A generalized Hardy-Rogers type with φ-best proximity point result

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***Abstract*—**The concept of a Hardy-Roger contraction self-mapping and its fixed point results have many applications in the various branches in mathematics. However, these results can not be applied in the global optimization problems. This is the motivation for improving the idea of a Hardy-Rogers contraction self-mapping to the sense of nonself-mappings since the best proximity point result for such mappings has some relation with the global optimization problems. The main aim of this paper is to define the new generalized Hardy-Rogers nonself-contraction mappings and prove the best proximity point result for such mappings.

***Keywords*—** φ-best proximity points, Hardy-Rogers contraction mappings, partially ordered metric spaces.

I. Introduction

HISTORICALLY, the idea of a complete metric space is interesting and it has important applications in the classical analysis, especially in the existence and uniqueness theories on one hand while on the other hand the Banach contraction mapping principle (BCP) by Banach [1]. This principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory.

Theorem 1.1. ([1]). Let (X,d) be a complete metric space and a mapping $T:X\to X$ be a Banach contraction mapping, that is, there exists $k\in[0,1)$ such that $d(Tx,Ty)\le kd(x,y)$ for all $x,y\in X$. Then T has unique a fixed point.

A generalization of the above principle has been heavily investigated in many branches of research. Several researchers extended this principle to many ways. In 1973, Hardy and Rogers [4] generalized the idea of a Banach contraction mapping and established a fixed point theorem as follows:

Theorem 1.2. Let (X,d) be a metric space and T be a self-mapping on X satisfying the following condition for all $x,y\in X$:

$$d(Tx,Ty)\le ad(x,Tx)+bd(y,Ty)+cd(x,Ty)+ed(y,Tx)+fd(x,y), \quad (1.1)$$

where a,b,c,e,f are nonnegative real numbers. Suppose that $\alpha:=a+b+c+e+f$. Then the following assertions hold.

(a) If X is complete and $\alpha<1$, then T has a unique fixed point.

(b) If (1.1) is modified to the condition $x,y\in X$ implies

$$d(Tx,Ty)< ad(x,Tx)+bd(y,Ty)+cd(x,Ty)+ed(y,Tx)+fd(x,y)$$

and in this case we assume X is compact, T is continuous and $\alpha=1$, then T has a unique fixed point.

On the other hand, many researchers investigated the fixed point problem in case of nonself-mapings, called the best proximity point problem, which is first introduced by Fan [3]. Here, we give the idea of the best proximity point.

Definition 1.3. Let A,B be two nonempty subsets of a metric space (X,d). A point $x\in A$ is said to be a **best proximity point** of a given mapping $T:A\to B$ if

$$d(x,Tx)=d(A,B),$$

where $d(A,B)=\inf\{d(a,b):a\in A \text{ and } b\in B\}$.

The aim of this paper is to generalize the fixed point theorem for Hardy-Rogers contraction mappings to the φ-best proximity point theorem. First, we introduce the new contraction mapping which is called a generalized Hardy-Rogers (F,φ)-proximal contraction mapping and then we establish the best proximity point theorem for such mappings in partially ordered metric spaces.

II. Preliminaries

First of all, we give some notations for using in this paper as follows:

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- (X,d) denotes a metric space,
- $\prec$ denotes a partial order on X ,
- $A,B \subseteq X$,
- the set A_0 and B_0 are defined by

$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

In 2012, Basha [2] introduced some definition as follows:

Definition 2.1 ([2]). Let A,B be nonempty subsets of a metric space (X,d) and $\prec$ be a partial order on X. A mapping $T: A \to B$ is called **proximally increasing** if for all $x_1, x_2, y_1, y_2 \in A$,

$$\left.\begin{array}{l} x_1 \prec x_2, \\ d(y_1,Tx_1) = d(A,B), \\ d(y_2,Tx_2) = d(A,B) \end{array}\right\} \Rightarrow y_1 \prec y_2.$$

Definition 2.2 ([2]). Let A,B be nonempty subsets of a metric space (X,d) and $\prec$ be a partial order on X. A mapping $T: A \to B$ is called **proximally increasing on** A_0 if for all $x_1, x_2, y_1, y_2 \in A_0$,

$$\left.\begin{array}{l} x_1 \prec x_2, \\ d(y_1,Tx_1) = d(A,B), \\ d(y_2,Tx_2) = d(A,B) \end{array}\right\} \Rightarrow y_1 \prec y_2.$$

We denote $\mathbb{F}$ by the set of all functions $F:[0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(F1) $\max\{a,b\} \le F(a,b,c)$ for all $a,b,c \in [0,\infty)$,

(F2) $F(0,0,0) = 0$,

(F3) F is continuous.

The set of all zeros of the function $\varphi: A \to [0,\infty)$ is denoted by

$$Z_\varphi = \{x \in A : \varphi(x) = 0\} .$$

The set of all best proximity points of a nonself-mapping $T: A \to B$ is denoted by

$$B_{est}(T) = \{x \in A : d(x,Tx) = d(A,B)\}.$$

In 2017, Isik *et al.* [5] introduced the notion of φ-best proximity points as follows:

Definition 2.3. Let A,B be nonempty subsets of a metric space (X,d) and $\varphi: A \to [0,\infty)$ be a function. An element $x^* \in A$ is called a φ-best proximity point of the nonself-mapping $T: A \to B$ if

$$x^* \in B_{est}(T) \cap Z_\varphi.$$

III. Main Results

First, we introduce the new contraction mapping as follows:

Definition 3.1. Let A,B be two nonempty subsets of a partially order metric space $(X,d,\prec)$, $\varphi: A \to [0,\infty)$ be a function and $F \in \mathbb{F}$. A mapping $T: A \to B$ is called **a generalized Hardy-Rogers (F,φ)-proximal contraction** if

$$\left.\begin{array}{l} x \prec y, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) \le\ & a_1 F(d(x,y),\varphi(x),\varphi(y)) \\ & + a_2 F(d(x,u),\varphi(x),\varphi(u)) \\ & + a_3 F(d(y,v),\varphi(y),\varphi(v)) \\ & + a_4 F(d(y,u),\varphi(y),\varphi(u)) \\ & + a_5 F(d(x,v),\varphi(x),\varphi(v)) \quad (3.1) \end{aligned}$$

for all $x,y,u,v \in A$, where $a_1,a_2,a_3,a_4,a_5 \in [0,\infty)$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

Next, we establish a new φ-best proximity point theorem for generalized Hardy-Rogers (F,φ)-proximal contraction mappings.

Theorem 3.2. Let A,B be two nonempty subsets of a partially order complete metric space $(X,d,\prec)$ and $T: A \to B$ be a generalized Hardy-Rogers (F,φ)-proximal contraction mapping with $T(A_0) \subseteq B_0$. Suppose that T is proximally increasing on A_0, $\varphi: A \to [0,\infty)$ is lower semi-continuous and there exist elements $x_0, x_1 \in A_0$ with $x_0 \prec x_1$ and $d(x_1,Tx_0) = d(A,B)$. If the following condition holds:

(F4) $F\left(d\left(a,b\right),\varphi(c),\varphi(e)\right)\leq F\left(d\left(a,\ f\right),\varphi(c),\varphi(g)\right)$
$+F\left(d\left(f,b\right),\varphi(g),\varphi(e)\right)$

for all $a,b,c,e,f,g\in X$, then T has a unique φ-best proximity point in A_0.

Proof. First of all, we assume that $\omega\in A$ is a best proximity point of T and then $d\left(\omega,T\omega\right)=d\left(A,B\right)$. Using (3.1) with $x=y=u=v=\omega$, we obtain

$$F\left(0,\varphi\left(\omega\right),\varphi\left(\omega\right)\right)$$
$$\leq\left(a_1+a_2+a_3+a_4+a_5\right)F\left(0,\varphi\left(\omega\right),\varphi\left(\omega\right)\right),$$

and so

$$F\left(0,\varphi\left(\omega\right),\varphi\left(\omega\right)\right)=0. \tag{3.2}$$

From (F1), we obtain

$$\varphi\left(\omega\right)\leq F\left(0,\varphi\left(\omega\right),\varphi\left(\omega\right)\right). \tag{3.3}$$

It follows from (3.2) and (3.3) that $\varphi\left(\omega\right)=0$ and hence

$$B_{est}\left(T\right)\subseteq Z_{\varphi}. \tag{3.4}$$

By assumption of this theorem, there exists $x_0,x_1\in A_0$ with $x_0\prec x_1$ such that

$$d\left(x_1,Tx_0\right)=d\left(A,B\right). \tag{3.5}$$

Since $Tx_1\in T\left(A_0\right)\subseteq B_0$, there exists an element $x_2\in A_0$ such that

$$d\left(x_2,Tx_1\right)=d\left(A,B\right). \tag{3.6}$$

As T is proximally increasing on A_0, we obtain

$$x_1\prec x_2.$$

By similarly process, we can construct a sequence $\left\{x_n\right\}\subseteq A_0$ such that

$$x_n\prec x_{n+1} \tag{3.7}$$

and

$$d\left(x_{n+1},Tx_n\right)=d\left(A,B\right) \tag{3.8}$$

for all $n\in\mathbb{N}\cup\left\{0\right\}$. If there exists $n'\in\mathbb{N}\cup\left\{0\right\}$ such that $x_{n'}=x_{n'+1}$, then

$$d\left(x_{n'},Tx_{n'}\right)=d\left(x_{n'+1},Tx_{n'}\right)=d\left(A,B\right). \tag{3.9}$$

Therefore, $x_{n'}$ is a best proximity point of T and we are done. So, we suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}\cup\left\{0\right\}$. By a generalized Hardy-Rogers $\left(F,\varphi\right)$-proximally contractive condition, we have

$$\begin{aligned}&F\left(d\left(x_{n+1},x_{n+2}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+2}\right)\right)\\&\leq a_1F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\\&+a_2F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\\&+a_3F\left(d\left(x_{n+1},x_{n+2}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+2}\right)\right)\\&+a_4F\left(d\left(x_{n+1},x_{n+1}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+1}\right)\right)\\&+a_5F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right)\end{aligned}$$

for all $n\in\mathbb{N}\cup\{0\}$. It follows that

$$\begin{aligned}&F\left(d\left(x_{n+1},x_{n+2}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+2}\right)\right)\\&\leq\left(\frac{a_1+a_2}{1-a_3}\right)F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\\&+\left(\frac{a_5}{1-a_3}\right)F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right)\end{aligned} \tag{3.10}$$

for all $n\in\mathbb{N}\cup\{0\}$. From (F4), we obtain

$$\begin{aligned}&F\left(d\left(x_{n+1},x_{n+2}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+2}\right)\right)\\&\geq F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right)\\&-F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\end{aligned} \tag{3.11}$$

for all $n\in\mathbb{N}\cup\{0\}$. From inequalities (3.4) and (3.5), we have

$$\begin{aligned}&F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right)\\&-F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\\&\leq\left(\frac{a_1+a_2}{1-a_3}\right)F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\\&+\left(\frac{a_5}{1-a_3}\right)F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right),\end{aligned} \tag{3.12}$$

that is,

$$\begin{aligned}&F\left(d\left(x_n,x_{n+2}\right),\varphi\left(x_n\right),\varphi\left(x_{n+2}\right)\right)\\&\leq\left(\frac{1+a_1+a_2-a_3}{1-a_3-a_5}\right)F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\end{aligned} \tag{3.13}$$

for all $n\in\mathbb{N}\cup\{0\}$. Substituting (3.13) into (3.10), we obtain

$$\begin{aligned}&F\left(d\left(x_{n+1},x_{n+2}\right),\varphi\left(x_{n+1}\right),\varphi\left(x_{n+2}\right)\right)\\&\leq\left(\frac{a_1+a_2+a_5}{1-a_3-a_5}\right)F\left(d\left(x_n,x_{n+1}\right),\varphi\left(x_n\right),\varphi\left(x_{n+1}\right)\right)\end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N} \cup \{0\}$. Similarly, we may exchange a_1 with a_3 and a_5 with a_4 to get

$$F\left(d\left(x_{n+1}, x_{n+2}\right), \varphi\left(x_{n+1}\right), \varphi\left(x_{n+2}\right)\right) \leq \left(\frac{a_2 + a_3 + a_4}{1 - a_1 - a_4}\right) F\left(d\left(x_n, x_{n+1}\right), \varphi\left(x_n\right), \varphi\left(x_{n+1}\right)\right). \tag{3.15}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (3.14) and (3.15), we can conclude that

$$F\left(d\left(x_{n+1}, x_{n+2}\right), \varphi\left(x_{n+1}\right), \varphi\left(x_{n+2}\right)\right) \leq \alpha F\left(d\left(x_n, x_{n+1}\right), \varphi\left(x_n\right), \varphi\left(x_{n+1}\right)\right) \tag{3.16}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\alpha := \min\left\{\left(\frac{a_1 + a_2 + a_5}{1 - a_3 - a_5}\right), \left(\frac{a_2 + a_3 + a_4}{1 - a_1 - a_4}\right)\right\}.$$

Therefore,

$$F\left(d\left(x_{n+1}, x_{n+2}\right), \varphi\left(x_{n+1}\right), \varphi\left(x_{n+2}\right)\right) \leq \alpha^{n+1} F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \tag{3.17}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (F1), we obtain

$$\max\left\{d\left(x_{n+1}, x_{n+2}\right), \varphi\left(x_{n+1}\right)\right\} \leq \alpha^{n+1} F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \tag{3.18}$$

and so

$$d\left(x_{n+1}, x_{n+2}\right) \leq \alpha^{n+1} F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \tag{3.19}$$

for all $n \in \mathbb{N} \cup \{0\}$. Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ such that $m > n$. From (3.19) and using triangular inequality, we obtain

$$\begin{aligned} &d\left(x_n, x_m\right) \\ &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+2}, x_{n+3}\right) + \cdots + d\left(x_{m-1}, x_m\right) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots) F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \\ &\leq \left(\frac{\alpha^n}{1 - a}\right) F\left(d\left(x_0, x_1\right), \varphi\left(x_0\right), \varphi\left(x_1\right)\right) \end{aligned} \tag{3.20}$$

Taking $n \to \infty$ in (3.20), we get $d(x_n, x_m) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since A_0 is complete, there exists $x \in A_0$ such that $x_n \to x$, that is,

$$\lim_{n \to \infty} d\left(x_n, x\right) = 0. \tag{3.21}$$

Now, we will show that x is a φ-best proximity point of T. From (3.18), we get

$$\lim_{n \to \infty} \varphi\left(x_{n+1}\right) = 0. \tag{3.22}$$

From (3.21), (3.22) and the lower-continuity of φ,

$$\varphi\left(x\right) = 0. \tag{3.23}$$

Since $x \in A_0$ and $T\left(A_0\right) \subseteq B_0$, there exists $x^* \in A_0$ such that

$$d(x^*, Tx) = d\left(A, B\right). \tag{3.24}$$

By (3.1), (3.8) and (3.24), we obtain

$$F(d(x_{n+1}, x^*), \varphi\left(x_{n+1}\right), \varphi(x^*)) \leq \alpha F\left(d\left(x_n, x\right), \varphi\left(x_n\right), \varphi\left(x\right)\right). \tag{3.25}$$

Taking limit as $n \to \infty$ in (3.25) and using (3.21)-(3.23), (F2) and the continuity of F, we get

$$F(d(x, x^*), 0, \varphi(x^*)) \leq \alpha F\left(0, 0, 0\right) = 0.$$

From (F1), it implies that $d(x, x^*) = 0$ and then $x = x^*$. By (3.24) we get

$$d\left(x, Tx\right) = d\left(A, B\right).$$

That is, x is a φ-best proximity point of T.

□

IV. Conclusions and Open problem

Building from the ideas of Hardy and Rogers [4], we define the new generalized contractive condition for nonself-mappings. Moreover, we establish new best proximity point for such a mapping in partially order metric spaces. Based on the fact that this result can be solved the problems for nonself-mappings, we can apply this result for solving some global optimization problems. This is advantage of the main result of this paper.

Finally, we give a question for readers as follows:

Problem: How to prove the uniqueness of a φ-best proximity point of T in Theorem 3.2?

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