On the functional equation

$F(x, y) \oplus z = F(x \oplus z, y \oplus z)$

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Abstract– This research belongs to the field of functional equations started by J.Aczel in 1969. The aim of this paper is to present a generalization of the functional equation $F(x, y) + z = F(x + z, y + z)$ on the pseudo-analysis. If the function F is unknown, the mentioned expression is called *distributivity equation* and many authors have given the solutions. Here, in order to generalize this equation, we introduce the pseudo-operations: called *pseudo-addition* $\oplus$, *pseudo- difference* $\ominus$ and pseudo-multiplication $\odot$. Then, we transform the mentioned equation in another equation in which the common addition $+$ has been replaced by the pseudo-addition $\oplus$. We study the new equation: $F(x, y) \oplus z = F(x \oplus z, y \oplus z)$ and we are able to find the solutions. Really, we give the class of all solutions, depending on two arbitrary functions.

Keywords– Pseudo-operations, distributivity property, functional equations

I. INTRODUCTION

The setting of this paper is the pseudo-analysis, introduced by Pap and his colligues [18, 16, 19].

Many authors have applied the pseudo-analysis in different setting: generalized derivatives, Laplace transform, functional equations, partial differential equations, fluid dynamics, theory of the pseudo-integration. In the bibliography we recall only the most important papers [20, 21, 22, 23, 24, 25, 28, 33, 34, 6, 7, 9, 10, 11, 13, 26, 27].

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Here we present only the application to the functional equations. This research continues a study on *Functional Equations* stared in 2002 with the contribution of P.Benvenuti and M.Divari [8].

Here, we shall introduce generalized operations, called pseudo-addition, pseudo-difference and pseudo-multiplication: $\oplus, \ominus$ and $\odot$. Through these operations we shall study the equality:

$$F(x, y) \oplus z = F(x \oplus z, y \oplus z). \tag{1}$$

In classical setting, when the pseudo-addition $\oplus$ is the common addition $+$, the (1) is

$$F(x, y) + z = F(x + z, y + z), \tag{2}$$

and it has different meanings.

If the function F is assigned, the equality (2) is the *distributivity property* of the common addition with respect to F, (see [14, 37]).

If the function F is unknown, the (2) is a functional equation [1] and it was been studied by many authors in different setting.

In general information theory without probability (see[12, 15]), the equation (2) has been called *compatibility equation*, because it is linked to the independence property among crisp sets [3]. I n this case, this equation (2) was solved by Benvenuti and others authors, (see [2, 4]). Later, we have introduced a generalization of independence property by using the pseudo-analysis [29, 31]. Then in [30] we have considered the measure of the union of two disjoint sets and in this setting the equation (1) appears as an equation which represent the independent property in pseudo-analysis.

The aim of this paper is to look for the solutions of the equation (1). Really, it is possible to find the class of solutions $F_{p,q}$, depending on two arbitrary functions p and q.

The paper is organized in the following way. In Sect.2 we recall some preliminaries about the pseudo-operartions; in Sect.3 we present our previuos results about Funtional Equations. We shall solve the given functional equation (2) in Sect.4 and some properties of solutions are given in Sect. 5. Sect.6 is devoted to the conclusion.

II. PRELIMINAIRES

Pseudo-operations

Here, we recall the pseudo-operations which we shall use (see [5, 16, 17]).

A. Pseudo-addition

The *pseudo-addition* $\oplus$ is a mapping

$$\oplus : [0, M]^2 \longrightarrow [0, M], \quad M \in (0, +\infty)$$

such that

(G_1) $a \leq a', b \leq b' \Longrightarrow a \oplus b \leq a' \oplus b'$;

(G_2) $a \oplus b = b \oplus a$;

(G_3) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(G_4) $a \oplus 0 = a$;

(G_5) $(a_n \longrightarrow b_n, a \longrightarrow b) \Longrightarrow a_n \oplus b_n \longrightarrow a \oplus b$.

B. Pseudo-difference

Moreover, following [35, 36], we can define a pseudo-difference $\ominus$:

Let a pseudo-addition $\oplus$ assigned.

The *pseudo-difference* $\ominus$: $[0, \infty[^2 \rightarrow [0, \infty]$ is given by

$$a \ominus b = inf\{t \in [0, \infty]\ b \oplus t \geq a\}. \tag{3}$$

We observe that

$$a \ominus b = 0 \text{ whenever } a \leq b.$$

First of all, we recognize that

Proposition n.1

The pseudo-operations seen above

$$\Big(\oplus, \ominus\Big)$$

satisfy the following properties:

(D_1) $b = a \ \oplus (b \ominus a), \qquad a < b$;

(D_2) $a = b \ \oplus (a \ominus b) \qquad a > b$;

(D_3) $a \ominus b = (a \oplus c) \ominus (b \oplus c) \quad \forall\, c \in [0, \infty)$.

□

C. Pseudo-multiplication

Following [5], we can define the pseudo-multiplication.

Let a pseudo-addition $\oplus$ assigned on $[0, M]$ and let $F \in]0, \infty]$. The $\oplus$-*fitting pseudo-multiplication* is a mapping $\odot : [0, M] \times]0, \infty] \longrightarrow [0, \infty]$, such that:

(M_1) $a \leq a', b \leq b' \Longrightarrow a \odot b \leq a' \odot b'$;

(M_2) $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$;

(M_3) $a \odot 0 = 0 \odot a = 0$;

(M_4) $(\sup_n a_n) \odot (\sup_m b_m) = \sup_{n,m}(a_n \odot b_m)$.

III. FUNCTIONAL EQUATIONS

For the first time, in 1966, Aczel studied and solved the famous *Cauchy equation* [1]:

$$F(x + y) = F(x) + F(y); \tag{4}$$

and later he considered and showed the solutions of the so called *remaining Cauchy equations*:

(–) $F(x + y) = F(x) \cdot F(y)$;

(–) $F(x \cdot y) = F(x) + F(y)$;

(–) $F(x \cdot y) = F(x) \cdot F(y)$.

When, we replace the common addition and multiplication with the pseudo-operations seen above, we find other functional equations. In [8], we have studied and solved the *generalized Cauchy equation*:

$$F(x \oplus y) = F(x) \oplus F(y),$$

Moreover, in [32] we have considered the equations, called by us *pseudo-remaining Cauchy equations*

(–) $F(x \oplus y) = F(x) \odot F(y)$;

(–) $F(x \odot y) = F(x) \oplus F(y)$;

(–) $F(x \odot y) = F(x) \odot F(y)$.

and we were able to give the solutions.

IV. SOLUTIONS OF THE GIVEN FUNCTIONAL EQUATION

In this paragraph, we come back to our equation and we are going to present the solution of the functional equation (1).

Before giving the solutions of the equation (1), we state first two propositions.

Proposition n.2
Let $F : [0, +infty) \to [0, +\infty)$ be any function which satisfies (1), then F is expressed by

$$F_{p,q}(x, y) = \begin{cases} x \oplus q(y \ominus x) & x < y \\ \\ y \oplus p(x \ominus y) & x > y \end{cases} \quad (5)$$

with $p, q : [0, +\infty) \to [0, +\infty)$ such that $p(0) = q(0)$.
Proof: Setting

$$F(0, t) = q(t) \text{ and } F(t, 0) = p(t), \quad (6)$$

then the functions p and q are well defined and

$$F(0, 0) = q(0) = p(0). \quad (7)$$

The equation (1) has satisfied for every x, y, z, putting $x = 0$ in (5), it is

$$F(0, y) \oplus z = F(0 \oplus z, y \oplus z);$$

by (G_4) and (6) we get

$$q(y) \oplus z = F(z, y \oplus z). \quad (8)$$

In the same manner, for $y = 0$ in (1), we get

$$F(x, 0) \oplus z = F(x \oplus z, 0 \oplus z)$$

and finally, by (G_4) and (6),

$$p(x) \oplus z = F(x \oplus z, z). \quad (9)$$

Now, we are going to prove the expression (5).
Let $x < y$. By (D_1), (8) and (G_2), we have

$$F(x, y) = F(x, x \oplus (y \ominus x)) = x \oplus q(g \ominus x).$$

If $x > y$, by (D_2), (G_2), and (9), it is

$$F(x, y) = F(y \oplus (x \ominus y), y) = y \oplus p(x \ominus y).$$

□

Moreover, we shall prove the viceversa.

Proposition n.3
Let p_0 and q_0 be two functions $p_0, q_0 : [0, +\infty) \to (0, +\infty)$ with $p_0(0) = q_0(0)$. Then, the function

$$F_0(x, y) = \begin{cases} x \oplus q_0(y \ominus x) & x < y \\ \\ y \oplus p_0(x \ominus y) & x > y \end{cases} \quad (10)$$

satisfies the *(1)*, i.e.

$$F_0(x, y) \oplus z = F_0(x \oplus z, y \oplus z), \quad (11)$$

$$\forall x, y, z \in [0, +\infty).$$

Proof: Let $x < y$, then $x \oplus z < y \oplus z, \ \ \forall z \in [0, \infty)$, by (G_1).
Taking into account (D_3) and (G_3)

$$F_0(x, y) \oplus z =$$
$$= x \oplus q_0(y \ominus x) \oplus z = (x \oplus z) \oplus q_0(y \ominus x) =$$
$$= (x \oplus z) \oplus q_0\Big[(y \oplus z) \ominus (x \oplus z)\Big] =$$
$$= F_0(x \oplus z, y \oplus z).$$

Under the same hypothesys for $x > y$ it is

$$F_0(x, y) \oplus z =$$
$$= x \oplus p_0(y \ominus x) \oplus z = (x \oplus z) \oplus p_0(x \ominus y) =$$
$$= (x \oplus z) \oplus p_0\Big[(x \oplus z) \ominus (y \oplus z)\Big] =$$
$$= F_0(x \oplus z, y \oplus z).$$

□

As a conseguence of Propp.n.2 and n.3, we are ready to give the *Main Theorem*.

Main Theorem
The *unique* class of functions $F_{p,q}$, depending on two functions $p, q : [0, +\infty) \to [0, +\infty)$ with $q(0) = p(0)$ is the solution of the (1) if and only if it has expressed by

$$F_{p,q}(x, y) = \begin{cases} x \oplus q(y \ominus x) & x < y \\ \\ y \oplus p(x \ominus y) & x > y \end{cases} \quad (12)$$

□

V. PROPERTIES OF SOLUTIONS

In this paragraph we are going to present some properties of the solutions (12).

Remark. If $x = y$, by (5) we get that all elements are idempotent [14, 37]:

$$F(x, x) = x \oplus q(0) = x \oplus p(0).$$

As $p(0) = q(0) \neq 0$, we have the idempotence, but with a translation of p(0)=q(0).

We recall from [14, 37] the definition of symmetric function.

A function F is *symmetric* if

$$F(x, y) = F(y, x),\ \forall\ (x, y) \in D.$$

Proposition n.4

Every function $F_{p,q}$ given by (12) is symmetric if and only if $p(x) = q(x), \forall\ x \in [0, \infty)$.

Proof: It is easy to see that $x < y$, then

$$F_{p,q}(x, y) = F_{q,p}(y, x) \Longleftrightarrow$$

$$x \oplus q(y \ominus x) = x \oplus q(y \ominus x) \Longleftrightarrow$$

$$q(x) = p(y), \forall\ x \in [0, \infty).$$

The same situation appears for $x > y$. □

Proposition n.5

Every function $F_{p,q}$ given by (12) is continuous if and only if the functions p and q are both continuous. □

VI. CONCLUSION

In this paper, in the setting of pseudo-analysis, we have considered the equality:

$$F(x, y) \oplus z = F(x \oplus z, y \oplus z).$$

If F is unknown, the previous equality is a functional equation.

We have found the unique class of solutions depending on two arbitrary functions p and q :

$$F_{p,q}(x, y) = \begin{cases} x\ \oplus q(y \ominus x) & x < y \\ & \\ y\ \oplus p(x \ominus y) & x > y \end{cases} \tag{13}$$

expressed by the pseudo-operation $\oplus$ and $\ominus$.

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