, pp. 88–89], we write

$$\frac{Z_S^{'}\left(s+\rho-\lambda,\tau\right)}{Z_S\left(s+\rho-\lambda,\tau\right)}=\frac{o_z^{p,\tau,\lambda}}{s-z}\left(1+\sum_{i=1}^{+\infty}a_{i,z}^{p,\tau,\lambda}\left(s-z\right)^i\right),$$

where z is a singularity of $Z_S\left(s+\rho-\lambda,\tau\right)$ and $o_z^{p,\tau,\lambda}$ is the order of z.

Now, for $s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}$,

$$\begin{aligned}&c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right)\\&=\lim_{s\to s^{p,\tau,\lambda}}\left(s-s^{p,\tau,\lambda}\right)\tfrac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\tfrac{x^{s+2n}}{s(s+1)\dots(s+2n)}\\&=\lim_{s\to s^{p,\tau,\lambda}}\left(s-s^{p,\tau,\lambda}\right)\frac{o_{s^{p,\tau,\lambda}}^{p,\tau,\lambda}}{s-s^{p,\tau,\lambda}}\cdot\\&\quad\left(1+\sum_{i=1}^{+\infty}a_{i,s^{p,\tau,\lambda}}^{p,\tau,\lambda}\left(s-s^{p,\tau,\lambda}\right)^i\right)\tfrac{x^{s+2n}}{s(s+1)\dots(s+2n)}\\&=o_{s^{p,\tau,\lambda}}^{p,\tau,\lambda}\frac{x^{s^{p,\tau,\lambda}+2n}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\dots\left(s^{p,\tau,\lambda}+2n\right)}.\end{aligned}\tag{14}$$

Let $-j\in I_{p,\tau,\lambda}$. We have

$$c_{-j}\left(p,\tau,\lambda\right)=\lim_{s\to-j}\tfrac{d}{ds}\left(\left(s+j\right)^2\tfrac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\tfrac{x^{s+2n}}{s(s+1)\dots(s+2n)}\right).$$

Since

$$\begin{aligned}&\left(s+j\right)^2\frac{Z_S^{'}\left(s+\rho-\lambda,\tau\right)}{Z_S\left(s+\rho-\lambda,\tau\right)}\frac{x^{s+2n}}{s\left(s+1\right)\dots\left(s+2n\right)}\\&=o_{-j}^{p,\tau,\lambda}\left(1+\sum_{i=1}^{+\infty}a_{i,-j}^{p,\tau,\lambda}\left(s+j\right)^i\right)\frac{x^{s+2n}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(s+l\right)}\\&=o_{-j}^{p,\tau,\lambda}\frac{x^{s+2n}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(s+l\right)}+o_{-j}^{p,\tau,\lambda}a_{1,-j}^{p,\tau,\lambda}\left(s+j\right)\frac{x^{s+2n}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(s+l\right)}+\dots\end{aligned}$$

and

$$\tfrac{d}{ds}\left(\left(s+j\right)^2\tfrac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\tfrac{x^{s+2n}}{s(s+1)\dots(s+2n)}\right)=$$

$$\begin{aligned}&\frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(s+l\right)}x^{s+2n}\log x-\frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(s+l\right)}\sum_{\substack{l=0\\l\neq j}}^{2n}\frac{1}{s+l}x^{s+2n}\\&+\tfrac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(s+l)}a_{1,-j}^{p,\tau,\lambda}x^{s+2n}+o_{-j}^{p,\tau,\lambda}a_{1,-j}^{p,\tau,\lambda}\left(s+j\right)\tfrac{d}{ds}\left(\tfrac{x^{s+2n}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(s+l)}\right)+\dots,\end{aligned}$$

we obtain

$$\begin{aligned}c_{-j}\left(p,\tau,\lambda\right)=&\tfrac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)}x^{-j+2n}\log x\\&+\tfrac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)}\left(-\sum_{\substack{l=0\\l\neq j}}^{2n}\tfrac{1}{-j+l}+a_{1,-j}^{p,\tau,\lambda}\right)x^{-j+2n}.\end{aligned}\tag{15}$$

Finally, let $-j\in I_{p,\tau,\lambda}^{'}$. Now,

$$\begin{aligned}&c_{-j}\left(p,\tau,\lambda\right)\\&=\lim_{s\to-j}\left(\left(s+j\right)\tfrac{Z_S^{'}(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\tfrac{x^{s+2n}}{s(s+1)\dots(s+2n)}\right)\\&=\frac{Z_S^{'}\left(-j+\rho-\lambda,\tau\right)}{Z_S\left(-j+\rho-\lambda,\tau\right)}\frac{x^{-j+2n}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}\left(-j+l\right)}.\end{aligned}\tag{16}$$

We denote:

$$\begin{aligned}I_{-2n}&=\left\{0,-1,...,-2n\right\},\\B_{p,\tau,\lambda}&=\left\{-j\in I_{-2n}\mid c_{-j}\left(p,\tau,\lambda\right)=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)\right\},\\B_{p,\tau,\lambda}^{'}&=I_{-2n}\setminus B_{p,\tau,\lambda},\\S_{\mathbb{R}}^{p,\tau,\lambda}&=S^{p,\tau,\lambda}\cap\mathbb{R},\\S_{-\rho+\lambda}^{p,\tau,\lambda}&=S^{p,\tau,\lambda}\setminus S_{\mathbb{R}}^{p,\tau,\lambda},\\C_{p,\tau,\lambda}^1&=\left\{s^{p,\tau,\lambda}\in S_{\mathbb{R}}^{p,\tau,\lambda}\mid s^{p,\tau,\lambda}\leq-2n-1\right\},\\C_{p,\tau,\lambda}^2&=\left\{s^{p,\tau,\lambda}\in S_{\mathbb{R}}^{p,\tau,\lambda}\mid-2n-1<s^{p,\tau,\lambda}\leq-2n+2\rho\tfrac{n+\rho-1}{n+2\rho-1}\right\},\\C_{p,\tau,\lambda}^3&=\left\{s^{p,\tau,\lambda}\in S_{\mathbb{R}}^{p,\tau,\lambda}\mid-2n+2\rho\tfrac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq2\rho\tfrac{n+\rho-1}{n+2\rho-1}\right\},\\C_{p,\tau,\lambda}^4&=\left\{s^{p,\tau,\lambda}\in S_{\mathbb{R}}^{p,\tau,\lambda}\mid2\rho\frac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq2\rho\right\}.\end{aligned}$$

Now, we can write

$$\begin{aligned}&\sum_{z\in A^{p,\tau,\lambda}}c_z\left(p,\tau,\lambda\right)\\&=\sum_{z\in B_{p,\tau,\lambda}}c_z\left(p,\tau,\lambda\right)+\sum_{z\in B_{p,\tau,\lambda}^{'}}c_z\left(p,\tau,\lambda\right)\\&\quad+\sum_{k=1}^{4}\sum_{z\in C_{p,\tau,\lambda}^k}c_z\left(p,\tau,\lambda\right)+\sum_{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}}c_z\left(p,\tau,\lambda\right).\end{aligned}\tag{17}$$

Consider the sum over $C_{p,\tau,\lambda}^1$ in (17).

Since $C^1_{p,\tau,\lambda} \subset S^{p,\tau,\lambda}_{\mathbb{R}} \subset S^{p,\tau,\lambda}$ and $z \leq 2n-1 < -2\rho + \lambda$ for $z \in C^1_{p,\tau,\lambda}$, it follows from (14) that

$$\begin{aligned}&\sum_{z \in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &= \sum_{z \in C^1_{p,\tau,\lambda}} o_z^{p,\tau,\lambda} \frac{x^{z+2n}}{z(z+1)\dots(z+2n)}\\ &= -2(-1)^{\frac{n}{2}} \tfrac{\mathrm{vol}(Y)}{\mathrm{vol}(X_d)} \sum_{k \geq \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} m_d(T(k-\epsilon_\tau),\gamma_\tau,\tau)\\ &\cdot \frac{x^{-T(k-\epsilon_\tau)-\rho+\lambda+2n}}{\prod_{l=0}^{2n}(-T(k-\epsilon_\tau)-\rho+\lambda+l)}.\end{aligned}$$

The fact that γ_τ is $\tau-$admissible element yields $m_d(s,\gamma_\tau,\tau) = P_\tau(s)$ for all $0 \leq s \in L(\tau) = T(\epsilon_\tau + \mathbb{Z})$. In particular, $m_d(T(k-\epsilon_\tau),\gamma_\tau,\tau) = P_\tau(T(k-\epsilon_\tau))$ for $k \geq \frac{1}{T}(2n+1-\rho+\lambda) + \epsilon_\tau$. We obtain

$$\begin{aligned}&\sum_{z \in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &= O\left(x^{-1} \sum_{k \geq \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} \tfrac{|P_\tau(T(k-\epsilon_\tau))|}{(T(k-\epsilon_\tau)+\rho-\lambda-2n)^{2n+1}}\right)\\ &= O\left(x^{-1} \sum_{k \geq \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} \tfrac{(2n+1-\rho+\lambda+T\epsilon_\tau)^{2n+1}|P_\tau(T(k-\epsilon_\tau))|}{T^{2n+1}k^{2n+1}}\right).\end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned}&\sum_{z \in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &= O\left(x^{-1} \sum_{k \geq \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} \tfrac{1}{k^{n+2}}\right) = O(x^{-1}).\end{aligned} \tag{18}$$

The sum over $B_{p,\tau,\lambda}$ in (17) is a finite one. Therefore, by the definition of $B_{p,\tau,\lambda}$,

$$\sum_{z \in B_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{19}$$

The sum over $C^2_{p,\tau,\lambda}$ is a finite one as well. Hence, by (14),

$$\begin{aligned}&\sum_{z \in C^2_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &= \sum_{z \in C^2_{p,\tau,\lambda}} o_z^{p,\tau,\lambda} \tfrac{x^{z+2n}}{z(z+1)\dots(z+2n)} = O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned} \tag{20}$$

Combining (13) and (17)–(20), we obtain

$$\begin{aligned}&\psi_{2n}(x)\\ &= \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in B'_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &+ \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in C^3_{p,\tau,\lambda}} c_z(p,\tau,\lambda) +\\ &\sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in C^4_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\ &+ \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in S^{p,\tau,\lambda}_{-\rho+\lambda}} c_z(p,\tau,\lambda)\\ &+ O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned} \tag{21}$$

Suppose $1 < h \leq \frac{x}{2}$.
We introduce the operator

$$\Delta^+_{2n} f(x) = \sum_{i=0}^{2n}(-1)^i \binom{2n}{i} f(x+(2n-i)h). \tag{22}$$

If f is at least $2n$ times differentiable function, then

$$\Delta^+_{2n} f(x) = \int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}\dots\int_{t_2}^{t_2+h} f^{(2n)}(t_1)\,dt_1\dots dt_{2n}. \tag{23}$$

The mean value theorem applied to (23) yields

$$\Delta^+_{2n} f(x) = h^{2n} f^{(2n)}(\tilde{x}), \tag{24}$$

where $\tilde{x} \in [x, x+2nh]$.
Since ψ_0 is nondecreasing, we obtain

$$\psi_0(x) \leq h^{-2n}\Delta^+_{2n}\psi_{2n}(x) \leq \psi_0(x+2nh). \tag{25}$$

Now, (21), (22) and the fact that $h \leq \frac{x}{2}$, imply

$$\begin{aligned}&h^{-2n}\Delta^+_{2n}\psi_{2n}(x)\\ &= \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in B'_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n} c_z(p,\tau,\lambda)\\ &+ \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in C^3_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n} c_z(p,\tau,\lambda)\\ &+ \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in C^4_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n} c_z(p,\tau,\lambda)\\ &+ \sum_{p=0}^{n-1}(-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z \in S^{p,\tau,\lambda}_{-\rho+\lambda}} h^{-2n}\Delta^+_{2n} c_z(p,\tau,\lambda)\\ &+ O\left(h^{-2n}x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned} \tag{26}$$

Consider the sum over $B'_{p,\tau,\lambda}$ on the right hand side of (26).
Let $z \in B'_{p,\tau,\lambda}$, $z = 0$.
Suppose that $0 \in I_{p,\tau,\lambda}$. Then, (15), (24) and the facts: $(x^n \log x)^{(n)} = n!\log x + n!\sum_{l=1}^n \frac{1}{l}$, $(x^n)^{(n)} = n!$, yield

$$h^{-2n}\Delta^+_{2n} c_0(p,\tau,\lambda) = o_0^{p,\tau,\lambda}\log\tilde{x}_{p,\tau,\lambda,0} + o_0^{p,\tau,\lambda} a_{1,0}^{p,\tau,\lambda}, \tag{27}$$

where $\tilde{x}_{p,\tau,\lambda,0} \in [x, x+2nh]$.
If $0 \in I'_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta_{2n}^{+}c_0(p,\tau,\lambda)=\frac{Z_S'(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)} \tag{28}$$

by (16). Let $z\in B'_{p,\tau,\lambda}$, $z=-j\leq -1$.

Suppose that $-j\in I_{p,\tau,\lambda}$.

Since $\left(x^k\log x\right)^{(n)}=k!\,(-1)^{n-k-1}\frac{(n-k-1)!}{x^{n-k}}$ and $\left(x^k\right)^{(n)}=0$ for $0\leq k<n$, $k\in\mathbb{N}$, we get

$$h^{-2n}\Delta_{2n}^{+}c_{-j}(p,\tau,\lambda)=o_{-j}^{p,\tau,\lambda}\frac{\tilde{x}_{p,\tau,\lambda,-j}^{-j}}{-j}, \tag{29}$$

where $\tilde{x}_{p,\tau,\lambda,-j}\in[x,x+2nh]$.

If $-j\in I'_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta_{2n}^{+}c_{-j}(p,\tau,\lambda)=0. \tag{30}$$

Now, (27)–(30) and the fact that $h\leq\frac{x}{2}$, imply

$$\sum_{z\in B'_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O(\log x). \tag{31}$$

Consider the sum over $C^3_{p,\tau,\lambda}$ on the right hand side of (26). Let $z\in C^3_{p,\tau,\lambda}$.

By (14) and (24),

$$\begin{aligned}\left|h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\right|&=\left|o_z^{p,\tau,\lambda}\frac{\tilde{x}^z_{p,\tau,\lambda,z}}{z}\right|\\&=\frac{\left|o_z^{p,\tau,\lambda}\right|}{|z|}\tilde{x}^z_{p,\tau,\lambda,z}\leq\frac{\left|o_z^{p,\tau,\lambda}\right|}{|z|}\tilde{x}^{2\rho\frac{n+\rho-1}{n+2\rho-1}}_{p,\tau,\lambda,z},\end{aligned}$$

where $\tilde{x}_{p,\tau,\lambda,z}\in[x,x+2nh]$. Hence, $h\leq\frac{x}{2}$ and the fact that $C^3_{p,\tau,\lambda}$ is a finite set, yield

$$\sum_{z\in C^3_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{32}$$

Similarly, the sum over $C^4_{p,\tau,\lambda}$ on the right hand side of (26) is a finite one. We have

$$h^{-2n}\Delta_{2n}^{+}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)=o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}\frac{\tilde{x}^{s^{p,\tau,\lambda}}_{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}$$

for $s^{p,\tau,\lambda}\in C^4_{p,\tau,\lambda}$, where $\tilde{x}_{s^{p,\tau,\lambda}}\in[x,x+2nh]$. Hence, reasoning as in [20, p. 246] and [19, p. 101], we obtain

$$\begin{aligned}&\sum_{z\in C^4_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&=\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}+O\left(h^{2\rho}\right),\end{aligned} \tag{33}$$

where $s^{p,\tau,\lambda}$ is counted $o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}$ times in the last sum.

Finally, we estimate the sum over $S^{p,\tau,\lambda}_{-\rho+\lambda}$ in (26). Let $z\in S^{p,\tau,\lambda}_{-\rho+\lambda}$. By (14),

$$c_z(p,\tau,\lambda)=o_z^{p,\tau,\lambda}\frac{x^{z+2n}}{z(z+1)\ldots(z+2n)}.$$

We derive two estimates for $h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)$.

Firstly, by (22),

$$\begin{aligned}&h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&=h^{-2n}\frac{o_z^{p,\tau,\lambda}}{z(z+1)\ldots(z+2n)}\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}(x+(2n-i)h)^{z+2n}.\end{aligned}$$

Since $h\leq\frac{x}{2}$, we obtain

$$h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(h^{-2n}|z|^{-2n-1}x^{-\rho+\lambda+2n}\right). \tag{34}$$

Secondly, by (23),

$$\begin{aligned}&\left|h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\right|\\&=\left|h^{-2n}\frac{o_z^{p,\tau,\lambda}}{z}\int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}\ldots\int_{t_2}^{t_2+h}t_1^z\,dt_1\ldots dt_{2n}\right|\\&\leq h^{-2n}\left|o_z^{p,\tau,\lambda}\right||z|^{-1}\int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}\ldots\int_{t_2}^{t_2+h}t_1^{-\rho+\lambda}\,dt_1\ldots dt_{2n}.\end{aligned}$$

Hence, by the mean value theorem and the fact that $h\leq\frac{x}{2}$,

$$h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(|z|^{-1}x^{-\rho+\lambda}\right). \tag{35}$$

Let $M>2\rho$. Now, using (34) and (35), we deduce

$$\begin{aligned}&\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&=\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|-\rho+\lambda|<|z|\leq M}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)+\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|z|>M}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\end{aligned} \tag{36}$$

$$\begin{aligned}&=O\left(x^{-\rho+\lambda}\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|-\rho+\lambda|<|z|\leq M}}|z|^{-1}\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|z|>M}}|z|^{-2n-1}\right)\\&=O\left(x^{-\rho+\lambda}\int_{|-\rho+\lambda|}^{M}t^{-1}dN_{p,\tau,\lambda}(t)\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}\int_M^{+\infty}t^{-2n-1}dN_{p,\tau,\lambda}(t)\right)\\&=O\left(x^{-\rho+\lambda}M^{n-1}\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}M^{-n-1}\right),\end{aligned}$$

where $N_{p,\tau,\lambda}(t)=O(t^n)$ denotes the number of singularities of $Z_S(s+\rho-\lambda,\tau)$ on the interval $-\rho+\lambda+\mathrm{i}\,x$, $0<x\leq t$.

Combining (26), (31)–(33) and (36), we obtain

$$\begin{aligned} & h^{-2n}\Delta_{2n}^{+}\psi_{2n}\left(x\right) \\ & =\sum_{p=0}^{n-1}\left(-1\right)^{p+1}\sum_{\left(\tau,\lambda\right)\in I_{p}}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} \\ & +O\left(h^{2\rho}\right) \\ & +O\left(x^{\rho}M^{n-1}\right)+O\left(h^{-2n}x^{\rho+2n}M^{-n-1}\right) \\ & +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{37}$$

Substituting $h=x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M=x^{\frac{\rho}{n+2\rho-1}}$ into (37) and taking into account (25), we get

$$\begin{aligned} \psi_{0}\left(x\right)\leq & \sum_{p=0}^{n-1}\left(-1\right)^{p+1}\sum_{\left(\tau,\lambda\right)\in I_{p}}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} \\ & +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{38}$$

Analogously, (see, e.g., [19, pp. 101–102]), one proves

$$\begin{aligned} \psi_{0}\left(x\right)\geq & \sum_{p=0}^{n-1}\left(-1\right)^{p+1}\sum_{\left(\tau,\lambda\right)\in I_{p}}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} \\ & +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{39}$$

Combining (38) and (39), we conclude that

$$\begin{aligned} \psi_{0}\left(x\right)= & \sum_{p=0}^{n-1}\left(-1\right)^{p+1}\sum_{\left(\tau,\lambda\right)\in I_{p}}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} \\ & +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{40}$$

Now, using (40) and following lines of [19, p. 102], we finally obtain

$$\begin{aligned} \pi_{\Gamma}\left(x\right)= & \sum_{p=0}^{n-1}\left(-1\right)^{p+1}\sum_{\left(\tau,\lambda\right)\in I_{p}}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\mathrm{li}\left(x^{s^{p,\tau,\lambda}}\right) \\ & +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\left(\log x\right)^{-1}\right) \end{aligned}$$

as $x\to+\infty$. This completes the proof. □

VI. Concluding remarks

Let us summarize the aspects in which Theorem 5 represents an improvement of (1).

As already mentioned in the Introduction, X is one of the following spaces:

$$H\mathbb{R}^{k}\left(k\text{ even},k\geq2\right),\ H\mathbb{C}^{m}\left(m\geq1\right),\ H\mathbb{H}^{l}\left(l\geq1\right),\ H\mathbb{C}a^{2}.$$

Hence, $n=k$, $2m$, $4l$, 16 and $\rho=\frac{1}{2}\left(k-1\right)$, m, $2l+1$, 11, respectively.

Since $H\mathbb{C}^{1}\cong H\mathbb{R}^{2}$ and $H\mathbb{H}^{1}\cong H\mathbb{R}^{4}$ (see, e.g., [15]), we may assume $m\geq2$ and $l\geq2$.

Now, $\alpha=n+q-1=k-1$, $2m$, $4l+2$, 22, respectively. Obviously, $\alpha=2\rho$.

The size of the error term in (1) is $O\left(x^{\left(1-\frac{1}{2n}\right)2\rho}\right)$. We compare this bound to our bound $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\left(\log x\right)^{-1}\right)$.

The factor $\left(\log x\right)^{-1}$ gives to our bound some advantage. However, let us have a look at the corresponding powers of x.

The inequality

$$2\rho\frac{n+\rho-1}{n+2\rho-1}\leq\left(1-\frac{1}{2n}\right)2\rho$$

always holds true since the corresponding equivalent inequality $\left(n-1\right)\left(2\rho-1\right)\geq0$ is always valid. Here, the equality sign occurs only if $X=H\mathbb{R}^{2}$.

Furthermore, the inequalities

$$2\rho\frac{n+\rho-1}{n+2\rho-1}\leq\frac{3}{2}\rho\leq\left(1-\frac{1}{2n}\right)2\rho$$

are always true.

Indeed, the left-hand inequality is valid, being equivalent to the inequality $n\leq2\rho+1$. The equality occurs only if $X=H\mathbb{R}^{k}$, $k\geq2$, k even.

On the other side, the right-hand inequality holds also true since it reduces to $n-2\geq0$. Clearly, the right-hand inequality becomes equality only if $X=H\mathbb{R}^{2}$.

Summarizing results derived above, we end up with the conclusion that the obtained bound $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\left(\log x\right)^{-1}\right)$ is of the form $O\left(x^{\theta}\left(\log x\right)^{-1}\right)$, where $\theta<\frac{3}{2}\rho$ if $X=H\mathbb{C}^{m}$, $(m\geq2)$, $H\mathbb{H}^{l}$, $(l\geq2)$, $H\mathbb{C}a^{2}$ and $\theta=\frac{3}{2}\rho$ if $X=H\mathbb{R}^{k}$, k even, $k\geq2$.

Note that our result coincides with the best known results for the compact Riemann surfaces [20] and the real hyperbolic manifolds with cusps [1].

Also, note that taking $k>2n$ in the proof of Theorem 5 does not yield a better result.

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