Horocycle trajectories and their limit-strings on a complex hyperbolic space

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Abstract—We take trajectory-harps for a Kähler magnetic field of strength $\sqrt{|c|}$ on a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. We show that distance functions between their limit-strings and their arch-trajectories are not bounded though their limit points in the ideal boundary are the same.

Keywords—Hadamard manifolds, ideal boundary, Kähler magnetic fields, limit-strings, trajectories, trajectory-harps.

I. INTRODUCTION

On a Kähler manifold with complex structure J we have a natural family of closed 2-forms which are constant multiples of its Kähler form $\mathbb{B}_J$. We say these 2-forms to be Kähler magnetic fields. A smooth curve γ parameterized by its arclength is said to be a *trajectory* for a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ ($\kappa \in \mathbb{R}$) if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Since its velocity vector and acceleration vector span a complex line at each point, this curve is deeply concerned with complex structure of the underlying Kähler manifold. It is hence natural to consider that properties of trajectories and those of the underlying manifold are closely related with each other.

In the preceding paper [9], Shi and the author studied asymptotic behaviors of trajectories for Kähler magnetic fields on a Hadamard Kähler manifold, a simply connected Kähler manifold of nonpositive sectional curvature. Two geodesic half-lines of unit speed are said to be asymptotic if the distance function between them is a bounded function. With the ideal boundary which is obtained as the set of asymptotic classes of all geodesic half-lines of unit speed, every Hadamard manifold is compactified ([7]). Since the geometry of the ideal boundary of a Hadamard manifold is closely related with the geometry of itself (see [6] and also [5], [8]), the author is interested in asymptotic behaviors of trajectories for Kähler magnetic fields. When sectional curvatures of tangent planes of a Hadamard Kähler manifold M are bounded from above by a negative constant c, every trajectory half-line $\gamma : [0,\infty) \to M$ for a Kähler magnetic field $\mathbb{B}_\kappa$ with $|\kappa| \le \sqrt{|c|}$ is unbounded and has a limit point $\gamma(\infty) = \lim_{t\to\infty}\gamma(t)$ in the ideal boundary ∂M of M. Moreover, if we take a geodesic half-line σ satisfying $\sigma(0) = \gamma(0)$ and $\sigma(\infty) = \gamma(\infty)$, then γ is contained in a tube of finite radius around σ if $|\kappa| < \sqrt{|c|}$. Thus, we are interested more in asymptotic behaviors of trajectories for $|\kappa| = \sqrt{|c|}$.

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In this paper we study trajectory half-lines for Kähler magnetic fields $\mathbb{B}_{\pm\sqrt{|c|}}$ on a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. We show that the distance from a point $\gamma(t)$ on such a trajectory half-line to its associated geodesic half-line σ grows logarithmically with respect to t.

II. TRAJECTORY-HARPS

Let M be a Kähler manifold with complex structure J. A smooth curve $\gamma : \mathbb{R} \to M$ parameterized by its arclength is said to be a *trajectory* for a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Since J is parallel, we see that this trajectory satisfies the Frenet formula $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$, $\nabla_{\dot\gamma}(J\dot\gamma) = -\kappa\dot\gamma$ with the Frenet frame $\{\dot\gamma, J\dot\gamma\}$ and the geodesic curvature $|\kappa|$. Thus, we may say that trajectories for Kähler magnetic fields are simplest curves showing complex structure of the underlying Kähler manifold from the viewpoint of classical curve theory. This suggests us importance of investigating trajectories for Kähler magnetic field to study Kähler manifolds.

In order to study properties of trajectories, we introduced trajectory-harps in [2]. We call a restriction of a trajectory on a finite interval a trajectory-segment, and call a restriction of a trajectory on a infinite interval a trajectory-half line. Let $\gamma : [0,T] \to M$ be a trajectory-segment or a trajectory half-line for a Kähler magnetic field $\mathbb{B}_\kappa$ satisfying $\gamma(t) \ne \gamma(0)$ for all $0 < t < T$. A smooth variation of geodesics α_γ : $[0,T]\times\mathbb{R} \to M$ is said to be a *trajectory-harp* associated with γ if it satisfies the following conditions:

i) $\alpha_\gamma(t,0) = \gamma(0)$ for each t,
ii) the curve $s \mapsto \alpha_\gamma(0,s)$ is the geodesic of initial vector $\dot\gamma(0)$,
iii) for each t, the curve $s \mapsto \alpha_\gamma(t,s)$ is a geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

When $\gamma([0,T])$ is contained in the geodesic ball centered at $\gamma(0)$ and of radius of the first conjugate value of $\gamma(0)$, we can define trajectory-harp uniquely. We call the geodesic $s \mapsto \alpha_\gamma(t,s)$ a *string* of α_γ at t, and call γ its *arch-trajectory*. We set $\ell_\gamma(t)$ as the length of the geodesic segment $s \mapsto \alpha_\gamma(t,s)$ from $\gamma(0)$ to $\gamma(t)$, and put $\delta_\gamma(t) = \langle \frac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big), \dot\gamma(t)\rangle$. We call them the *string-length* and the *string-cosine* of α_γ at t, respectively. The differential of the string-length coincides with the string-cosine ($\ell'_\gamma(t) = \delta_\gamma(t)$). String-lengths show lengths of trajectory-harps. In [2] and in [10], Shi and the author estimated string-lengths and string-cosines under an assumption on sectional curvatures of underlying Kähler manifolds. For a negative c and for κ with $\kappa^2 \le |c|$, we set

$\ell_\kappa(t;c)$ by the following relations

$$\begin{cases} \sqrt{|c|-\kappa^2}\,\sinh\bigl(\sqrt{|c|}\,\ell_\kappa(t;c)/2\bigr) \\ \qquad = \sqrt{|c|}\,\sinh\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr), & \text{when } |\kappa|<\sqrt{|c|}, \\ 2\sinh\bigl(\sqrt{|c|}\,\ell_\kappa(t;c)/2\bigr) = \sqrt{|c|}\,t, & \text{when } \kappa=\pm\sqrt{|c|}. \end{cases}$$

Also, we define a function $\delta_\kappa(t;c)$ by

$$\delta_\kappa(t;c) = \begin{cases} \dfrac{\sqrt{|c|-\kappa^2}\,\cosh\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)-\kappa^2}}, & \text{when } |\kappa|<\sqrt{|c|}, \\ 2\big/\sqrt{|c|t^2+4}, & \text{when } \kappa=\pm\sqrt{|c|}. \end{cases}$$

These functions show string-lengths and string-cosines of trajectory-harps associated with trajectories for $\mathbb{B}_\kappa$ on a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. The string-length $\ell_\kappa(t;c)$ is monotone increasing with respect to t and satisfies $\lim_{t\to\infty}\ell_\kappa(t;c)=\infty$. Trajectories on a Hadamard Kähler manifold satisfy a similar property. We call the set

$$\mathcal{H}_\gamma = \bigl\{\alpha_\gamma(t,s)\ \big|\ 0\le t\le T,\ 0\le s\le \ell_\gamma(t)\bigr\}$$

the *harp-body* of a trajectory-harp α_γ.

Proposition 1 ([2]): Let $\gamma:[0,\infty)\to M$ be a trajectory half-line for a Kähler magnetic field $\mathbb{B}_\kappa$ on a Hadamard Kähler manifold M. Suppose sectional curvatures of planes tangent to the harp-body $\mathcal{H}_\gamma$ are not greater than a negative constant c. If $|\kappa|\le\sqrt{|c|}$, then the function ℓ_γ of string-lengths is monotone increasing and satisfies $\ell_\gamma(t)\ge\ell_\kappa(t;c)$.

This proposition guarantees that on a Hadamard Kähler manifold M every trajectory γ for $\mathbb{B}_\kappa$ with $|\kappa|\le\sqrt{|c|}$ is unbounded in both directions when sectional curvatures of M satisfy $\mathrm{Riem}^M\le c<0$. Here, we say γ is unbounded in both directions if both $\gamma([0,\infty))$ and $\gamma(-\infty,0])$ are unbounded sets.

In order to show widths of trajectory-harps we have zenith angles and lengths of sector-arcs for harp-sectors. For a trajectory-harp α_γ, given a, b with $0\le a<b\le T$, we call its restriction $\alpha_\gamma|_{[a,b]\times[0,\ell_\gamma(a)]}$ onto $[a,b]\times[0,\ell_\gamma(a)]$ a harp-sector. We call the curve $[a,b]\ni t\mapsto\alpha_\gamma\bigl(t,\ell_\gamma(a)\bigr)\in M$ its *sector-arc*, and call the length $\vartheta_\gamma(a,b)$ of the curve $[a,b]\ni t\mapsto\frac{\partial\alpha_\gamma}{\partial s}(t,0)\in T_{\gamma(0)}M$ its *zenith angle*. In [3], [9] and [10] we gave estimates of lengths of sector-arcs and zenith angles. We here recall some of the results which are closely related to our present paper.

Proposition 2 ([9], [3]): Let $\gamma:[0,\infty)\to M$ be a trajectory half-line for a Kähler magnetic field $\mathbb{B}_\kappa$ on a Hadamard Kähler manifold M. Suppose sectional curvatures of planes tangent to the harp-body $\mathcal{H}_\gamma$ are not greater than a negative constant c. If $|\kappa|\le\sqrt{|c|}$, for arbitrary a,b with $0\le a<b$, the zenith angle $\vartheta_\gamma(a,b)$ and the length $s\ell_\gamma(a,b)$ of the sector-arc are estimated as follows:

1) When $|\kappa|<\sqrt{|c|}$, we have

$$\vartheta_\gamma(a,b)\le\int_a^b\frac{\sqrt{|c|-\kappa^2}}{\sinh\sqrt{|c|-\kappa^2}\,t}\,dt = \log\biggl(\frac{(e^{bk}-1)(e^{ak}+1)}{(e^{bk}+1)(e^{ak}-1)}\biggr)$$

and

$$s\ell_\gamma(a,b)\le\frac{|\kappa|\pi}{2\sqrt{|c|(|c|-\kappa^2)}};$$

2) When $\kappa=\pm\sqrt{|c|}$, we have

$$\vartheta_\gamma(a,b)\le\int_a^b\frac{2}{\sqrt{|c|}\,t^2}\,dt=\frac{2}{\sqrt{|c|}}\Bigl(\frac{1}{a}-\frac{1}{b}\Bigr).$$

Since we have $\angle\bigl(\frac{\partial\alpha_\gamma}{\partial s}(a,0),\frac{\partial\alpha_\gamma}{\partial s}(b,0)\bigr)\le\vartheta_\gamma(a,b)$, by the estimates on zenith angles we find that for each trajectory-harp α_γ associated with a trajectory half-line γ for $\mathbb{B}_\kappa$ with $|\kappa|\le\sqrt{|c|}$ on a Hadamard Kähler manifold M satisfying $\mathrm{Riem}^M\le c<0$ the limit $\lim_{t\to\infty}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\in U_{\gamma(0)}M$ of initial vectors of strings exists. We call the geodesic half-line $\sigma_\gamma:[0,\infty)\to M$ having this vector as initial vector the *limit-string* of this trajectory-harp α_γ. Propositions 1, 2 guarantee that the point $\gamma(\infty):=\lim_{t\to\infty}\gamma(t)$ at infinity exists in the ideal boundary ∂M and it coincides with the point $\sigma_\gamma(\infty):=\lim_{t\to\infty}\sigma_\gamma(t)$ at infinity of its limit-string. Moreover, when $|\kappa|<\sqrt{|c|}$, the estimate on sector-arcs shows that the trajectory γ is contained in the tube around the limit-string σ_γ of radius $|\kappa|\pi/\bigl(2\sqrt{|c|(|c|-\kappa^2)}\bigr)$, and that σ_γ is contained in the tube around γ of the same radius. We are hence interested in the relationship between trajectories for $\mathbb{B}_{\pm\sqrt{|c|}}$ and their limit-strings of their trajectory-harps.

Proposition 3: Let $\gamma:[0,\infty)\to M$ be a trajectory half-line for a Kähler magnetic field $\mathbb{B}_\kappa$ with $k=\pm\sqrt{|c|}$ on $\mathbb{C}H^n(c)$. For arbitrary a,b with $0\le a<b$, the length of sector-arc of $\alpha_\gamma^{a,b}$ is estimated as

$$s\ell_\gamma(a,b)\le\pi\sqrt{|c|}a(b-a)\big/\bigl(2\sqrt{|c|b^2+4}\bigr).$$

In particular, we have $\lim_{b\to\infty}s\ell_\gamma(t,b)\le\pi t/2$.

Proof: We note that each trajectory γ lies on a totally geodesic $\mathbb{C}H^1(c)$. Thus, the trajectory-harp α_γ is a variation of geodesics on $\mathbb{C}H^1(c)$. We therefore find that

$$s\ell_\gamma(a,b)=\frac{1}{\sqrt{|c|}}\sinh\bigl(\sqrt{|c|}\ell_\kappa(a;c)\bigr) \times\angle\Bigl(\frac{\partial\alpha_\gamma}{\partial s}(a,0),\frac{\partial\alpha_\gamma}{\partial s}(b,0)\Bigr),$$

where $\bigl(1/\sqrt{|c|}\,\bigr)\sinh\bigl(\sqrt{|c|}t\bigr)$ shows the norm of the Jacobi field $\frac{\partial\alpha_\gamma}{\partial s}(t,s)$. Since we have

$$\angle\Bigl(\frac{\partial\alpha_\gamma}{\partial s}(t,0),\frac{\partial\alpha_\gamma}{\partial s}(0,0)\Bigr)=\cos^{-1}\delta_\kappa(t;c)$$

and $\theta \leq (\pi/2)\sin\theta$ for $0 \leq \theta \leq \pi/2$, by trigonometric addition formula, we obtain

$$\begin{aligned}
&\angle\Big(\frac{\partial\alpha_\gamma}{\partial s}(a,0), \frac{\partial\alpha_\gamma}{\partial s}(b,0)\Big)\\
&\quad = \cos^{-1}\delta_\kappa(b;c) - \cos^{-1}\delta_\kappa(a;c)\\
&\quad \leq \frac{\pi}{2}\sin\big(\cos^{-1}\delta_\kappa(b;c) - \cos^{-1}\delta_\kappa(a;c)\big)\\
&\quad = \frac{\pi\sqrt{|c|}\,(b-a)}{\sqrt{(|c|b^2+4)(|c|a^2+4)}},
\end{aligned}$$

because $\sin\big(\cos^{-1}\delta_\kappa(t;c)\big) = \sqrt{|c|}\,t/\sqrt{|c|t^2+4}$. On the other hand, we have

$$\begin{aligned}
&\sinh\big(\sqrt{|c|}\ell_\kappa(a;c)\big)\\
&\quad = 2\sinh\big(\sqrt{|c|}\ell_\kappa(a;c)/2\big)\cosh\big(\sqrt{|c|}\ell_\kappa(a;c)/2\big)\\
&\quad = \frac{1}{2}a\sqrt{|c|(|c|a^2+4)},
\end{aligned}$$

hence we get the estimate. Since $s\ell_\gamma(a,b)$ is monotone increasing with respect to b, we get the conclusion. ■

By using the comparison theorem on lengths of sector-arc, we obtain the following.

Corollary 1: Let $\gamma : [0,\infty) \to M$ be a trajectory half-line for a Kähler magnetic field $\mathbb{B}_\kappa$ on a Hadamard Kähler manifold M. Suppose sectional curvatures of planes tangent to the harp-body $\mathcal{H}_\gamma$ are not greater than a negative constant c. For arbitrary a, b with $0 \leq a < b$, we take positive $\hat{a}, \hat{b}$ so that they satisfy $\ell_\gamma(a) = \ell_\kappa(\hat{a};c)$ and $\ell_\gamma(b) = \ell_\kappa(\hat{b};c)$. When $\kappa = \pm\sqrt{|c|}$, the length of sector-arc of $\alpha_\gamma^{a,b}$ is estimated as

$$s\ell_\gamma(a,b) \leq \pi\sqrt{|c|}\hat{a}(\hat{b}-\hat{a})\Big/\Big(2\sqrt{|c|\hat{b}^2+4}\Big)$$

and $\limsup_{b\to\infty} s\ell_\gamma(a,b) \leq \pi\hat{a}/2$.

Proof: We take a trajectory half-line $\hat{\gamma}$ for $\mathbb{B}_{\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$. We then have $s\ell_\gamma(a,b) \leq s\ell_{\hat{\gamma}}(\hat{a},\hat{b})$ (see [9], [4]). We hence get our estimates. ■

We note that $\hat{a} \geq a$ by Proposition 1. Thus the above does not tell on the growth order of lengths of sector-arcs.

III. Horocycle trajectories and their limit-strings

According to Propositions 2, 3, it seems trajectories for $\mathbb{B}_{\pm\sqrt{|c|}}$ on a complex hyperbolic space $\mathbb{C}H^n(c)$ has different properties compared with those for $\mathbb{B}_\kappa$ with $|\kappa| < \sqrt{|c|}$, though all of them are unbounded in both directions. We study more on these trajectories in this section. It is known that each trajectory for $\mathbb{B}_{\pm\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$ lies on a horosphere (see [1]). We hence call it a horocycle trajectory.

We take a trajectory half-line $\gamma : [0,\infty) \to \mathbb{C}H^n(c)$ for $\mathbb{B}_\kappa$ with $\kappa = \pm\sqrt{|c|}$ and consider its trajectory-harp α_γ. We here study the distance between γ and the limit-string σ_γ of α_γ. First, we give σ_γ explecitely. For each positive a, the curve $\gamma_a : [0,a] \to \mathbb{C}H^n(c)$ defined by $\gamma_a(t) = \gamma(a-t)$ is a trajectory segment for $\mathbb{B}_{-\kappa}$. Hence we see

$$\Big\langle \dot{\gamma}(0), \frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\rangle = \delta_{-\kappa}(t:c) = 2/\sqrt{|c|t^2+4}.$$

This shows that the angle between $\dot{\gamma}(0)$ and the initial vector $\dot{\sigma}_\gamma(0)$ of the limit-string σ_γ is $\pi/2$. Since γ lies on a totally geodesic $\mathbb{C}H^1(c)$, we find $\dot{\sigma}_\gamma(0) = \pm J\dot{\gamma}(0)$.

Proposition 4: Let γ be a trajectory half-line for $\mathbb{B}_{\pm\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$. The distance from an arbitrary point $\sigma_\gamma(s)$ on the limit-string σ_γ of its trajectory-harp α_γ to γ is $d\big(\sigma_\gamma(s),\gamma\big) = s$.

Proof: Since we have $\lim_{s\to\infty}\sigma_\gamma(s) = \lim_{t\to\infty}\gamma(t) \in \partial\mathbb{C}H^n(c)$, we see γ lies on a horosphere HS which is a level set of the Busemann function defined by σ_γ and that passes through $\sigma_\gamma(0)$. For $s > 0$, the geodesic ball $B_s\big(\sigma_\gamma(s)\big)$ centered at $\sigma_\gamma(s)$ passing through $\gamma(0) = \sigma_\gamma(0)$ is contained in the horoball HB whose boundary is HS. They satisfy $B_s\big(\sigma_\gamma(s)\big)\cap HB = \{\gamma(0)\}$. Therefore, we see that the distance from $\sigma_\gamma(s)$ to γ is s which is attained as $d(\sigma_\gamma(s),\gamma(0))$. ■

Next we study the distance from $\gamma(t)$ to σ_γ. For each positive s we take the geodesic μ_s satisfying $\mu_s(0) = \sigma_\gamma(s)$ and $\dot{\mu}_s(0) = \mp J\dot{\sigma}_\gamma(s)$. It also lies on that totally geodesic $\mathbb{C}H^1(c)$ containing γ. Thus we have positive u_s and t_s satisfying $\mu_s\big(u_s\big) = \gamma(t_s)$. For each $t \geq 0$, if there is s satisfying $t_s = t$, then we find the distance $d_\gamma(t)$ from $\gamma(t)$ to σ_γ is given by u_s.

Theorem 1: Let γ be a trajectory half-line for $\mathbb{B}_{\pm\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$. The distance $d_\gamma(t)$ from a point $\gamma(t)$ to the limit-string σ_γ of the trajectory-harp α_γ is give by

$$d_\gamma(t) = \big(1/\sqrt{|c|}\big)\,\log\big(\sqrt{|c|}\,t + \sqrt{|c|t^2+1}\,\big).$$

Hence the function d_γ is unbounded, monotone increasing and concave.

Proof: First we study the case that $c = -4$ and γ is a trajectory for $\mathbb{B}_2$. Let $\varpi : H_1^{2n+1} \to \mathbb{C}H(-4)$ denote a Hopf fibration of a anti-de Sitter space $H_1^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \langle\!\langle z,z\rangle\!\rangle = -1\}$, where $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes a Hermitian form on $\mathbb{C}^{n+1}$ given by

$$\langle\!\langle z,w\rangle\!\rangle = -z_0\overline{w}_0 + z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$$

for $z = (z_0,\ldots,z_n),\ w = (w_0,\ldots,w_n) \in \mathbb{C}^{n+1}$.

We take $z \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$ and a horizontal vector $v \in T_zH_1^{2n+1} \subset T_z\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ satisfying $\varpi(z) = \gamma(0)$, $d\varpi(v) = \dot{\gamma}(0)$. Regarding v as an element of $\mathbb{C}^{n+1}$ we find that the horizontal lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = z$ is expressed as

$$\hat{\gamma}(t) = e^{\sqrt{-1}t}\big\{(1-\sqrt{-1}\,t)z + tv\big\}$$

(see [1]). Since the horizontal lift $\hat{\sigma}_\gamma$ of the limit-string σ_γ with $\hat{\sigma}_\gamma(0) = z$ is expressed as $\hat{\sigma}_\gamma(s) = \cosh sz + \sinh sJv$, we see that the horizontal lift $\hat{\mu}_s$ of μ_s with $\hat{\mu}_s(0) = \hat{\sigma}_\gamma(s)$ is given as

$$\begin{aligned}\hat{\mu}_s(u) &= \cosh u\hat{\sigma}(s) - \sinh uJ\dot{\hat{\sigma}}(s) \\ &= \big(\cosh u \cosh s - \sqrt{-1}\sinh u \sinh s\big)z \\ &\qquad + \big(\sinh u \cosh s + \sqrt{-1}\cosh u \sinh s\big)v \\ &= \frac{\sinh u \cosh s + \sqrt{-1}\cosh u \sinh s}{\sqrt{\sinh^2 u \cosh^2 s + \cosh^2 u \sinh^2 s}} \\ &\qquad \times \Bigg\{ \frac{\cosh u \sinh u}{\sqrt{\sinh^2 u \cosh^2 s + \cosh^2 u \sinh^2 s}}\, z \\ &\qquad\quad - \frac{(\cosh^2 u + \sinh^2 u)\cosh s \sinh s}{\sqrt{\sinh^2 u \cosh^2 s + \cosh^2 u \sinh^2 s}}\, \sqrt{-1}z \\ &\qquad\quad + \sqrt{\sinh^2 u \cosh^2 s + \cosh^2 u \sinh^2 s}\, v \Bigg\}.\end{aligned}$$

As $\mu_s(u_s) = \gamma(t_s)$ there is a real θ with $e^{\sqrt{-1}\theta}\hat{\mu}_s(u_s) = \hat{\gamma}(t_s)$. By comparing these expressions of $\hat{\gamma}$ and $\hat{\mu}_s$ we obtain

$$t_s = \sqrt{\sinh^2 u_s \cosh^2 s + \cosh^2 u_s \sinh^2 s} \tag{III.1}$$

$$1 = \frac{\cosh u_s \sinh u_s}{\sqrt{\sinh^2 u_s \cosh^2 s + \cosh^2 u_s \sinh^2 s}}, \tag{III.2}$$

$$t_s = \frac{\{\cosh^2 u_s + \sinh^2 u_s\}\cosh s \sinh s}{\sqrt{\sinh^2 u_s \cosh^2 s + \cosh^2 u_s \sinh^2 s}}. \tag{III.3}$$

By (III.2) we get $\cosh^2 u_s = (\cosh 2s + \sinh 2s + 1)/2$, which is equivalent to $\cosh 2u_s = \cosh 2s + \sinh 2s = e^{2s}$. Thus we obtain

$$u_s = \frac{1}{2}\log\{e^{2s} + \sqrt{e^{4s}-1}\} \quad \text{and} \quad t_s = \frac{1}{2}\sqrt{e^{4s}-1},$$

which show

$$d_\gamma(t) = \frac{1}{2}\log\big(2t + \sqrt{4t^2+1}\big).$$

Clearly, this function is monotone increasing and is concave.

To study the general case, we consider a homothetic change of metrics. When we change the metric $\langle\ ,\ \rangle$ on a Kähler manifold homothetically to a metric $\lambda^2\langle\ ,\ \rangle$ with some positive λ, then for a trajectory γ for $\mathbb{B}_\kappa$ with respect to the original metric we see that the curve $\tilde{\gamma}(t) = \gamma(t/\lambda)$ is a trajectory for $\mathbb{B}_{\kappa/\lambda}$ with respect to the new metric. We note that sectional curvatures change λ^{-2}-times of the original sectional curvatures. Let γ be a trajectory for $\mathbb{B}_{\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$. Considering the homothetic change of metric with $\lambda = \sqrt{|c|}/2$, we find that $\tilde{\gamma}(t) = \gamma(t/\lambda)$ is a trajectory for $\mathbb{B}_2$ on $\mathbb{C}H^n(-4)$. Thus we have

$$d_\gamma(t) = \frac{2}{\sqrt{|c|}}\,\tilde{d}_{\tilde{\gamma}}(\lambda t) = \frac{1}{\sqrt{|c|}}\ \log\big(\sqrt{|c|}\,t + \sqrt{|c|t^2+1}\big).$$

For trajectories for $\mathbb{B}_{-\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$, we have the same computations. This complete the proof. ■

This theorem shows that even each horocycle trajectory and its limit-string have the same limit points at infinity their distance function grows logarithmically.

We define a variation of geodesics $\mu : \mathbb{R}\times[0,\infty) \to \mathbb{C}H^n(c)$ by $\mu(s,u) = \mu_s(u)$. Here, μ_0 denotes the geodesic of initial vector $\dot{\gamma}(0)$. By definition, we have $\mu(s,0) = \sigma_\gamma(s)$ and $\mu(s,u_s) = \gamma(t_s)$. To study more on the relationship between trajectories and their limit-strings, we set $\eta_\gamma(t) = \big\langle \dot{\gamma}(t), \frac{\partial\mu}{\partial u}\big(s(t), u_{s(t)}\big)\big\rangle$, where $s(t)$ denotes the inverse function of the function $s \mapsto t_s$. We shall call the angle $\cos^{-1}\eta_\gamma(t)$ between $\dot{\gamma}(t)$ and $\frac{\partial\mu}{\partial u}\big(s(t), u_{s(t)}\big)$ the *arch-angle* of a trajectory-harp α_γ at t. Setting $\tilde{\mu}$ by $\tilde{\mu}(s,u) = \mu(s, u_s u)$, we have $u_s^2 = \int_0^1 \big\|\frac{\partial\tilde{\mu}}{\partial u}(s,u)\big\|^2 du$. Hence we find

$$\begin{aligned}u_s\frac{du_s}{ds} &= \int_0^1 \Big\langle \frac{\partial\tilde{\mu}}{\partial u}(s,u), \Big(\nabla_{\frac{\partial\tilde{\mu}}{\partial s}}\frac{\partial\tilde{\mu}}{\partial u}\Big)(s,u)\Big\rangle\, du \\ &= \int_0^1 \frac{d}{du}\Big\langle \frac{\partial\tilde{\mu}}{\partial u}(s,u), \frac{\partial\tilde{\mu}}{\partial s}(s,u)\Big\rangle\, du \\ &= \Big\langle \frac{\partial\tilde{\mu}}{\partial u}(s,1), \frac{\partial\tilde{\mu}}{\partial s}(s,1)\Big\rangle - \Big\langle \frac{\partial\tilde{\mu}}{\partial u}(s,0), \frac{\partial\tilde{\mu}}{\partial s}(s,0)\Big\rangle \\ &= u_s\frac{dt_s}{ds}(s)\Big\langle \frac{\partial\mu}{\partial u}(s,u_s), \dot{\gamma}(t_s)\Big\rangle \\ &= u_s\frac{dt_s}{ds}(s)\,\eta_\gamma(t_s).\end{aligned}$$

Since $d_\gamma(t) = u_{s(t)}$, we obtain

$$\eta_\gamma(t) = d'_\gamma(t) = 1/\sqrt{|c|t^2+1}.$$

Therefore we have the following:

Theorem 2: For a trajectory hal-line γ for $\mathbb{B}_{\pm\sqrt{|c|}}$ on $\mathbb{C}H^n(c)$, we have $\eta_\gamma(t) = 1/\sqrt{|c|t^2+1}$. Hence the arch-angle $\cos^{-1}\eta_\gamma(t)$ is monotone increasing and satisfies $\lim_{t\to\infty}\cos^{-1}\eta_\gamma(t) = \pi/2$.

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