Analytical Solutions of a 1D Time-fractional Coupled Burger Equation via Fractional Complex Transform

S. O. Edeki, and G. O. Akinlabi

***Abstract*—**In this paper, we obtain analytical solutions of a system of time-fractional coupled Burger equation of one-dimensional form via the application of Fractional Complex Transform (FCT) coupled with a modified differential transform method (MDTM). The associated fractional derivatives are in terms of Jumarie's sense. Illustrative cases are considered in clarifying the effectiveness of the proposed technique. The method requires minimal knowledge of fractional calculus. Neither linearization nor discretization is involved. The results are also presented graphically for proper illustration and efficiency is ascertained. Hence, the recommendation of the method for linear and nonlinear space-fractional models.

***Keywords*—**Fractional calculus; fractional complex transform; MDTM; coupled Burger equation.

Mathematics Subject Classification— 83C15, 37N30, 26A33

I. Introduction

Buuger's equation appears to be a basic partial differential equation with copious applications in applied mathematics viz: modelling, gas dynamics, traffic flow, nonlinear acoustics and so on [1-3]. As regards stochastic dynamics, we the application of stochastic Burgers equation in mathematical finance, quantum physics, and financial physics [4-6]. The integer one-dimensional form of the coupled nonlinear Burger equation follows:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+\xi_1\frac{\partial^2 u}{\partial x^2}+\xi_2 u\frac{\partial u}{\partial x}+\gamma\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0\\ \frac{\partial v}{\partial t}+\mu_1\frac{\partial^2 v}{\partial x^2}+\mu_2 v\frac{\partial v}{\partial x}+\eta\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0\end{aligned}\right\} \tag{1.1}$$

subject to the following conditions (1.2) and (1.3) (that is, initial and Dirichlet boundary conditions respectively):

$$\left.\begin{aligned}u(x,0)=f_1(x)\\ v(x,0)=f_2(x)\end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned}u(x,t)=e_1(x,t)\\ v(x,t)=e_2(x,t)\end{aligned}\right\} \tag{1.3}$$

for $x\in\Omega,\ t>0$ where $\Omega=\{x:x\in[c,d]\}$ signifies a domain of computational interval while the constants $\xi_1,\ \xi_2,\ \mu_1,$ and μ_2 are real, while γ and η are arbitrary constants subject to the system's constraints.

A lot of analytical, semi-analytical, and numerical methods of solution appear in literature for solving PDEs such as the one-dimensional Burger, coupled Burger equations (1.1) and the likes [7-24].

Sequel to fractional calculus, this work considers a non-integer ordered form of (1,1) as an extension which is regarded as time-fractional order coupled nonlinear Burger equation of the form:

$$\left.\begin{aligned}&\frac{\partial^\alpha u}{\partial t^\alpha}+\xi_1\frac{\partial^2 u}{\partial x^2}+\xi_2 u\frac{\partial u}{\partial x}+\gamma\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0\\ &\frac{\partial^\alpha v}{\partial t^\alpha}+\mu_1\frac{\partial^2 v}{\partial x^2}+\mu_2 v\frac{\partial v}{\partial x}+\eta\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0,\\ &\alpha\in(0,1]\end{aligned}\right\}, \tag{1.4}$$

Recent work on fractional Burgers' equation include that of Momani [25] via the application of a semi-analytical approach: Adomian Decomposition Method (ADM).

II. Fractional Derivative in the Sense of Jumarie

It is noted here that Jumarie's Fractional Derivative (JFD) is a modified form of the Riemann-Liouville derivatives [26]. Hence, the definition of JFD and its basic properties as follows:

Suppose $\sigma(z)$ is a continuous real valued function of z, and $D_z^\alpha\sigma=\dfrac{\partial^\alpha\sigma}{\partial z^\alpha}$ denoting JFD of h, of order α w.r.t. z.

Then,

This work was supported by Covenant University, Ota, Nigeria.

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$$D_z^{\alpha}\sigma=\begin{cases}\left(\begin{array}{l}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dz}\displaystyle\int_0^z(z-\zeta)^{-\alpha}\left(\sigma(\zeta)-\sigma(0)\right)d\zeta,\\ \text{for } \alpha\in(0,1)\end{array}\right),\\ \left(\begin{array}{l}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dz}\displaystyle\int_0^z(z-\zeta)^{-\alpha}\left(\sigma(\zeta)-\sigma(0)\right)d\zeta,\\ \text{for } \alpha\in(0,1)\end{array}\right),\\ \left(\begin{array}{l}\left(\sigma^{(\alpha-\phi)}(z)\right)^{(\phi)},\ \alpha\in[\phi,\phi+1),\\ \phi\geq 1\end{array}\right),\end{cases}\tag{2.1}$$

where $\Gamma(\cdot)$ represents a gamma function. The main features of JFD [23] as follows:

(i) $D_z^{\alpha}c=0,\ \alpha>0$, for a constant c

(ii) $D_z^{\alpha}\left(c\sigma(z)\right)=cD_z^{\alpha}\sigma(z),\ \alpha>0$,

(iii) $D_z^{\alpha}z^{\beta}=\dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}z^{\beta-\alpha},\ \beta\geq\alpha>0,$

(iv) $D_z^{\alpha}\left(\sigma_1(z)\sigma_2(z)\right)=\left(\begin{array}{l}D_z^{\alpha}\sigma_1(z)\left(\sigma_2(z)\right)\\ +\sigma_1(z)D_z^{\alpha}\sigma_2(z)\end{array}\right),$

(v) $D_z^{\alpha}\left(\sigma\left(z(g)\right)\right)=D_z^{1}\sigma\cdot D_g^{\alpha}z$,

The features (i)-(v) are fractional derivative of: constant function, constant multiple function, power function, product function, and function of function respectively. Though, (v) can be associated to Jumarie's chain rule in terms of fractional derivative.

III. THE REDUCED DIFFERENTIAL TRANSFORM [27-30]

Suppose $m(x,t)$ is an analytic and continuously differentiable function, defined on D a given domain, then the differential transformation form of $m(x,t)$ is defined and expressed as:

$$M_k(x)=\frac{1}{k!}\left[\frac{\partial^k m(x,t)}{\partial t^k}\right]_{t=0}\tag{3.1}$$

where $M_k(x)$ and $m(x,t)$ are referred to as the transformed and the original functions respectively. Thus, the differential inverse transform (DIT) of $M_k(x)$ is defined and denoted as:

$$m(x,t)=\sum_{k=0}^{\infty}M_k(x)t^k.\tag{3.2}$$

A. *The fundamentals properties of the DTM*

D1: If $m(x,t)=\alpha p(x,t)\pm\beta q(x,t)$, then $M_k(x)=\alpha P_k(x)\pm\beta Q_k(x)$.

D2: If $m(x,t)=\dfrac{\alpha\partial^{\eta}h(x,t)}{\partial t^{\eta}},\ \eta\in\mathbb{N}$, then

$$M_k(x)=\frac{\alpha(k+\eta)!}{k!}H_{k+\eta}(x).$$

D3: If $m(x,t)=\dfrac{g(x)\partial^{\eta}h(x,t)}{\partial x^{\eta}},\ \eta\in\mathbb{N}$, then

$$M_k(x)=\frac{g(x)\partial^{\eta}H_k(x)}{\partial x^{\eta}},\ \eta\in\mathbb{N}.$$

D4: If $m(x,t)=p(x,t)q(x,t)$, then

$$M_k(x)=\sum_{\eta=0}^{k}P_{\eta}(x)Q_{k-\eta}(x).$$

D5: If $m(x,t)=x^{n_1}t^{n_2}$, then

$$M_c=x^{n_1}\delta(c-n_2),\delta(c)=\begin{cases}0, & c\neq 0,\\ 1, & c=0\ .\end{cases}$$

B. *The Fractional Complex Transform [26, 31]*

Suppose we consider a general fractional differential equation of the form:

$$h\left(\upsilon,D_t^{\alpha}\upsilon,D_x^{\beta}\upsilon,D_y^{\lambda}\upsilon,D_z^{\gamma}\upsilon\right)=0,\ \upsilon=\upsilon(t,x,y,z),\tag{3.3}$$

and define the Fractional Complex Transform (FCT) as follows:

$$T=\frac{at^{\alpha}}{\Gamma(1+\alpha)},\ \alpha\in(0,1],\tag{3.4}$$

where a is an unknown constant, then from (iii), we have:

$$D_v^{\alpha}z^{\beta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}z^{\beta-\alpha},\ \beta\geq\alpha>0,$$

$$\therefore\quad D_t^{\alpha}T==\frac{a}{\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\alpha)}\right]t^{\alpha-\alpha}=a.\tag{3.5}$$

Hence,

$$D_t^{\alpha}\upsilon=D_t^{\alpha}\upsilon\left(T(t)\right)=D_T^{1}\upsilon\cdot D_t^{\alpha}T=a\frac{\partial\upsilon}{\partial T}.\tag{3.6}$$

IV. EXAMPLES/APPLICATIONS

Here, the concerned method of solution is used for a nonlinear time-fractional coupled Burger equation as follows.

Suppose we take $\xi_1=-1,\ \xi_2=-2,\ \mu_1=-1,\ \mu_2=-2$, & $\gamma=\eta=1$, then we consider (1.4) of the form:

$$\left.\begin{array}{l}\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}}-\dfrac{\partial^2 u}{\partial x^2}-2u\dfrac{\partial u}{\partial x}+\left(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial u}{\partial x}\right)=0\\ \dfrac{\partial^{\alpha}v}{\partial t^{\alpha}}-\dfrac{\partial^2 v}{\partial x^2}-2v\dfrac{\partial v}{\partial x}+\left(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial u}{\partial x}\right)=0\end{array}\right\}\tag{4.1}$$

subject to:

$$u(x,0)=v(x,0)=\sin(x). \quad (4.2)$$

Solution Steps:

By FCT, $T=\dfrac{at^{\alpha}}{\Gamma(1+\alpha)}$, which according to section 3 gives $\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}}=\dfrac{\partial u}{\partial T}$ and $\dfrac{\partial^{\alpha}v}{\partial t^{\alpha}}=\dfrac{\partial v}{\partial T}$ for $a=1$. Hence, (21) becomes:

$$\left.\begin{aligned}&\frac{\partial u}{\partial T}-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0\\&\frac{\partial v}{\partial T}-\frac{\partial^2 v}{\partial x^2}-2v\frac{\partial v}{\partial x}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0\end{aligned}\right\} \quad (4.3)$$

subject to:

$$u(x,0)=\sin x=v(x,0).$$

By the RDTM in section 3, we have the recurrence relation from (4.3) as:

$$\begin{cases}U_{\upsilon+1}=\dfrac{1}{(1+\upsilon)}\left(U''_{x,\upsilon}+2\displaystyle\sum_{r=0}^{\upsilon}U_{x,p}U'_{x,\upsilon-p}-\frac{\partial}{\partial x}\sum_{p=0}^{\upsilon}U_pV_{\upsilon-p}\right),\\ V_{\upsilon+1}=\dfrac{1}{(1+\upsilon)}\left(V''_{x,\upsilon}+2\displaystyle\sum_{r=0}^{\upsilon}V_{x,p}V'_{x,\upsilon-p}-\frac{\partial}{\partial x}\sum_{p=0}^{\upsilon}U_pV_{\upsilon-p}\right),\ \upsilon\ge 0.\end{cases} \quad (4.4)$$

Hence, using the initial condition: $u(x,0)=\sin x=v(x,0)$ we obtain:

$$\begin{cases}U_0=V_0=\sin x,\ U_2=\dfrac{\sin x}{2!}=V_2,\\ U_4=\dfrac{\sin x}{4!}=V_4,\cdots\\ U_1=-\sin x=V_1,\ U_3=\dfrac{-\sin x}{3!}=V_3,\\ U_5=\dfrac{-\sin x}{5!}=V_5,\cdots\end{cases} \quad (4.5)$$

In general, we have:

$$U_3=V_3=\frac{(-1)^p\sin x}{p!},\ p\in\mathbb{N}\cup(0). \quad (4.6)$$

$$\therefore\quad u(x,T)=\sum_{\hbar=0}^{\infty}U_{\hbar}T^{\alpha\hbar}$$

$$=\left(\begin{aligned}&\sin x-(\sin x)T+\frac{\sin x}{2!}T^2\\&-\frac{\sin x}{3!}T^3+\frac{\sin x}{4!}T^4+\cdots\end{aligned}\right)$$

$$=\sin x\sum_{n=0}^{\infty}\frac{(-1)^nT^n}{n!}=\sin x\sum_{n=0}^{\infty}\frac{(-T)^n}{n!}=\sin(x)\exp(-T). \quad (4.7)$$

Similarly,

$$v(x,T)=\sum_{\hbar=0}^{\infty}V_{\hbar}T^{\hbar}=\sin(x)\exp(-T). \quad (4.8)$$

Hence, the exact solution of (4.1) is:

$$\begin{cases}u(x,t)=\sin(x)\exp\left(-\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right),\\ v(x,t)=\sin(x)\exp\left(-\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right).\end{cases} \quad (4.9)$$

Note**:** when $\alpha=1$ in (4.9), we have $u(x,t)=\sin(x)\exp(-t)=v(x,t)$ yielding the exact solution of the classical coupled nonlinear Burgers equation in line the result in [1], [7], and [23].

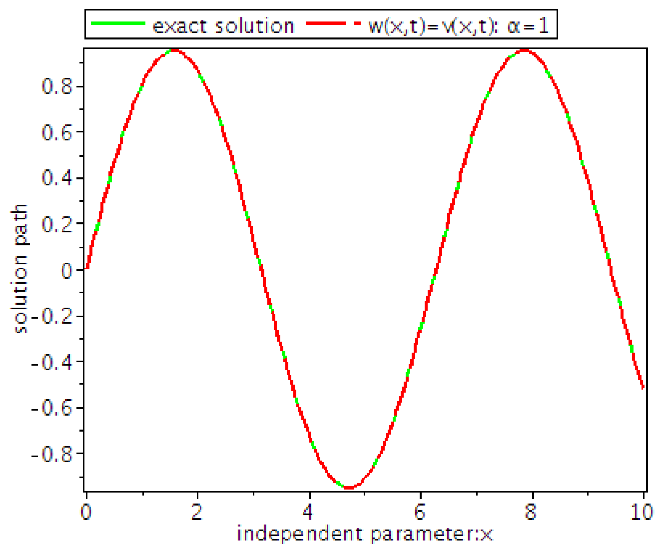

Fig. 1: Graphical solution for at $\alpha=1$, $(t=1)$

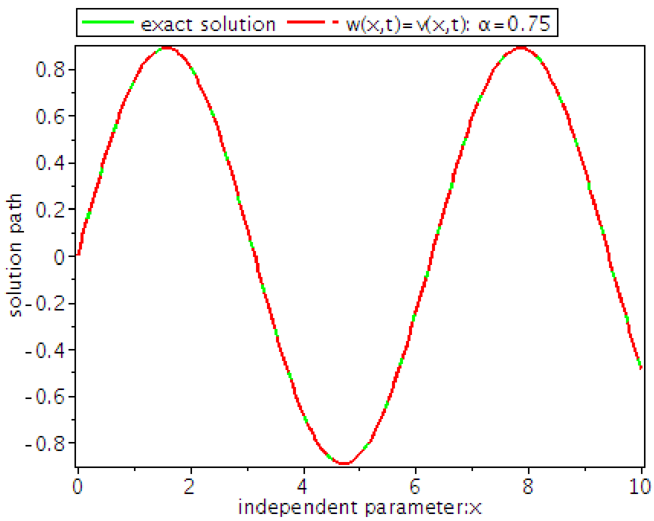

Fig. 2: **Fig. 1**: Graphical solution for at $\alpha = 0\cdot 75,\ (t=1)$

V. Concluding Remarks

We obtained exact solutions of solutions of a system type of time-fractional nonlinear coupled Burger equations via the application of FCT coupled with reduced differential transform method. The FCT is indeed simple but effective and accurate for the solutions of fractional differential equations. The associated derivatives were defined in terms of Jumarie's sense. It is noted that basic knowledge of advanced calculus is more required than that of fractional calculus while obtaining exact solutions of fractional equations with high level of accuracy not being compromised. This can therefore be extended to space-fractional derivatives of higher orders both in linear and nonlinear forms.

Acknowledgment

The authors are grateful to Covenant University for enabling working environment and financial support. Sincere thanks to the anonymous referee(s) for their productive comments.

Conflict of Interests

The authors declare that they have no conflict of interest regarding the publication of this paper.

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