Semi-symmetric Metric Connection On Doubly Warped Product Manifolds

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***Abstract*—**We find relations between Levi-Civita connection and semi-symmetric metric connection of a doubly warped product manifold $M={}_f B\times_b F$. We also obtain some results of Einstein doubly warped product manifolds with respect to a semi-symmetric metric connection.

***Keywords*—**Doubly warped product manifold, semi-symmetric metric connection, Einstein manifold.

I. INTRODUCTION

THE idea of a semi-symmetric linear connection on a Riemannian manifold was introduced by A. Friedmann and J. A. Schouten in [1]. Later, H. A. Hayden [4] gave the definition of a semi-symmetric metric connection. In 1970, K. Yano [15] considered semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold admitting the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. Then, the generalization of this result for vanishing Ricci tensor of the semi-symmetric metric connection was given by T. Imai ([5], [6]).

On the other hand, doubly warped product submanifolds are studied by several authors in ([2], [8] and [13]).

Motivated by the above studies, we study doubly warped product manifolds with a semi-symmetric metric connection and find relations between the Levi-Civita connection and the semi-symmetric metric connection.

Furthermore, in [3], A. Gebarowski studied Einstein warped product manifolds. As an application, in this study we consider Einstein doubly warped product manifolds endowed with a semi-symmetric metric connection.

There are also various studies on doubly warped product manifolds as [2], [10], [11]. We have examined these studies and have comparisons of the features of doubly warped product manifolds endowed with Levi-Civita connection and semi-symmetric metric connection.

II. SEMI-SYMMETRIC METRIC CONNECTION

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g. A linear connection ∇ on a Riemannian manifold M is called a *semi-symmetric connection* if the torsion tensor T of the connection $\overset{\circ}{\nabla}$

$$T(X,Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y] \tag{1}$$

satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2}$$

where π is a 1-form associated with the vector field P on M defined by

$$\pi(X) = g(X,P). \tag{3}$$

$\overset{\circ}{\nabla}$ is called a *semi-symmetric metric connection* if it satisfies

$$\overset{\circ}{\nabla} g=0.$$

If ∇ is the Levi-Civita connection of a Riemannian manifold M, the semi-symmetric metric connection $\overset{\circ}{\nabla}$ is given by

$$\overset{\circ}{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P, \tag{4}$$

(see [15]).

Let R and $\overset{\circ}{R}$ be curvature tensors of ∇ and $\overset{\circ}{\nabla}$ of a Riemannian manifold M, respectively. Then R and $\overset{\circ}{R}$ are related by

$$\begin{aligned}\overset{\circ}{R}(X,Y)Z &= R(X,Y)Z+g(Z,\nabla_X P)Y- g(Z,\nabla_Y P)X \\ &+g(X,Z)\nabla_Y P - g(Y,Z)\nabla_X P \\ &+\pi(P)[g(X,Z)Y- g(Y,Z)X] \\ &+[g(Y,Z)\pi(X) - g(X,Z)\pi(Y)]P \\ &+\pi(Z)[\pi(Y)X - \pi(X)Y],\end{aligned} \tag{5}$$

for any vector fields X,Y,Z on M [15]. For a general survey of different kinds of connections see also [12].

III. DOUBLY WARPED PRODUCT MANIFOLDS

Let (B,g_B) and (F,g_F) be two Riemannian manifolds and $b:B\to(0,\infty)$ and $f:F\to(0,\infty)$ smooth functions. Consider the product manifold $B\times F$ with its projections $\pi:B\times F\to B$ and $\sigma:B\times F\to F$. The *doubly warped product* ${}_f B\times_b F$ is the manifold $B\times F$ with the Riemannian structure such that

$$g = (f\circ\sigma)^2\pi_*(g_B)\oplus(b\circ\pi)^2\sigma_*(g_F),$$

which implies that

$$g = f^2 g_B + b^2 g_F \tag{6}$$

The functions $b:B\to(0,\infty)$ and $f:F\to(0,\infty)$ are called *warping functions* of the doubly warped product [7].

We need the following three lemmas from [7], for the later use :

Lemma 3.1: Let us consider $M={}_f B\times_b F$ and denote by ∇, ${}^B\nabla$ and ${}^F\nabla$ the Riemannian connections on M, B and F, respectively. If X,Y are vector fields on B and V,W on F, then:

(i) $\nabla_X Y = {}^B\nabla_X Y - (1/(fb^2))g(X,Y)\mathrm{grad}_F f$,

(ii) $\nabla_X V = \nabla_V X = ((V(f))/f)X+((X(b))/b)V$,

(iii) $\nabla_V W = {}^F\nabla_X Y - (1/(bf^2))g(V,W)\mathrm{grad}_B b$.

Lemma 3.2: Let $M={}_f B\times_b F$ be a doubly warped product, with Riemannian curvature ${}^MR$. Given fields X,Y,Z on B and U,V,W on F, then:

(i) ${}^MR(X,Y)Z = {}^BR(X,Y)Z$
$+(1/(fb^3))[g(Y,Z)X(b) - g(X,Z)Y(b)]\ \mathrm{grad}_F f$
$- (1/(b^2))[g_B(Y,Z)X - g_B(X,Z)Y](\mathrm{grad}_F f)(f)$,

(ii) ${}^MR(X,V)Y = ((H_B{}^b(X,Y))/b)V- ((H^{f\circ\sigma}(Y,V))/f)X$
$+((g_B(X,Y))/b)[((f^F)/b)\nabla_V \mathrm{grad}_F f-((V(f))/f)\mathrm{grad}_B b]$,

(iii) ${}^MR(X,Y)V = ((H^{f\circ\sigma}(Y,V))/f)X+((H^{f\circ\sigma}(X,V))/f)Y$,

(iv) ${}^MR(V,W)X =-((H^{b\circ\pi}(X,W))/b)V+((H^{b\circ\pi}(X,V))/b)W$,

(v) ${}^MR(X,V)W = -((H_F{}^f(V,W))/f)X+((H^{b\circ\pi}(X,W))/b)V$
$- ((g_F(V,W))/f)[((b^B)/f)\nabla_X \mathrm{grad}_B b-((X(b))/b)\mathrm{grad}\ f]$

(vi) ${}^MR(V,W)U = {}^FR(V,W)U$
$+(1/(bf^3))[g(V,W)U(f)-g(U,W)V(f)]\mathrm{grad}_B b$
$-(1/(f^2))[g_F(V,W)U-g_F(U,W)V](\mathrm{grad}_B b)(b)$.

Lemma 3.3: Let $M={}_f B\times_b F$ be a doubly warped product with Ricci tensor ${}^MS$. Given fields X,Y on B and V,W on F, then:

(i) ${}^MS(X,Y) = {}^BS(X,Y)$
$-(1/(b^2))[(r-1)(\mathrm{grad}_F f)(f)+f\Delta_F(f)]g_B(X,Y)$
$-(s/b)H_B{}^b(X,Y)$,

where r = dimB and s = dimF,

(ii) ${}^MS(X,V) = (n-2)((V(f)X(b))/(fb))$,

(iii) ${}^MS(V,W) = {}^FS(V,W)$
$-(1/(f^2))[(s-1)(\mathrm{grad}_B b)(b)+b\Delta_B(b)]g_F(V,W)$
$-(r/f)H_F{}^f(V,W)$.

Moreover, the scalar curvature ${}^M\tau$ of M satisfies the condition

$$\begin{aligned}{}^M\tau = {} & ({}^B\tau)/(b^2)+({}^B\tau)/(f^2)\\ & - 2s((\Delta_B(b))/(bf^2))-2r((\Delta_F(f))/(fb^2))\\ & - s(s-1)(((\mathrm{grad}_B b)(b))/(f^2b^2))\\ & - r(r-1)(((\mathrm{grad}_F f)(f))/(f^2b^2)),\end{aligned} \tag{7}$$

where ${}^B\tau$ and ${}^F\tau$ are scalar curvatures of B and F, respectively.

IV. DOUBLY WARPED PRODUCT MANIFOLDS WITH A SEMI-SYMMETRIC METRIC CONNECTION

In this section, we consider doubly warped product manifolds with respect to the semi-symmetric metric connection and find new expressions concerning with curvature tensor, Ricci tensor and the scalar curvature admitting this connection where the associated vector field $P\in\chi(M)$ that

$$P = P_B+P_F\ , \tag{8}$$

where P_B (resp. P_F) is the component of P on B (resp. on F).

Now, let begin with the following lemma:

Lemma 4.1: Let us consider $M={}_f B\times_b F$ and denote by ∇ the semi-symmetric metric connection on M, ${}^M\nabla$ and ${}^F\nabla$ be connections on B and F, respectively. If $X,Y\in\chi(B)$, $V,W\in\chi(F)$, then:

(i) ${}^\circ\nabla_X Y = {}^{B\circ}\nabla_X Y- (1/(fb^2))g(X,Y)(\mathrm{grad}_F f)(f)$
$- g(X,Y)P_F$,

(ii) ${}^\circ\nabla_X V = ((V(f))/f)X+((X(b))/b)V+\pi(V)X$,

(iii) ${}^\circ\nabla_V X = ((V(f))/f)X+((X(b))/b)V+\pi(X)V$,

(iv) ${}^\circ\nabla_V W = {}^{F\circ}\nabla_X Y-(1/(bf^2))g(V,W)(\mathrm{grad}_B b)(b)$
$-g(V,W)P_B$.

Proof : In view of the Koszul formula from [7] we can write

$$2g(\nabla_X Y,Z) = Xg(Y,Z)+Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z])+g(Z,[X,Y]), \quad (9)$$

for all vector fields X,Y,Z on M, where ∇ is the Levi-Civita connection of M. By the use of (4) for the semi-symmetric metric connection, the equation (9) turns into

$$2g({}^\circ\nabla_X Y,V) = Xg(Y,V)+Yg(X,V) -Vg(X,Y) - g(X,[Y,V]) - g(Y,[X,V])+g(V,[X,Y]) +2\pi(Y)g(X,V) - 2\pi(V)g(X,Y), \quad (10)$$

for any vector fields $X,Y\in\chi(B)$ and $V\in\chi(F)$.

Since X,Y and [X,Y] are lifts from B and V is vertical, we know from [7] we can write

$$g(Y,V)=g(X,V)=0 \quad (11)$$

and

$$[X,V]=[Y,V]=0. \quad (12)$$

Hence, the equation (10) reduces to

$$2g({}^\circ\nabla_X Y,V) =-Vg(X,Y)-2\pi(V)g(X,Y). \quad (13)$$

By the definition of the doubly warped product metric from (6), we have

$$g(X,Y) = (f\circ \sigma)^2 g_B(X,Y).$$

Then by making use of the function f instead of $(f\circ \sigma)$, we get

$$g(X,Y) = f^2(g_B(X,Y)\circ \pi).$$

Hence, we can write

$$Vg(X,Y) = V[f^2(g_B(X,Y)\circ \pi)] = 2fV(f)(g_B(X,Y)\circ \pi)+f^2V(g_B(X,Y)\circ \pi).$$

Since the term $(g_B(X,Y)\circ \pi)$ is constant on fibers, by the use of (6), the above equation turns into

$$Vg(X,Y) = 2((V(f))/f)g(X,Y). \quad (14)$$

By making use of (14) in (13), we obtain

$$g({}^\circ\nabla_X Y,V) = -[((V(f))/f)+\pi(V)]g(X,Y). \quad (15)$$

Since $V(f) = (1/(b^2))g(\mathrm{grad}_F f,V)$ on F, by making use of (6) and (8) in (15) we get (i).

By the use of the definition of the covariant derivative with respect to the semi-symmetric metric connection, we can write

$$g({}^\circ\nabla_X V,Y) = Xg(Y,V) - g(V, {}^\circ\nabla_X Y),$$

for all vector fields X,Y on B and V on F. By making use of (11) and (15), the above equation turns into

$$g({}^\circ\nabla_X V,Y) = [((V(f))/f)+\pi(V)]g(X,Y). \quad (16)$$

On the other hand, from Koszul formula and the definition of and the semi-symmetric metric connection we can write

$$2g({}^\circ\nabla_X V,W) = Xg(V,W)+Vg(X,W) - Wg(X,V) - g(X,[V,W]) - g(V,[X,W])+g(W,[X,V]) +2\pi(V)g(X,W) - 2\pi(W)g(X,V),$$

for any vector fields X on B and V,W on F. In view of (11) and (12), the last equation reduces to

$$2g({}^\circ\nabla_X V,W) = Xg(V,W) - g(X,[V,W]).$$

Since X is horizontal and [V,W] is vertical, $g(X,[V,W])=0$, thus we obtain

$$2g({}^\circ\nabla_X V,W) = Xg(V,W). \quad (17)$$

By the use of the equation (6), we have

$$g(V,W) = (b\circ \pi)^2 g_F(V,W).$$

Then by making use of the function b instead of $(b\circ \pi)$, we get

$$g(V,W) = b^2(g_F(V,W)\circ \sigma).$$

Hence, we can write

$$Xg(V,W) = X[b^2(g_F(V,W)\circ \sigma)] = 2bX(b)(g_F(V,W)\circ \sigma)+b^2X(g_F(V,W)\circ \sigma).$$

Since the term $(g_F(V,W)\circ \sigma)$ is constant on leaves, by the use of (6), the above equation turns into

$$Xg(V,W) = 2((X(b))/b)g(V,W). \quad (18)$$

By making use of (18) in (17), we obtain

$$g({}^\circ\nabla_X V,W) = ((X(b))/b)g(V,W). \quad (19)$$

Then in view of the equations (16) and (19), we get (ii).

Now, by the use of (1) we can write

$${}^\circ\nabla_V X = {}^\circ\nabla_X V - [X,V] - T(X,V).$$

Using (2) and (12), the above equation reduces to

$${}^\circ\nabla_V X = {}^\circ\nabla_V V- \pi(V)X+\pi(X)V. \quad (20)$$

By virtue of the equation (ii), we get

$$ {}^{\circ}\nabla_V X = ((X(b))/b)V+((V(f))/f)X+\pi(X)V. \qquad (21)$$

Hence we obtain (iii).

On the other hand, by the definition of the covariant derivative with respect to the semi-symmetric metric connection, we can write

$$Vg(X,W) = g({}^{\circ}\nabla_V X,W)+g({}^{\circ}\nabla_V W,X),$$

for any vector fields X on B and V,W on F. From (12), the above equation reduces to

$$g({}^{\circ}\nabla_V W,X) = -g({}^{\circ}\nabla_V X,W). \qquad (22)$$

By the use of (22), we get

$$g({}^{\circ}\nabla_V W,X) = -[((X(b))/b)+\pi(X)]g(V,W),$$

which implies that

$${}^{\circ}\nabla_V W = {}^{F\circ}\nabla_V W- (1/(bf^2))g(V,W)grad_B b - g(V,W)P_B,$$

where $X(b) = (1/(f^2))g(grad_B b,X)$ for any vector field X on B. Thus, the proof of the lemma is completed.

Lemma 4.2 : Let $M={}_f B\times_b F$ be a doubly warped product and R and ${}^{\circ}R$ denote the Riemannian curvature tensors of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively . If $X,Y,Z \in\chi(B)$ and $U,V,W\in\chi(F)$, then:

(i) $${}^{B\circ}R(X,Y)Z = {}^{B}R(X,Y)Z - (1/(b^2))[g_B(Y,Z)X-g_B(X,Z)Y](grad_F f)(f) +g(Z,{}^{B}\nabla_X P_B)Y-g(Z,{}^{B}\nabla_Y P_B)X +2((P_F(f))/f)[g(X,Z)Y-g(Y,Z)X] +g(X,Z){}^{B}\nabla_Y P_B-g(Y,Z){}^{B}\nabla_X P_B +\pi(P)[g(X,Z)Y-g(Y,Z)X] +[g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]P_B +\pi(Z)[\pi(Y)X-\pi(X)Y],$$

(ii) $${}^{F\circ}R(X,Y)Z = (1/(fb^2))[g(Y,Z)((X(b))/b) - g(X,Z)((Y(b))/b) +g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]grad_F f - [g(Y,Z)((X(b))/b)-g(X,Z)((Y(b))/b) - g(Y,Z)\pi(X)+g(X,Z)\pi(Y)]P_F,$$

(iii) $${}^{B\circ}R(V,X)Y = ((H^{f\circ\sigma}(Y,V))/f)X +((V(f))/(fb))g_B(X,Y)grad_B b +((V(f))/f)[\pi(Y)X-g(X,Y)P_B] - ((Y(b))/b)\pi(V)X+(1/(bf^2))g(X,Y)P_B +g(X,Y)\pi(V)P_B-\pi(Y)\pi(V)X,$$

(iv) $${}^{F\circ}R(V,X)Y = ((H_B{}^{b}(X,Y))/b)V - (f/(b^2))g_B(X,Y){}^{F}\nabla_V grad_F f - g(Y,{}^{B}\nabla_X P_B)V - [((P_F(f))/f)+((P_B))/b)]g(X,Y)V - g(X,Y){}^{F}\nabla_V P_F-g(X,Y)\pi(P)V +g(X,Y)\pi(V)P_F+\pi(X)\pi(Y)V,$$

(v) $${}^{B\circ}R(X,Y)V = -((H^{f\circ\sigma}(Y,V))/f)X+((H^{f\circ\sigma}(X,V))/f)Y - ((V(f))/f)[\pi(X)Y-\pi(Y)X] +[((X(b))/b)Y-((Y(b))/b)X]\pi(V) - [\pi(X)Y-\pi(Y)X]\pi(V),$$

(vi) $${}^{F\circ}R(X,Y)V=0,$$

(vii) $${}^{B\circ}R(V,W)X=0,$$

(viii) $${}^{F\circ}R(V,W)X=((H^{b\circ\pi}(X,W))/b)V+((H^{b\circ\pi}(X,V))/b)W - ((X(b))/b)[\pi(V)W-\pi(W)V] +\pi(X)[((V(f))/f)W-((W(f))/f)V] - \pi(X)[\pi(V)W-\pi(W)V],$$

(ix) $${}^{B\circ}R(X,V)W = - ((H_F{}^{f}(V,W))/f)X - (b/(f^2))g_F(V,W){}^{B}\nabla_X grad_B b - [((P_B(b))/b)+((P_F(f))/f)]g(V,W)X - g(W,{}^{F}\nabla_V P_F)X-g(V,W){}^{B}\nabla_X P_B - g(V,W)\pi(P)X+g(V,W)\pi(X)P_B +\pi(V)\pi(W)X,$$

(x) $${}^{F\circ}R(X,V)W = ((H^{b\circ\pi}(X,W))/b)V +((X(b))/(bf))g_F(V,W)grad_F f - ((W(f))/f)\pi(X)V+((X(b))/b)\pi(W)V +(1/(fb^2))g(V,W)\pi(X)grad_F f - ((X(b))/b)g(V,W)P_F +g(V,W)\pi(X)P_F - \pi(X)\pi(W)V,$$

(xi) $${}^{B\circ}R(U,V)W = (1/(bf^2))[g(V,W)((U(f))/f) - g(U,W)((V(f))/f) +g(V,W)\pi(U)-g(U,W)\pi(V)]grad_B b - [g(V,W)((U(f))/f)-g(U,W)((V(f))/f) -g(V,W)\pi(U)-g(U,W)\pi(V)]P_B,$$

(xii) $${}^{F\circ}R(U,V)W = {}^{F}R(U,V)W$$

$$-(1/(f^2))[g_F(V,W)U-g_F(U,W)V](grad_B b)(b)$$
$$+g(W,{}^F\nabla_U P_F)V-g(W,{}^F\nabla_V P_F)U$$
$$+2((P_B(b))/b)[g(U,W)V-g(V,W)U]$$
$$+g(U,W){}^F\nabla_V P_F-g(V,W){}^F\nabla_U P_F$$
$$+\pi(P)[g(U,W)V-g(V,W)U]$$
$$+[g(V,W)\pi(U)-g(U,W)\pi(V)]P_F$$
$$+[\pi(V)U-\pi(U)V]\pi(W).$$

Proof : Assume that $M={}_f B\times_b F$ is a doubly warped product and R and $^{\circ}R$ denote the curvature tensors of the Levi-Civita connection and the semi-symmetric metric connection, respectively.
By the use of equation (5), we have

$$\begin{aligned} {}^{\circ}R(X,Y)Z &= R(X,Y)Z+g(Z,\nabla_X P)Y-g(Z,\nabla_Y P)X \\ &\quad - g(X,Z)\nabla_Y P+g(Y,Z)\nabla_X P \\ &\quad +\pi(P)[g(X,Z)Y-g(Y,Z)X] \\ &\quad +[g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]P \\ &\quad +\pi(Z)[\pi(Y)X-\pi(X)Y], \qquad (23) \end{aligned}$$

for any vector fields X,Y,Z on B.
In view of the equation (8), Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} {}^{\circ}R(X,Y)Z &= {}^B R(X,Y)Z \\ &\quad +(1/(fb^2))[g(Y,Z)((X(b))/b)-g(X,Z)((Y(b))/b) \\ &\quad +g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]grad_F f \\ &\quad - (1/(b^2))[g_B(Y,Z)X-g_B(X,Z)Y](grad_F f)(f) \\ &\quad +g(Z,{}^B\nabla_X P_B)Y-g(Z,{}^B\nabla_Y P)X \\ &\quad +2((P_F(f))/f)[g(X,Z)Y-g(Y,Z)X] \\ &\quad +g(X,Z){}^B\nabla_Y P_B-g(Y,Z){}^B\nabla_X P_B \\ &\quad +\pi(P)[g(X,Z)Y-g(Y,Z)X] \\ &\quad +[g(Y,Z)\pi(X)-g(X,Z)\pi(Y)]P_B \\ &\quad +\pi(Z)[\pi(Y)X-\pi(X)Y] \\ &\quad - [g(Y,Z)((X(b))/b)-g(X,Z)((Y(b))/b) \\ &\quad - g(Y,Z)\pi(X)+g(X,Z)\pi(Y)]P_F, \end{aligned}$$

which gives us (i) and (ii).
In view of the equation (5), we can write

$$\begin{aligned} {}^{\circ}R(V,X)Y &= R(V,X)Y+g(Y,\nabla_V P)X-g(Y,\nabla_X P)V \\ &\quad - g(X,Y)[\nabla_V P+\pi(P)V-\pi(V)P] \\ &\quad +\pi(Y)[\pi(X)V-\pi(V)X], \qquad (24) \end{aligned}$$

for all vector fields $X,Y\in\chi(B)$ and $V\in\chi(F)$, respectively. By making use of the equation (8), Lemma 3.1 and Lemma 3.2 again we obtain (iii) and (iv).
Putting Z = V in equation (5), we get

$$\begin{aligned} {}^{\circ}R(X,Y)V &= R(X,Y)V+g(V,\nabla_X P)Y-g(V,\nabla_Y P)X \\ &\quad +\pi(V)[\pi(Y)X-\pi(X)Y], \end{aligned}$$

where $X,Y\in\chi(B)$ and $V\in\chi(F)$. By virtue of the equation (8), Lemma 3.1 and Lemma 3.2, the above equation can be written as follows

$$\begin{aligned} {}^{\circ}R(X,Y)V &= - ((H^{f\circ\sigma}(Y,V))/f)X+((H^{f\circ\sigma}(X,V))/f)Y \\ &\quad - ((V(f))/f)[\pi(X)Y-\pi(Y)X] \\ &\quad +[((X(b))/b)Y-((Y(b))/b)X]\pi(V) \\ &\quad - [\pi(X)Y-\pi(Y)X]\pi(V), \end{aligned}$$

which shows us (v) and (vi).
By making use of (8) and Lemma 3.2, we can write

$$\begin{aligned} {}^{\circ}R(V,W)X &= R(V,W)X+g(X,\nabla_V P)W-g(X,\nabla_W P)V \\ &\quad +\pi(X)[\pi(W)V-\pi(V)W], \end{aligned}$$

for any vector fields X on B and V,W on F, respectively. Similarly using (8), Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} {}^{\circ}R(V,W)X &= - ((H^{b\circ\pi}(X,W))/b)V+((H^{b\circ\pi}(X,V))/b)W \\ &\quad - ((X(b))/b)[\pi(V)W-\pi(W)V] \\ &\quad +\pi(X)[((V(f))/f)W-((W(f))/f)V] \\ &\quad - \pi(X)[\pi(V)W-\pi(W)V]. \end{aligned}$$

Hence we obtain (vii) and (viii).
From the equation (5), we have

$$\begin{aligned} {}^{\circ}R(X,V)W &= R(X,V)W+g(W,\nabla_X P)V-g(W,\nabla_V P)X \\ &\quad - g(V,W)[\nabla_X P+\pi(P)X-\pi(X)P] \\ &\quad +\pi(W)[\pi(V)X-\pi(X)V], \end{aligned}$$

for all vector fields $X\in\chi(B)$ and $V,W\in\chi(F)$. In view of Lemma 3.1 and Lemma 3.2 and by the use of (8), we obtain (ix) and (x), respectively.
In view of the equation (5) again, we have

$$\begin{aligned} {}^{\circ}R(U,V)W &= R(U,V)W+g(W,\nabla_U P)V-g(W,\nabla_V P)U \\ &\quad +g(U,W)\nabla_V P-g(V,W)\nabla_U P \\ &\quad +\pi(P)[g(U,W)V-g(V,W)U] \\ &\quad +[g(U,W)\pi(U)-g(V,W)\pi(V)]P \\ &\quad +\pi(W)[\pi(V)U-\pi(U)V], \end{aligned}$$

for any vector fields U,V,W on F. Similarly by making use of (8), Lemma 3.1 and Lemma 3.2 we get (xi) and (xii). Hence, the proof of the lemma is completed.

As a consequence of Lemma 4.2, by a contraction of the curvature tensors we obtain the Ricci tensors of doubly warped product manifold with respect to the semi-symmetric metric connection as follows:

Corollary 4.3: Let $M={}_f B\times_b F$ be a doubly warped product and S and $^{\circ}S$ denote the Ricci tensors of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively, where dimB=r and dimF=s. If $X,Y\in\chi(B)$, $V,W\in\chi(F)$, then:

(i) $\overset{\circ}{S}(X,Y) = {}^{B}\overset{\circ}{S}(X,Y) -((r-1))/(b^2f^2))g(X,Y)(grad_F f)(f)$
$- \sum[g(Y,{}^{B}\nabla_{ei}P_B)g(X,e_i) - g(X,Y)g({}^{B}\nabla_{ei}P_B,e_i)]$
$- (n-1)g(Y,{}^{B}\nabla_X P_B)-s((H_B{}^{b}(X,Y))/b)$
$- [((\Delta_F(f))/(fb^2))+(2r+s-2)((P_F(f))/f)$
$+(n-2)\pi(P)+s((P_B(b))/b)]g(X,Y) +(n-2)\pi(X)\pi(Y)$
$- \sum g(X,Y)g({}^{F}\nabla_{ei}P_F,e_i)$,

(ii) $\overset{\circ}{S}(X,V) = -(r-1)((H^{f\circ\ \sigma}(X,V))/f -(s-1)((H^{b\circ\ \pi}(X,V))/b)$
$+(n-2)((V(f))/f)\pi(X)-(n-2)((X(b))/b)\pi(V)$
$+(n-2)\pi(V)\pi(X)$,

(iii) $\overset{\circ}{S}(V,X) =-(r-1)((H^{f\circ\ \sigma}(X,V))/f)-(s-1)((H^{b\circ\ \pi}(X,V))/b)$
$- (n-2)((V(f))/f)\pi(X)+(n-2)((X(b))/b)\pi(V)$
$+(n-2)\pi(V)\pi(X)$,

(iv) $\overset{\circ}{S}(V,W) ={}^{F}\overset{\circ}{S}(V,W)- ((s-1))/(b^2f^2))g(V,W)(grad_B b)(b)$
$- \sum[g(W,{}^{F}\nabla_{ei}P_F)g(V,e_i)-g(V,W)g({}^{F}\nabla_{ei}P_F,e_i)]$
$- (n-1)g(W,{}^{F}\nabla_V P_F)-r((H_F{}^{f}(V,W))/f)$
$- [((\Delta_B(b))/(bf^2))+(2s+r-2)((P_B(b))/b)$
$+(n-2)\pi(P)+r((P_F(f))/f)]g(V,W)+(n-2)\pi(V)\pi(W)$
$-\sum g(V,W)g({}^{B}\nabla_{ei}P_B,e_i)$.

As a consequence of Corollary 4.3, by a contraction of the Ricci tensors we get scalar curvatures of doubly warped product with respect to the semi-symmetric metric connection as follows:

Corollary 4.4 : Let $M={}_f B\times_b F$ be a doubly warped product and τ and $\overset{\circ}{\tau}$ denote the scalar curvatures of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. Then, we have

$$\overset{\circ}{\tau} = ({}^{B\circ}\tau)/(f^2))+(({}^{F\circ}\tau)/(b^2)) - ((r(r-1))/(b^2f^2))(grad_F f)(f)$$
$$- ((s(s-1))/(b^2f^2))(grad_B b)(b)$$
$$- 2(n-1)\sum g({}^{B}\nabla_{ei}P_B,e_i) - 2(n-1)\sum g({}^{F}\nabla_{ei}P_F,e_i)$$
$$- 2s(n-1)((P_B(b))/b) - 2r(n-1)((P_F(f))/f)-(n-1)(n-2)\pi(P)$$
$$- (r/f)[1+(1/(b^2))]\Delta_F(f) - (s/b)[1+(1/(f^2))]\Delta_B(b).$$

V. EINSTEIN DOUBLY WARPED PRODUCT MANIFOLDS ENDOWED WITH THE SEMI-SYMMETRIC METRIC CONNECTION

In this section we consider Einstein doubly warped products endowed with the semi-symmetric metric connection.

Now, let us begin with the following theorem:

Theorem 5.1: Let (M,g) be a doubly warped product ${}_f I\times_b F$, where dimI=1 and dimF=n-1 ($n\geq 3$). Then (M,g) is an Einstein manifold with respect to the semi-symmetric metric connection, $P_F\in\chi(F)$ is parallel on F with respect to the Levi-Civita connection and f is constant on F, then b is constant on I and F is a quasi-Einstein manifold with respect to the Levi-Civita connection.

Proof : Let denote by g_I the metric on I. By making use of Corollary 4.3, we can write

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = -(n-2)f^2\pi(P)+(n-2)f^4 - (n-1)((b'')/b)-(n-1)((b')/b), \quad (25)$$

$$\overset{\circ}{S}((\partial/(\partial t)),V) = -(n-2)((H^{b\circ\ \pi}((\partial/(\partial t)),V))/b) - (n-2)[((b')/b)-f^2]\pi(V), \quad (26)$$

$$\overset{\circ}{S}(V,(\partial/(\partial t))) = -(n-2)((H^{b\circ\ \pi}((\partial/(\partial t)),V))/b) +(n-2)[((b')/b)+f^2]\pi(V) \quad (27)$$

and

$$\overset{\circ}{S}(V,W) = {}^{F}S(V,W)-(n-2)g(V,W)\pi(P) +(n-2)\pi(V)\pi(W), \quad (28)$$

for any vector fields V,W on F.

Since M is an Einstein manifold with respect to the semi-symmetric metric connection, we have

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = \alpha g((\partial/(\partial t)),(\partial/(\partial t))), \quad (29)$$

$$\overset{\circ}{S}((\partial/(\partial t)),V) = \overset{\circ}{S}(V,(\partial/(\partial t))) = \alpha g(V,(\partial/(\partial t))) \quad (30)$$

and

$$\overset{\circ}{S}(V,W) = \alpha g(V,W). \quad (31)$$

Comparing the right hand sides of the equations (26) and (27) and by the use of (30), we get

$$2(n-2)((b')/b)\pi(V)=0,$$

which gives us $b'=0$ ($n\geq 3$). So, b is constant on I.

On the other hand by making use of (6), the equations (29) and (31) reduce to

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = \alpha f^2 \quad (32)$$

and

$$\overset{\circ}{S}(V,W) = \alpha b^2 g_F(V,W). \quad (33)$$

Comparing the right hand sides of (25) and (32), we get

$$\alpha = (n-2)[f^2-\pi(P)]. \quad (34)$$

Similarly, comparing the right hand sides of (28) and (31) and by the use of (34), we obtain

$$^{F}S(V,W) = (n-2)b^2f^2g_F(V,W)-(n-2)\pi(V)\pi(W),$$

which implies that F is a quasi-Einstein manifold with respect to the Levi-Civita connection. Thus, the proof of the theorem is completed.

Theorem 5.2 : Let (M,g) be a doubly warped product ${}_{f}B\times_{b}I$, where dimI=1 and dimB=n-1 (n≥3), $P_B\in\chi(B)$ is parallel on B with respect to the Levi-Civita connection on B and b and f are both constant on B and I, respectively. Then

(i) If (M,g) is an Einstein manifold with respect to the semi-symmetric metric connection, then:

$$^{B}\tau = f^2(n-2)[(n-1)\pi(P)-g(P_B,P_B)].$$

(ii) If B is an Einstein manifold with respect to the the Levi-Civita connection, then M is a quasi-Einstein manifold endowed with a semi-symmetric metric connection.

Proof : (i) Assume that (M,g) is an Einstein manifold with respect to the semi-symmetric metric connection. Then we can write

$$^{\circ}S(X,Y) = (^{\circ}\tau/n)g(X,Y), \tag{35}$$

for any vector fields $X,Y\in\chi(B)$. By the use of the equation (6) and Corollary 4.4, the equation (35) reduces to

$$^{\circ}S(X,Y)=(1/n)[((^{B}\tau)/(f^2))-(n-1)(n-2)\pi(P)]g(X,Y).$$

By a contraction from the above equation over X and Y, we get

$$^{\circ}\tau = ((n-1)/n)[((^{B}\tau)/(f^2))-(n-1)(n-2)\pi(P)]. \tag{36}$$

On the other hand, by making use of Corollary 4.3, we can write

$$^{\circ}S(X,Y) = {}^{B}S(X,Y)-(n-2)g(X,Y)\pi(P) +(n-2)\pi(X)\pi(Y).$$

Similarly, by a contraction from the last equation over X and Y, it can be easily seen that

$$^{\circ}\tau=((^{B}\tau)/(f^2))-(n-1)(n-2)\pi(P)+(n-2)g(P_B,P_B). \tag{37}$$

Comparing the right hand sides of the equations (36) and (37), we get

$$((n-1)/n)[((^{B}\tau)/(f^2))-(n-1)(n-2)\pi(P)] = ((^{B}\tau)/(f^2))-(n-1)(n-2)\pi(P)+(n-2)\, g(P_B,P_B),$$

which gives us

$$^{B}\tau = f^2(n-2)[(n-1)\pi(P)- g(P_B,P_B)].$$

(ii) Assume that B is an Einstein manifold with respect to the Levi-Civita connection. Then we have

$$^{B}S(X,Y)=\alpha g_B(X,Y), \tag{38}$$

for any vector fields X,Y on B. In view of (6) in the last equation, we obtain

$$^{B}S(X,Y)=(\alpha/(f^2))g(X,Y). \tag{39}$$

On the other hand, in view of Corollary 4.3, we can write

$$^{\circ}S(X,Y) = {}^{B}S(X,Y)-(n-2)\pi(P)g(X,Y)+(n-2)\pi(X)\pi(Y).$$

By the use of (39) in the last equation, we obtain

$$^{\circ}S(X,Y)=[(\alpha/(f^2))-(n-2)\pi(P)]g(X,Y)+(n-2)\pi(P)g(X,Y),$$

which shows us ${}_{f}B\times_{b}I$ is a quasi-Einstein manifold with respect to the semi-symmetric metric connection. Therefore, we complete the proof of the theorem.

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