Volume 5, 2018

Coupled Fixed Point Theorems in Ordered Non-Archimedean Intuitionistic Fuzzy Metric Space Using K-Monotone Property

Akhilesh Jain, R.S. Chandel, Kamal Badhwa, Rajesh Tokse

***Abstract*:** In this paper we define k-monotone property and proved the coupled fixed point theorem in ordered non-Archimedean Intuitionistic fuzzy metric space. Our result is an extension of the results of Mohinta S., Samanta T.K. [15].

Key words: Non- Archimedean property, k-monotone property, mixed monotone mappings, coupled fixed point, Fuzzy metric space, Intuitionistic Fuzzy metric space, Cauchy sequence, complete fuzzy metric space.

1. INTRODUCTION

Fuzzy set theory, a generalization of crisp set theory, was first introduced by Zadeh [21] in 1965 to describe situations in which data are imprecise or vague or uncertain. Kramosil and Michalek [11] introduced the concept of fuzzy metric spaces in 1975, which opened an avenue for further development of analysis in such spaces. Later on it is modified that a few concepts of mathematical analysis have been generalized by George and Veeramani [9].

Afterwards, many articles have been published on fixed point theorems under different contractive condition in fuzzy metric spaces.

Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [3] introduced the concepts of the so called "Intuitionistic fuzzy topological spaces". Park [18], using the idea of intuitionistic fuzzy sets, define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [9].

Bhaskar and Lakshmikantham [3] discussed the mixed monotone mappings and gave some coupled fixed point theorems which can be used to discuss the existence and uniqueness of solution for a periodic boundary value problem.

Hu[10] studied common coupled fixed point theorems for contractive mappings in fuzzy metric space, and Park et.al.[18] defined an IFMS and proved a fixed point theorem in IFMS. Chandok alt el. [4], Choudhury at. Al[5],Cric and Laxmikantam [6], Nguyen at. Al.[16] studied and give the results on common coupled fixed point theorems in different metric spaces. Berinde [2] Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Recently Luong et.al.[12] proved coupled fixed points in partially ordered metric spaces . Mohinta and Samanta[15] and Park [19] prove the coupled fixed point theorem in non-Archimedean intuitionistic fuzzy metric space.

In this paper, we define non-Archimedean intuitionistic fuzzy metric space, and prove a coupled fixed point theorems for map satisfying the mixed monotone property in partially ordered complete non-Archimedean intuitionistic fuzzy metric space.

Akhilesh Jain is with the Dept. of Mathematics, Corporate Institute of Science and Technology, Bhopal, India. R.S. Chandel is with the Dept of Mathematics, Govt. Geetanjali Girls P.G. College, Bhopal, India.
Kamal Badhwa is with the Dept. of Mathematics, Government P.G. College, Pipariya, Hosangabad, India. Rajesh Tokse is with the Dept. of Mathematics, Corporate Institute of Research and Technology, Bhopal, India

2. PRELIMINARIES

Dentition 2.1[20] A binary operation $*[0,1]\times[0,1] \to[0,1]$ is a continuous t-norms if " * " is satisfying conditions:

(i) $*$ is an commutative and associative

(ii) $*$ is continuous

(iii) $a * 1 = a$ for all $a\in [0, 1]$

(iv) $a * b \le c*d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0, 1]$.

Basic example of t – norm are the Lukasiewicz t – norm T_1, where $T_1 (a, b) = \max (a+b-1, 0)$, t –norm T_p, where $T_p (a,b) = ab$, and t – norm T_M, where $T_M (a,b) = \min \{a,b\}$.

Definition 2.2[14] A 3-tuple $(X, M,*)$ is said to be non-Archimedean fuzzy metric space if X is an arbitrary set,$*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions, for all $x, y, z\in X$ and $s, t >0$,

(F_1) $M(x, y, t) > 0$

(F_2) $M(x, y, t) = 1$ if and only if $x = y$

(F_3) $M(x, y, t) = M(y, x, t)$

(F_4) $M(x, y, t) *M(y, z, s) \le M(x, z, t + s)$

(F_5) $M(x, y, \cdot)$**:** $(0, \infty) \to (0, 1]$ is continuous.

Then M is called a non-Archimedean fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Lemma 2.1. Let (X, M, *) non-Archimedean fuzzy metric space, then M is a continuous function on $X^2 \times (0, \infty)$.

Remark 2.1.Since * is continuous, if follows from (F_4) that the limit of the sequence in fuzzy metric space is uniquely determined.

Let (X, M, *) be a fuzzy metric space with the following condition:

(F_6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$

Remark 2.2. In the above definition 2.2, the triangular inequality (F_4) is replaced by

$$M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s) \quad \text{for all } x, y, z\in X \text{ and } s, t > 0$$

More equivalently $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$

$$\text{for all } x, y, z\in X,\ s, t > 0 \qquad \text{(NA)}$$

Then the triple (X, M,*) is called a non-Archimedean fuzzy metric space.

It is easy to check that the triangular inequality (NA) implies (F_4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Definition 2.3[20] A binary operation $\diamond:[0,1]\times[0,1] \to[0,1]$ is a continuous t-co norms if "$\diamond$" is satisfying conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a$ for all $a\in [0, 1]$
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Note. The concepts of *triangular norms* (*t*-norms) and *triangular conorms* (*t*-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [13] in his study of statistical metric spaces.

Definition-2.4[17]: A 5-tuple (X, M, N, *, $\diamond$) is said to be *non Archimedean intuitionistic fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$,

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$

(IFM-2) $M(x, y, t) > 0$

(IFM-3) $M(x, y, t) = 1$ if and only if $x = y$

(IFM-4) $M(x, y, t) = M(y, x, t)$

(IFM-5) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z\in X$, $s, t > 0$

(IFM-6) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous

(IFM-7) $N(x, y, t) > 0$

(IFM-8) $N(x, y, t) = 0$ if and only if $x = y$

(IFM-9) $N(x, y, t) = N(y, x, t)$

(IFM-10) $N(x, z, \min\{t, s\}) \leq N(x, y, t) \diamond N(y, z, s)$ for all $x, y, z\in X$, $s, t > 0$

(IFM-11) $N(x, y,.): (0,\infty) \to (0, 1]$ is continuous

Then (M, N) is called an *non Archimedean intuitionistic fuzzy metric* on X , the function $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non nearness between x and y with respect to 't' respectively.

Remark 2.3: In the above definition the triangular inequality (IFM5) and (IFM10) are equivalent to

$M(x, z, t) \geq M(x, y, t) * M(y, z, t)$

and $N(x, z, t) \leq N(x, y, t) \diamond N(y, z, t)$

$$\text{for all } x, y, z\in X,\ s, t > 0 \qquad \text{(NA)}$$

Then the triple (X, M, N, *, $\diamond$) is called a *non-Archimedean Intuitionistic fuzzy metric space*.

Remark 2.4. It is easy to check that the triangular inequality (NA) implies, that every non-Archimedean Intuitionistic fuzzy metric space is intuitionistic fuzzy metric space.

Definition 2.5[18] Let (X, M, N, *, $\diamond$) be a non-Archimedean Intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$in X is called an *Cauchy sequence*, if for each $\varepsilon \in (0,1)$ and $t>0$ there exists $n_0\in N$ such that
$\lim_{n\to\infty} M(x_n, x_{n+p}, t)=1$ and $\lim_{n\to\infty} N(x_n, x_{n+p}, t) =0$ for all p=0,1,2…..

(b) A sequence $\{x_n\}$ in a non-Archimedean Intuitionistic fuzzy metric space (X, M, N, *, $\diamond$) is said to be *convergent* to $x\in X$
$\lim_{n\to\infty} M(x_n, x, t)=1$, $\lim_{n\to\infty} N(x_n, x, t)= 0$ for all $t>0$.

(c) A non-Archimedean Intuitionistic fuzzy metric space (X, M, N, *, $\diamond$) is called *complete* if every Cauchy sequence is convergent in X.

Definition 2.6. [15] A partially ordered set is a set P and a binary relation $\preccurlyeq$, denoted by $(X, \preccurlyeq)$ such that for all $a, b, c \in P$,

(a) $a \preccurlyeq a$(reflexivity),

(b) $a\preccurlyeq b$ and $b\preccurlyeq c$ implies $a\preccurlyeq c$(transitivity),$\preccurlyeq$

(c) a≼ b and b≼ a implies a = b(anti-symmetry).

Definition 2.7[15]: Let (X, ≼) be a partially ordered *set* and F: X×X→X .The mapping F is said to have *k-monotone property* if

$$x_0 \preccurlyeq x_1 ,\ y_0 \succcurlyeq y_1 \Rightarrow F(x_0, y_0) \preccurlyeq F(x_1, y_1)$$

$$\& \ F(y_0, x_0) \preccurlyeq F(y_1, x_1) \text{ for all } x_0,x_1,y_0,y_1 \in X$$

Definition 2.8.[15]. Let (X, ≼) be a partially ordered *set* and F: X×X→X .The mapping F is said to have mixed monotone property if F(x, y) is monotone non-decreasing in first coordinate and is monotone non-increasing in second coordinate . i.e. for any x,y ∈ X ,

$$x_0 \preccurlyeq x_1 \Rightarrow F(x_0, y) \preccurlyeq F(x_1, y)$$

$$\& \ y_0 \preccurlyeq y_1 \Rightarrow F(x, y_0) \succcurlyeq F(x, y_1) \text{ for all } x_0,x_1,y_0,y_1 \in X$$

Remark 2.5. Thus mixed monotone property is particular case of k-monotone property.

Example 2.1. Let X=[2, 64] on the set X, we consider following relation x≼ y ⇔ x≤ y ,Where ≼ is a usual ordering, (X, ≼) a partial order set .
We define F: X×X→X. as F(x,y) = x+[1/y] ,Where [k] represents greatest integer just less than or equal to k.
One can verify that F(x,y) follows k-monotone property.

Definition 2.9 [19]. An element (x,y)∈X×X→X is called a *coupled fixed point* of the mapping F: X×X→X if F(x, y)=x & F(y, x)=y .

3. MAIN RESULTS

Theorem 3.1: Let (X, ≼) be a partially ordered set and (X, M, N, *, ◇) is a complete Non-Archimedean Intuitionistic fuzzy metric space. Let F: X×X→X be a continuous mapping having k-monotone property on X. Assume that for every ε∈ (0, 1) with

$$M(F(x,y),F(u,v),t) \geq 1-\frac{\varepsilon}{2}\max\left\{\begin{matrix} M(F(x,y),x,t),M(x,F(u,v),t), \\ M(F(x,y),u,t),M(u,F(u,v),t) \end{matrix}\right\}$$

$$N(F(x,y),F(u,v),t) \leq 1-\frac{\varepsilon}{2}\min\left\{\begin{matrix} N(F(x,y),x,t),N(x,F(u,v),t), \\ N(F(x,y),u,t),N(u,F(u,v),t) \end{matrix}\right\} \qquad \text{(I)}$$

for all x,y,u,v ∈ X with x≽u and y≼ v .

If there exists $x_0, y_0, x_1, y_1 \in X$, such that $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, where $x_1=F(x_0, y_0)$ & $y_1 = F(y_0,x_0)$ then there exists x, y, ∈ X such that F(x, y) = x & F(y, x)=y.

Proof: Let $x_0, x_1, y_0, y_1 \in X$ be such that $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$ .where $x_1=F(x_0, y_0)$ & $y_1 = F(y_0,x_0)$
We construct sequences $\{x_n\}$ & $\{y_n\}$ in X as follows

$$x_{n+1}=F(x_n, y_n) \ \& \ y_{n+1} = F(y_n, x_n) \quad \text{for all } n\geq 0$$

we shall show that $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n\geq 0$
Since $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, therefore by k-monotone property

$$x_1=F(x_0, y_0) \preccurlyeq F(x_1,y_1)=x_2$$

and $$y_1=F(y_0, x_0) \succcurlyeq F(y_1,x_1)=y_2$$

i. e. $x_1 \preccurlyeq x_2$, $y_1 \succcurlyeq y_2$,
again applying the same property we have

$$x_2=F(x_1, y_1) \preccurlyeq F(x_2,y_2)=x_3$$

and $$y_2=F(y_1, x_1) \succcurlyeq F(y_2,x_2)=y_3$$

Continue in this manner we shall have,

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \ldots\ldots\ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots\ldots\ldots$$

and $$y_0 \succcurlyeq y_1 \succcurlyeq y_2 \ldots\ldots\ldots \succcurlyeq y_n \succcurlyeq y_{n+1} \succcurlyeq \ldots\ldots\ldots$$

Since $x_{n-1} \preccurlyeq x_n$ and $y_{n-1} \succcurlyeq y_n$, from (1) we have,

$$M(F(x_n,y_n),F(x_{n-1},y_{n-1}),t)$$

$$\geq 1-\frac{\varepsilon}{2}\max\left\{\begin{matrix} M(F(x_n,y_n),x_n,t),M(x_n,F(x_{n-1},y_{n-1}),t), \\ M(F(x_n,y_n),x_{n-1},t),M(x_{n-1},F(x_{n-1},y_{n-1}),t) \end{matrix}\right\}$$

$$=1-\frac{\varepsilon}{2}\max\left\{\begin{matrix} M(x_{n+1},x_n,t),M(x_n,x_n,t), \\ M(x_{n+1},x_{n-1},t),M(x_{n-1},x_n,t) \end{matrix}\right\}$$

$$=1-\frac{\varepsilon}{2}\max\left\{\begin{matrix} M(x_{n+1},x_n,t),1, \\ M(x_{n+1},x_{n-1},t),M(x_{n-1},x_n,t) \end{matrix}\right\}$$

$$=1-\frac{\varepsilon}{2}>1-\varepsilon$$

i.e. $M(x_{n+1},x_n,t)>1-\varepsilon$

and $N(F(x_n,y_n),F(x_{n-1},y_{n-1}),t)$

$$\leq 1-\frac{\varepsilon}{2}\min\left\{\begin{matrix} N(F(x_n,y_n),x_n,t), \\ N(x_n,F(x_{n-1},y_{n-1}),t), \\ N(F(x_n,y_n),x_{n-1},t), \\ N(x_{n-1},F(x_{n-1},y_{n-1}),t) \end{matrix}\right\}$$

$$=1-\frac{\varepsilon}{2}\min\left\{\begin{matrix} N(x_{n+1},x_n,t),N(x_n,x_n,t), \\ N(x_{n+1},x_{n-1},t),N(x_{n-1},x_n,t) \end{matrix}\right\}$$

$$=1-\frac{\varepsilon}{2}\min\left(N(x_{n+1},x_n,t),0,N(x_{n+1},x_{n-1},t),N(x_{n-1},x_n,t)\right)$$

$$=1-\frac{\varepsilon}{2}<1-\varepsilon$$

i.e. $N(x_{n+1},x_n,t) < 1-\varepsilon$

Similarly we can show that $M(x_{n+1},x_{n+2},t) > 1-\varepsilon$

So for all $\varepsilon>0$, there exists $n_0\in N$ such that for all $m > n > n_0$ and $t>0$ we have

$M(x_n,x_m,t) \geq M(x_n,x_{n+1},t)* M(x_{n+1},x_{n+2},t)*.. M(x_{m-1},x_m,t)$

$M(x_n,x_m,t) \geq (1-\varepsilon)*(1-\varepsilon)*(1-\varepsilon)*\ldots\ldots*(1-\varepsilon)$

$\Rightarrow M(y_{n+1},y_{n+2},t) > 1-\varepsilon$

And

$N(x_n,x_m,t) \leq N(x_n,x_{n+1},t)\Diamond N(x_{n+1},x_{n+2},t)\Diamond\ldots \Diamond N(x_{m-1},x_m,t)$

$N(x_n,x_m,t) \leq (1-\varepsilon)\Diamond(1-\varepsilon)\Diamond(1-\varepsilon)\Diamond\ldots\ldots\Diamond(1-\varepsilon)$

$\Rightarrow N(y_{n+1},y_{n+2},t) < 1-\varepsilon$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in X and since X is complete non-Archimedean Intuitionistic fuzzy metric space , it converges to a point $x\in X$

i.e. $\lim_{n\to\infty} x_n = x$

Again , since $y_{n-1} \succcurlyeq y_n$, $x_{n-1}\preccurlyeq x_n$, from (1) we have,

$$M(F(y_{n-1},x_{n-1}),F(y_n,x_n),t)$$

$$\geq 1-\frac{\varepsilon}{2}\max\left\{\begin{array}{l} M(F(y_{n-1},x_{n-1}),y_{n-1},t), \\ M(y_{n-1},F(y_n,x_n),t), \\ M(F(y_{n-1},x_{n-1}),y_n,t), \\ M(y_n,F(y_n,x_n),t) \end{array}\right\}$$

$$=1-\frac{\varepsilon}{2}\max\left\{\begin{array}{l} M(y_n,y_{n-1},t), M(y_{n-1},y_{n+1},t), \\ M(y_n,y_n,t), M(y_n,y_{n+1},t) \end{array}\right\}$$

$$=1-\frac{\varepsilon}{2}\max\left\{\begin{array}{l} M(y_n,y_{n-1},t), M(y_{n-1},y_{n+1},t), \\ 1, M(y_n,y_{n+1},t) \end{array}\right\}$$

$$=1-\frac{\varepsilon}{2} > 1-\varepsilon$$

$$M(y_{n+1}, y_n, t) > 1-\varepsilon$$

similarly we can show that $M(y_{n+1}, y_{n+2}, t) > 1-\varepsilon$

So for all $\varepsilon > 0$, there exists $n_0\in N$ such that for all $m> n> n_0$ and $t>0$ we have

$M(y_n, y_m,t) \geq M(y_n,y_{n+1},t)* M(y_{n+1},y_{n+2},t)*\ldots\ldots* M(y_{m-1},y_m,t)$

$M(y_n, y_m,t) > (1-\varepsilon)*(1-\varepsilon)*\ldots\ldots*(1-\varepsilon)$

And $y_1= F(y_0,x_0)$

$N(y_n, y_m,t) \geq N(y_n,y_{n+1},t)\Diamond N(y_{n+1},y_{n+2},t)\Diamond\ldots\ldots\Diamond N(y_{m-1},y_m,t)$

$N(y_n,y_m,t) < (1-\varepsilon)\Diamond(1-\varepsilon)\Diamond\ldots\ldots\Diamond(1-\varepsilon)$

This shows that the sequence $\{y_n\}$ is Cauchy sequence in X and since X is complete fuzzy metric space it converges to a point $y\in X$ i.e. $\lim_{n\to\infty}y_n=y$

Since F is given continuous therefore using convergence of $\{x_n\}$ and $\{y_n\}$ we have, $F(x,y)=x$ & $F(x,y)=y$.

Now we shall define a partial order relation over non-Archimedean fuzzy metric space and prove a coupled fixed point theorem using that relation.

Lemma 3.2: Let (X, M, N, *, $\Diamond$) be a a non-Archimedean Intuitionistic fuzzy metric space with $a*b\geq \max\{a+b-1, 0\}$ and $a \Diamond b \leq \min\{a+b-1,0\}$ with ϕ: $X\times X\times[0,\infty)\to R$, define the relation "$\preccurlyeq$" on X as follows

$x \preccurlyeq u , y \succcurlyeq v \Leftrightarrow M(x, u, t)M(y, v, t)\geq 1+\phi(x, y, t)-\phi(u, v, t)$

for all $t>0$ then "$\preccurlyeq$" is partial order on X, called the partial order induced by ϕ.

Proof: The relation "$\preccurlyeq$" is a reflexive relation: let $x , y \in X$ be any element

Since $M(x, x, t)M(y, y, t)=1=1+\phi(x, y, t)-\phi(u, v, t)$ for all $x ,y \in X$

Therefore "$\preccurlyeq$"is a reflexive relation (i)

For any $x, y, u, v \in X$ suppose that $x \preccurlyeq u, y \succcurlyeq v ,x \succcurlyeq u, y \preccurlyeq v$ then we have.

$x \preccurlyeq u , y \succcurlyeq v \Leftrightarrow M(x, u, t)M(y, v, t)\geq 1+\phi(x, y, t)-\phi(u, v, t)$ (I)

and $x \succcurlyeq u , y \preccurlyeq v \Leftrightarrow$

$M(u, x, t)M(v, y, t)\geq 1+\phi(u, v, t)-\phi(x,y,t)$ (II)

Adding (I) & (II), we get,

$2M(x, u, t) M(y, v, t) \geq 2$

Or $M(x, u, t) M(y, v, t) \geq 1$

$M(x, u, t) M(y, v, t)=1 \Rightarrow M(x, u, t)=1 ,M(y, v, t)=1$

i.e. $x=u$ & $y=v$

Therefore "$\preccurlyeq$" is antisymmetric relation. (ii)

If $x \preccurlyeq u , y \succcurlyeq v , u \preccurlyeq u , v \succcurlyeq v$,

We have,

$M(x ,u', t) M(y, v', t)$

$\geq M(x, u, t)M(y, v, t)*M(u, u', t) M(v, v', t)$

$=\max[M(x, u, t)M(y, v, t)+ M(u, u', t) M(v, v', t)-1,0]$

$=\max[1+\phi(x, y, t)-\phi(u, v, t)+1+\phi(u, v, t)-\phi(u',v', t)-1,0]$

$=\max[1+\phi(x, y, t)-\phi(u', v', t),0]$

$=1+\phi(x, y, t)-\phi(u', v', t)$ i.e. $x \preccurlyeq u'$, $y \succcurlyeq v'$

And

$N(x, u', t) N(y, v', t)$

$\leq N(x, u, t)N(y, v, t)\Diamond N(u, u', t) N(v, v', t)$

$=\max[N(x, u, t)N(y, v, t)+ N(u, u' ,t) N(v, v', t)-1,0]$

$=\max[1+\phi(x, y, t)-\phi(u, v, t)+1+\phi(u, v, t)-\phi(u', v', t)-1,0]$

$=\max[1+\phi(x, y, t)-\phi(u', v', t),0]$

$=1+\phi(x, y, t)-\phi(u', v', t)$ i.e. $x \preccurlyeq u'$, $y \succcurlyeq v'$

Thus "$\preccurlyeq$" is transitive relation. (iii)

Theorem 3.3: Let (X, M, N, *, $\Diamond$) be a a non-Archimedean Intuitionistic fuzzy metric space With $a*b \geq \max\{a+b-1,0\}$ and $a\Diamond b\leq \min\{a+b-1,0\}$ with ϕ: $X\times X\times[0,\infty) \to R$, bounded from above "$\preccurlyeq$" the partial order induced by ϕ if $F:X\times X\to X$ follows k-monotone property over X and there are $x_0 , y_0 ,x_1,y_1 \in X$,such that $x_0\preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, where $x_1=F(x_0, y_0)$ & $y_1 = F(y_0,x_0)$

then there exists $x, y, \in X$ such that $F(x, y) = x$ & $F(y, x)=y$.

Proof: Let $x_0, y_0, x_1, y_1 \in X$,such that $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, where $x_1=F(x_0, y_0)$ & $y_1 = F(y_0,x_0)$.we construct sequences $\{x_n\}$ & $\{y_n\}$ in X as follows

$$x_{n+1}= F(x_n, y_n) \ \& y_{n+1}= F(y_n, x_n) \text{ for all } n \geq 0.$$

we shall show that $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n \geq 0$

Since $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, therefore by k-monotone property $x_1=F(x_0, y_0) \preccurlyeq F(x_1,y_1)=x_2$ and $y_1=F(y_0, x_0) \succcurlyeq F(y_1,x_1)=y_2$ i. e. $x_1 \preccurlyeq x_2$, $y_1 \succcurlyeq y_2$,

again applying the same property we have

$$x_2=F(x_1, y_1) \preccurlyeq F(x_2,y_2)=x_3$$

and
$$y_2=F(y_1, x_1) \succcurlyeq F(y_2,x_2)=y_3$$

Continue in this manner we shall have,

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \ldots\ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots\ldots$$

and $y_0 \succcurlyeq y_1 \succcurlyeq y_2 \ldots\ldots \succcurlyeq y_n \succcurlyeq y_{n+1} \succcurlyeq \ldots\ldots$

By the definition of "$\preccurlyeq$" we have , for all $t>0$ $\phi(x_0, y_0,t) \preccurlyeq \phi(x_1, y_1,t) \preccurlyeq \phi(x_3, y_3,t) \preccurlyeq \ldots\ldots$ In other words, for all $t>0$, the sequence $\{\phi(x_n ,y_n ,t)\}$ is non decreasing in R. Since $\phi$is bounded above, and $\{\phi(x_n , y_n , t)\}$ is convergent and hence it is a Cauchy sequence . So, for all $\varepsilon>0$, there exists $n_0 \in N$ so that for all $m>n>n_0$ and $t>0$ we have,

$$\left| \phi(x_m , y_m ,t)- \phi(x_n ,y_n ,t) \right| <\varepsilon$$

Since $x_n \preccurlyeq x_m$ & $y_n \succcurlyeq y_m$, we have

$$x_n \preccurlyeq x_m \ \& \ y_n \succcurlyeq y_m \Leftrightarrow M(x_n , x_m ,t) M(y_n , y_m ,t)\geq 1+\phi(x_n , y_n ,t)- \phi(x_m , y_m ,t) \text{ for all } t>0$$

$$1-[\phi(x_m , y_m ,t) -\phi(x_n ,y_n ,t)] >1-\varepsilon$$

$$x_n \preccurlyeq x_m \ \& \ y_n \succcurlyeq y_m \Leftrightarrow N(x_n, x_m, t) N(y_n, y_m, t) \leq 1+\phi(x_n, y_n, t)- \phi(x_m ,y_m ,t) \text{ for all } t>0$$

$$1-[\phi(x_m , y_m ,t) -\phi(x_n , y_n ,t)] <1-\varepsilon$$

We claim that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X, if not then there exists some ε_1 , ε_2 such that$\varepsilon_1 < \varepsilon_2$ and

$$M (x_n, x_m, t)\leq(1-\varepsilon_1) \ \& \ M (y_n, y_m, t)\leq(1-\varepsilon_2)$$

Then

$$M (x_n, x_m, t) M (y_n, y_m, t)\leq(1-\varepsilon_1))(1-\varepsilon_2) < (1-\varepsilon_1))^2 < (1-\varepsilon_1))$$

And
$$N (x_n,x_m,t)\leq(1-\varepsilon_1) \ \& \ N(y_n,y_m,t)\leq(1-\varepsilon_2)$$

Then $N (x_n,x_m,t) N (y_n,y_m,t)\leq(1-\varepsilon_1)) (1-\varepsilon_2) < (1-\varepsilon_1))^2 < (1-\varepsilon_1))$

Which is a contradiction.

This shows that the sequence $\{x_n\}$ & $\{y_n\}$ a Cauchy sequence in X, since X is complete , these converges to points x, y respectively in X consequently, by the continuity of F, we have $F(x,y)=x$ & $F(y,x)=y$.

REFERENCES

[1]. Atanassov K., Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986) 87-96.

[2]. Berinde V., Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 7347-7355.

[3]. Bhaskar T.G. and Lakshmikantham V., Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal: Theory Methods Appl. 65 (2006) 1379-1393.

[4]. Chandok S., Mustafa Z., Postolache M., Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces, U. Politeh. Buch. Ser. A 75(2013), No. 4, 11-24.

[5]. Choudhury B.S., Metiya N., PostolacheM., A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. Volume 21 (2013)

[6]. Ciric L. C. and Lakshmikantham V., Coupled random fixed point theorems for nonlinear contraction partially ordered metric spaces, Stochastic Analysis and Applications, vol. 27, (2009) 1246–1259

[7]. Coker D., An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997), 81–99

[8]. Deng Z., Fuzzy pseudo-metric space, J. Math. Anal. Appl. 86 (1982) 74-95.

[9]. George A., Veeramani P., On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems, 65(1994), 395–399

[10] Hu X.Q., Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, Fixed Point Theory and Appl., (2011), ID 363716

[11] Kramosil O. and Michalek J. ,Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326-334.

[12] Luong N.V., Thuan N.X., Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal., 74(2011), 983–992.

[13] Menger K. , Statistical metrics, Proc. Nat. Acad. Sci. 28 (1942), 535-537.

[14] Mihet D., Fuzzy contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems 159 (2008) 739-744.

[15] Mohinta S., Samanta T.K., Coupled fixed point theorems in partially ordered non-Archimedean complete fuzzy metric spaces, Annals of Fuzzy Mathematics and Informatics, Volume 11, No. 5, (May 2016), pp. 829-840

[16] Nguyen Van Luong and Nguyen Xuan Thuan, Coupled Fixed Point Thermos in Partially Ordered Metric Spaces, Bulletin of Mathematical Analysis and Applications 2 (4) (2010) 16-24.

[17] Park J.H., Park J.S., Kwun Y.C., A common fixed point theorem in the intuitionistic fuzzy metric space, Advances in Natural Comput. Data Mining (Proc. 2nd ICNC and 3rd FSKD), (2006) 293–300.

[18] Park J.S. , Fixed point theorem for common property(E.A.) and weak compatible functions in intuitionistic fuzzy metric space, Int. J. F. I. S., 11(3)(2011), 149–153.

[19] Park J.S., Coupled Fixed Point for Map Satisfying the Mixed Monotone Property in Partially Ordered Complete NIFMS, Applied Mathematical Sciences, 8 (43) (2014) 2105-2111.

[20] Schweizer B. and Sklar A., Statistical metric space, Pacific journal of mathematics 10 (1960)314{334.

[21] Zadeh L.A., Fuzzy sets, Information and control 8 (1965) 338-353.

***Corresponding Author:** Akhilesh Jain
Department of Mathematics, Corporate Institute of Science and Technology, Bhopal (India)-462021
(e-mail: akhiljain2929@gmail.com) Mobile No. *+919827353835*