On new orthogonal contractions in *b*-metric spaces

O. Yamaod and W. Sintunavarat

Abstract—In this paper, we introduce the concept of an *s*-orthogonal contraction in the sense of *b*-metric spaces by using the notion of the orthogonal sets. We also establish some fixed point theorem for the purposed contraction and state some illustrative example to claim that our results properly generalize some results in the literature. Further, by using the main results, we prove some fixed point results for orthogonal contraction in metric spaces. Our results generalize and improve the result of Gordji *et al.* [3] and several well-known results given by some authors in metric and *b*-metric spaces.

Keywords— *b*-metric space, *O*-set, *s*-orthogonal contraction, orthogonal preserving.

I. Introduction

THE famous fixed point result called **Banach contraction principle** is one of the most important results in mathematical analysis. It is the most widely applied fixed point result in many branches of mathematics and it is generalized in many different directions. One usual way for improving the Banach contraction principle is to replace the metric space by certain generalized metric spaces.

The concept of a *b*-metric space was introduced by Bakhtin [2] and Czerwik [5]. They also established the fixed point result in the setting of *b*-metric spaces which is a generalization of the Banach contraction principle in metric spaces. Since then several papers have dealt with fixed point theorems for single-valued and multi-valued mappings in *b*-metric spaces (see [1,3,4,6,8]).

On the other hand, the concept of an orthogonal set has many applications in several branches in mathematics and it has many types of the orthogonality. Recently, Gordji *et al*. [7] introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with the new orthogonality. Furthermore, they gave the application of this results for claming the existence and uniqueness of solutions of the first-ordinary differential equation while the Banach contraction mapping can not be applied in this problem

The purpose of this paper is to improve and generalize the concept of an orthogonal contraction in the sense of metric spaces due to Gordji *et al*. [7]. We also establish some fixed point theorem for the purposed contractions in the sense of *b*-metric spaces. We also state some illustrative example to claim that our results properly generalize some results in the literature.

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O. Yamaod is with the Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, 12121, Thailand (corresponding author to provide e-mail: oratai_eve@hotmail.com).

W. Sintunavarat is with the Division of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, 12121, Thailand (e-mail: wutiphol@mathstat.sci.tu.ac.th).

II. Preliminaries

Throughout this work, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

The concept of a *b*-metric space was introduced by Bakhtin [1] and Czerwik [2] as follows:

Definition 2.1. [2] Let X be a nonempty set and $s \geq 1$. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions for all $x, y, z \in X$:

- $d(x,y) = 0$ if and only if $x = y$;
- $d(x,y) = d(y,x)$;
- $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then (X,d) is called a *b*-metric space with the coefficient s.

Any metric space is a *b*-metric space with $s = 1$ and so the class of *b*-metric spaces is larger than the class of metric spaces.

Now, we give some known examples of *b*-metric spaces as follows:

Example 2.2. Define a mapping $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$d(x,y) = |x - y|^2$$

for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a *b*-metric space with the coefficient $s = 2$.

The most recently, Gordji *et al*. [3] introduced the notion of the orthogonal set as follows:

Definition 2.3. [3] Let X be a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 : (\forall y \in X, y \perp x_0) \text{ or } (\forall y \in X, x_0 \perp y),$$

then it is called an orthogonal set (briefly *O*-set). We denote this *O*-set by $(X, \perp)$.

As an illustration, let us consider the following examples:

Example 2.4. [3] Let X be the set of all people in the world. Define the binary relation $\perp$ on X by $x \perp y$ if x can give blood to y. According to the Table 1, if x_0 is a person such that his (her) blood type is *O*-, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an *O*-set. In this *O*-set, x_0 (in Definition 2.3) is not unique. Note that, in this example, x_0 may be a person with blood type $AB+$. In this case, we have $y \perp x_0$ for all $y \in X$.

Table 1

Type	You can give blood to	You can receive blood from
$A+$	$A+\ AB+$	$A+\ A-\ O+\ O-$
$O+$	$O+\ A+\ B+\ AB+$	$O+\ O-$
$B+$	$B+\ AB+$	$B+\ B-\ O+\ O-$
$AB+$	$AB+$	Everyone
$A-$	$A+\ A-\ AB+\ AB-$	$A-\ O-$
$O-$	Everyone	$O-$
$B-$	$B+\ B-\ AB+\ AB-$	$B-\ O-$
$AB-$	$AB+\ AB-$	$AB-\ B-\ O-\ A-$

Example 2.5. [3] Let $X = \mathbb{Z}$. Define the binary relation $\perp$ on X by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an *O*-set.

Definition 2.6. [3] Let $(X, \perp)$ be an *O*-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, *O*-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 2.7. [3] Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an *O*-set and (X, d) is a metric space). Then $f : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) at $a \in X$ if, for each *O*-sequence $\{a_n\}$ in X with $a_n \to a$, we have $f(a_n) \to f(a)$. Also, f is said to be $\perp$-continuous on X if f is $\perp$-continuous at each $a \in X$.

Definition 2.8. [3] Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be orthogonally complete (briefly, *O*-complete) if every Cauchy *O*-sequence is convergent.

Definition 2.9. [3] Let $(X, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is called an orthogonal contraction (briefly, $\perp$-contraction) with the Lipschitz constant λ if

$$d(fx, fy) \le \lambda d(x, y)$$

for all $x, y \in X$ with $x \perp y$.

Definition 2.10. [3] Let $(X, \perp)$ be an *O*-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $fx \perp fy$ whenever $x \perp y$. Also, $f : X \to X$ is said to be weakly $\perp$-preserving if $fx \perp fy$ or $fy \perp fx$ whenever $x \perp y$.

III. MAIN RESULTS

Let us begin with new important definitions.

Definition 3.1. Let $(X, \perp, d)$ be an orthogonal *b*-metric space with the coefficient $s \ge 1$ ($(X, \perp)$ is an *O*-set and (X, d) is a *b*-metric space) and $0 < \lambda < 1$. A mapping $f : X \to X$ is called an *s*-orthogonal contraction (briefly, $\perp_s$-contraction) with the Lipschitz constant λ if

$$sd(fx, fy) \le \lambda d(x, y)$$

for all $x, y \in X$ with $x \perp y$.

Definition 3.2. Let $(X, \perp, d)$ be an orthogonal *b*-metric space. Then $f : X \to X$ is said to be orthogonally continuous (or $\perp_s$-continuous) at $a \in X$ if, for each *O*-sequence $\{a_n\}$ in X with $a_n \to a$, we have $f(a_n) \to f(a)$. Also, f is said to be $\perp_s$-continuous on X if f is $\perp_s$-continuous at each $a \in X$.

Definition 3.3. Let $(X, \perp, d)$ be an orthogonal *b*-metric space with the coefficient $s \ge 1$. Then X is said to be *s*-orthogonally complete (briefly, O_s-complete) if every Cauchy *O*-sequence is convergent.

Remark 3.4. Definitions 3.1-3.3 generalize orthogonal contractions, orthogonally continuity and orthogonally completeness which are introduced by Gordji *et al*. [7].

Now, we are ready to prove the main theorem of this paper which can be considered as a real extension of Banach contraction principle and fixed point result in [3].

Theorem 3.5. Let $(X, \perp, d)$ be an O_s-complete *b*-metric space with the coefficient $s \ge 1$. Suppose that $f : X \to X$ is $\perp_s$-continuous, $\perp$-preserving and $\perp_s$-contraction with the Lipschitz constant $\lambda \in (0,1)$. Then f has a unique fixed point $x^* \in X$. Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point x^*.

Proof. By the definition of orthogonality, there exists $x_0 \in X$ such that

$$(\forall y \in X,\ x_0 \perp y) \text{ or } (\forall y \in X,\ y \perp x_0).$$

It follows that $x_0 \perp fx_0$ or $fx_0 \perp x_0$. Let

$$x_1 := fx_0,\ x_2 = fx_1 = f^2 x_0, \ldots, x_{n+1} = fx_n = f^{n+1} x_0$$

for all $n \in \mathbb{N}$. Since f is $\perp$-preserving, $\{x_n\}$ is an *O*-sequence. On the other hand, f is an $\perp_s$-contraction. Then we have

$$d(x_{n+1}, x_n) \le \lambda^n d(x_1, x_0)$$

for all $n \in \mathbb{N}$. For any $m \ge 1$ and $p \ge 1$, it follows that

$$
\begin{aligned}
d(x_{m+p}, x_m) &\leq s[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)] \\
&= sd(x_{m+p}, x_{m+p-1}) + sd(x_{m+p-1}, x_m) \\
&\leq sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + s^2 d(x_{m+p-2}, x_m) \\
&\ \ \vdots \\
&\leq sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + s^3 d(x_{m+p-2}, x_{m+p-3}) + \ldots + s^{p-1} d(x_{m+2}, x_{m+1}) \\
&\quad + s^{p-1} d(x_{m+1}, x_m) \\
&\leq s\lambda^{m+p-1} d(x_1, x_0) + s^2 \lambda^{m+p-2} d(x_1, x_0) \\
&\quad + s^3 \lambda^{m+p-3} d(x_1, x_0) + \ldots \\
&\quad + s^{p-1} \lambda^{m+1} d(x_1, x_0) + s^{p-1} \lambda^m d(x_1, x_0) \\
&\leq (s\lambda^{m+p-1} + s^2 \lambda^{m+p-2} + s^3 \lambda^{m+p-3} \\
&\quad + \ldots + s^{p-1} \lambda^{m+1} + s^{p-1} \lambda^m) d(x_1, x_0) \\
&\leq \frac{s^p \lambda^m}{s - \lambda} d(x_1, x_0).
\end{aligned}
$$

Taking limit as $m \to \infty$, we have

$$\lim_{m\to\infty} d(x_{m+p}, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy *O*-sequence. Since X is *O*-complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. From f is $\perp_s$-continuous, we get

$$fx^* = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Hence, x^* is a fixed point of f.

To prove the uniqueness property of fixed point, let $y^* \in X$ be a fixed point of f. Then we have

$$f^n x^* = x^* \quad \text{and} \quad f^n y^* = y^*$$

for all $n \in \mathbb{N}$. By the choice of x_0 in the first part of the proof, we obtain

$$\left[x_0 \perp x^* \text{ and } x_0 \perp y^*\right] \text{ or } \left[x^* \perp x_0 \text{ and } y^* \perp x_0\right].$$

Since f is $\perp$-preserving, we have

$$\left[f^n x_0 \perp f^n x^* \text{ and } f^n x_0 \perp f^n y^*\right]$$

or

$$\left[f^n x^* \perp f^n x_0 \text{ and } f^n y^* \perp f^n x_0\right]$$

for all $n \in \mathbb{N}$. Therefore, by the triangle inequality, we get

$$
\begin{aligned}
d(x^*, y^*) &= d(f^n x^*, f^n y^*) \\
&= s[d(f^n x^*, f^n x_0) + d(f^n x_0, f^n y^*)] \\
&\leq s\lambda^n d(x^*, x_0) + s\lambda^n d(x_0, y^*).
\end{aligned}
$$

Taking limit as $n \to \infty$ in above inequality, we obtain

$$d(x^*, y^*) = 0$$

and so

$$x^* = y^*.$$

Finally, let $x \in X$ be arbitrary. Similarly, we have

$$\left[x_0 \perp x^* \text{ and } x_0 \perp x\right] \text{ or } \left[x^* \perp x_0 \text{ and } x \perp x_0\right]$$

and

$$\left[f^n x_0 \perp f^n x^* \text{ and } f^n x_0 \perp f^n x\right]$$

or

$$\left[f^n x^* \perp f^n x_0 \text{ and } f^n x \perp f^n x_0\right]$$

for all $n \in \mathbb{N}$. Hence

$$
\begin{aligned}
d(x^*, f^n x) &= d(f^n x^*, f^n X) \\
&= s[d(f^n x^*, f^n x_0) + d(f^n x_0, f^n x)] \\
&\leq s\lambda^n d(x^*, x_0) + s\lambda^n d(x_0, x)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(x^*, f^n x) = 0$$

and so

$$\lim_{n\to\infty} f^n x = x^*$$

for all $x \in X$.

Example 3.6. Let $X = [0,12]$ and $d : X \times X \to [0,\infty)$ be given by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Define the binary relation $\perp$ on X by $x \perp y$ if $xy \leq (x \vee y)$, where $x \vee y = x$ or y. Then (X, d) is O_s-complete *b*-metric space with $s = 2$. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \dfrac{x}{3}, & 0 \leq x \leq 3; \\ 0, & 3 < x \leq 12. \end{cases}$$

Let $x \perp y$. Without loss of generality, we may assume that $xy \leq x$. Then the following cases are satisfied:

Case I: If $x = 0$ and $0 \leq y \leq 3$, then $fx = 0$ and $fy = \dfrac{y}{3}$.

Case II: If $x = 0$ and $3 < y \leq 12$ then $fx = fy = 0$.

Case III: If $0 \leq y \leq 1$ and $0 < x \leq 3$, then $fy = \dfrac{y}{3}$ and $fx = \dfrac{x}{3}$.

Case IV: If $0 \leq y \leq 1$ and $3 < x \leq 12$ then $fy = \dfrac{y}{3}$ and $fx = 0$.

From Case I – IV, we obtain

$$2|fx - fy|^2 \leq \frac{2}{9}|x - y|^2$$

for all $x, y \in X$ with $x \perp y$. So f is $\perp_s$-contraction with $\lambda = \frac{2}{9}$. It is easy to see that f is $\perp_s$-continuous and $\perp$-preserving. Therefore, all the conditions of Theorem 3.4 are satisfied. Hence we can conclude that f has a unique fixed point in X, that is, a point 0.

Next, we show that Theorem 3.4 is a real extension of the result of Gordji *et al*. [3].

Corollary 3.7. [3] Let $(X, \perp, d)$ be an *O*-complete metric space. Suppose that $f : X \to X$ is $\perp$-continuous, $\perp$-preserving and $\perp$-contraction with the Lipschitz constant $\lambda \in (0,1)$. Then f has a unique fixed point $x^* \in X$. Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point x^*.

IV. CONCLUSIONS AND FUTURE WORKS

The study of fixed points of mappings satisfying orthogonal sets has been focused to the vigorous research activity in the recent decade. As a consequence, many mathematicians obtained more results in this direction. In this paper, the concept of new generalized orthogonal contractive condition in *b*-metric spaces was introduced. Based on this concept, we have studied the existence fixed point results for the purposed contraction in *b*-metric spaces. Some illustrative examples are furnished which demonstrate the validity of the hypotheses and degree of utility of our results. Also, we can derive some fixed points results for mappings satisfying a orthogonal contractive condition in metric spaces from our main results. These results improve and generalize the main results of Gordji *et al.* [7]. In the future, we will apply our result to investigate many nonlinear problems which are related real-world problems.

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