On Methods used in Oscillation and Nonoscillation Criteria for Second Order Differential Equations

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Abstract—Second order differential equations play essential role in describing various physical, chemical and biological phenomena. In this paper, the main attention is devoted to half-linear second order differential eqaution and its special case which is Sturm-Liouville linear differential equation. In particular, oscillatory properties of solutions of these types of equations are investigated. We formulate Hille-Nehari type criteria that guarantee oscillation or nonoscillation of linear and half-linear differential equation. Results given in this paper can be applied in further investigation of oscillatory and nonoscillatory properties of solutions of both linear and half-linear second order differential equations.

Keywords—Half-linear differential equation, nonoscillation, oscillation, perturbation, Ricatti technique, Sturm-Liouville differential equation, variational principle.

I. Introduction

THE aim of this paper is to investigate oscillatory properties of the half-linear second-order differential equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1}$$

where $\Phi(x) := |x|^{p-2}x$, $p > 1$, $t \in I := [T, \infty)$ and r, c are real-valued continuous functions and $r(t) > 0$.

Oscillation theory of (1) attracted considerable attention in the past years and it was shown that solutions of (1) behave in many aspects like those of the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \tag{2}$$

which is the special case $p = 2$ of (1).

The aim of this paper is to present some results of the investigation oscillatory properties of equation (1) in comparison with that one of (2).

Note that the term *half-linear* equations is motivated by the fact that the solution space of (1) has just one half of the properties which characterize linearity, namely homogeneity, but not additivity.

Half-linear equations are closely related to the partial differential equations with the so called p-Laplacian. In fact, (HL) is sometimes called the *differential equation with the one-dimensional p-Laplacian*.

The paper is organized as follows. In Section 2 we present basic concepts and properties of solutions of (1) and (2). Section 3 is devoted to the investigation of properties of solutions of (1) and (2), in particular, we present oscillation criteria for (1) and (2). Section 4 gives some nonoscillation criteria for (1) and (2). Section 5 gives concluding remarks and comments.

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II. Preliminary results

In this section we define basic concepts concerning the half-linear differential equation (1), see [2], [5], [8].

Definition 1: Two points $t_1, t_2 \in \mathbb{R}$ are said to be *conjugate relative to* (1) if there exists a nontrivial solution x of this equation such that $x(t_1) = x(t_2) = 0$.

Definition 2: Equation (1) is said to be *disconjugate* on a closed interval $[a, b]$ if this interval contains no pair of points conjugate relative to (1) (i.e., every nontrivial solution has at most one zero in I). In the opposite case, (1) is said to be *conjugate* on I (i.e., there exists a nontrivial solution with at least two zeros in I).
Note that by a zero of a solution x we mean such a $t_0 \in \mathbb{R}$ that $x(t_0) = 0$.

Proposition 1: Equation (1) is disconjugate on an interval $I = [a, b]$ if and only if every its nontrivial solution has at most one zero in $[a, b]$.

The following property of zeros of linearly independent solutions is one of the most characteristic properties which justifies the definition of oscillation/nonoscillation of the equation. It is known as the Sturmian separation theorem and reads as follows (see [5]).

Proposition 2: Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of (1). Then any other solution of this equation which is not proportional to x has exactly one zero on (t_1, t_2).

Along with (1) consider another equation of the same form

$$(R(t)\Phi(y'))' + C(t)\Phi(y) = 0, \tag{3}$$

where the functions R and C satisfy the same assumptions as r and c in (1). The next theorem is known as the Sturmian comparison theorem and reads as follows (see [5]).

Proposition 3: Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of (1) and suppose that

$$C(t) \geq c(t), \quad r(t) \geq R(t) > 0$$

for $t \in [t_1, t_2]$. Then any solution of (3) has a zero in (t_1, t_2) or it is a multiple of the solution x.

Definition 3: Equation of (1) is said to be *nonoscillatory* at ∞, if there exists $T_0 \in \mathbb{R}$ such that (1) is disconjugate on $[T_0, T_1]$ for every $T_1 > T_0$. In the opposite case, (1) is said to be *oscillatory*, i.e., if every nontrivial solution has infinitely many zeros tending to ∞.

The previous definition says that one solution of (1) is oscillatory if and only if any other solution of (1) is oscillatory.

Oscillation of a nontrivial solution of (1) means the existence of zeros of this solution tending to ∞.

The following methods of oscillation theory are used in this paper: the variational principle and the Riccati technique. Basic properties of these methods are as follows.

A. Variational principle

The investigation of nonoscillation of (2) using the variational principle is based on the following statement.

Proposition 4: Equation (2) is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that

$$\mathcal{F}(y;T,\infty) = \int_T^\infty [r(t)y'^2 - c(t)y^2]\,dt > 0$$

for every nontrivial solution y such that $y(T) = 0$ and $y(t) \equiv 0$ on $[T_1, \infty)$ for some $T_1 > T$.

Similar nonoscillation criterion holds for equation (1):

Proposition 5: Equation (1) is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that

$$\mathcal{F}(y;T,\infty) = \int_T^\infty [r(t)|y'|^p - c(t)|y|^p]\,dt > 0$$

for every nontrivial solution y such that $y(T) = 0$ and $y(t) \equiv 0$ on $[T_1, \infty)$ for some $T_1 > T$.

On the other hand, to prove that (1), (2), respectively, is oscillatory, one needs to construct a nontrivial function y for which $\mathcal{F}(y;T,\infty) \le 0$.

B. Riccati technique

Let x be a solution of (2) such that $x(t) \neq 0$ in an interval I. Then $w(t) = \frac{r(t)x'}{x}$ is a solution of the associated Riccati differential equation

$$w' + c(t) + \frac{w^2}{r(t)} = 0. \tag{4}$$

Nonoscillation of (2) via the Riccati technique is proved using the following statement.

Proposition 6: Equation (2) is nonoscillatory if and only if there exists $T_0 \in \mathbb{R}$ and a (continuously differentiable) function $w : [T_0, \infty) \to \mathbb{R}$ such that

$$w' + c(t) + \frac{w^2}{r(t)} \le 0 \quad \text{for} \quad t \in [T_0, \infty)\,.$$

Now we introduce the half-linear version of the Riccati type equation associated with equation (1).

Let x be a solution of (1) such that $x(t) \neq 0$ in an interval I. Then

$$w(t) = \frac{r(t)\Phi(x'(t))}{\Phi(x(t))}$$

is a solution of the Riccati type differential equation of the form

$$w' + c(t) + (p-1)\,r^{1-q}(t)|w|^q = 0, \tag{5}$$

where q is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Base on the Riccati substitution, the following holds.

Proposition 7: Equation (1) is nonoscillatory if and only if there exists $T_0 \in \mathbb{R}$ and a (continuously differentiable) function $w : [T_0, \infty) \to \mathbb{R}$ such that

$$w' + c(t) + (p-1)\,r^{1-q}(t)|w|^q \le 0 \quad \text{for} \quad t \in [T_0, \infty)\,.$$

III. Oscillation criteria

In this section we present oscillation criteria for equations (2) and (1), respectively.

A. Oscillation criteria for Sturm-Liouville equation

Hille-Nehari type oscillation criteria are criteria formulated in terms of the asymptotic behavior of the functions

$$\int^t r^{-1}(s)\,ds \int_t^\infty c(s)\,ds \tag{6}$$

or

$$\int_t^\infty r^{-1}(s)\,ds \int^t c(s)\,ds$$

depending on the convergence/divergence of the integrals appearing in these formulas. Note that if both integrals $\int^\infty r^{-1}(t)\,dt = \infty, \int^\infty c(t)\,dt = \infty$, equation (2) is oscillatory by the Leighton-Wintner oscillation criterion, see [12]. For the case (6), the Hille-Nehari criterion reads as follows (see, e.g. [1, Chap. 2]).

Theorem 1: Suppose that

$$\int^\infty r^{-1}(t)\,dt = \infty \tag{7}$$

and the integral $\int^\infty c(t)\,dt$ is convergent. Equation (2) is oscillatory provided one of the following conditions holds:

(i) $c(t) \ge 0$ for large t and

$$\limsup_{t\to\infty} \left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty c(s)\,ds\right) > 1, \tag{8}$$

(ii)

$$\liminf_{t\to\infty} \left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty c(s)\,ds\right) > \frac{1}{4}. \tag{9}$$

PROOF. (i) We prove this statement using the variational principle, i.e., we find, for every $T \in \mathbb{R}$, a function $y \in W^{1,2}(T,\infty)$ with a compact support in (T,∞) such that the functional $\mathcal{F}(y;T,\infty) \le 0$. To this end, let $T \in \mathbb{R}$ be arbitrary, $T < t_0 < t_1 < t_2 < t_3$, and let

$$y(t) = \begin{cases} 0 & T \le t \le t_0, \\ \int_{t_0}^t r^{-1}(s)\,ds \left(\int_{t_0}^{t_1} r^{-1}(s)\,ds\right)^{-1} & t_0 \le t \le t_1, \\ 1 & t_1 \le t \le t_2, \\ \int_t^{t_3} r^{-1}(s)\,ds \left(\int_{t_2}^{t_3} r^{-1}(s)\,ds\right)^{-1} & t_2 \le t \le t_3, \\ 0 & t_3 \le t < \infty. \end{cases}$$

By a direct computation, using the fact that $c(t) \ge 0$ for large t, we obtain

$$\mathcal{F}(y;T,\infty) \le \left(\int_{t_0}^{t_1} r^{-1}(s)\,ds\right)^{-1} \left(\int_{t_2}^{t_3} r^{-1}(s)\,ds\right)^{-1} - \int_{t_1}^{t_2} c(s)\,ds$$

$$= \left(\int_{t_0}^{t_1} r^{-1}(s)\,ds\right)^{-1} \left[1 + \left(\int_{t_0}^{t_1} r^{-1}(s)\,ds\right)^{-1} \left(\int_{t_2}^{t_3} r^{-1}(s)\,ds\right)^{-1}\right.$$

$$-\int_{t_0}^{t_1} r^{-1}(s)\,ds\left(\int_{t_1}^{t_2} c(s)\,ds\right)\Bigg].$$

Now, by (8), there exists $\varepsilon > 0$ such that the last term in the brackets is less than $-1-\varepsilon$ for t_1 and t_2 sufficiently large, and (7) implies that the middle term is less than ε if t_3 is sufficiently large. Hence $\mathcal{F}(y;T,\infty) \le 0$ for t_1, t_2, t_3 chosen in this way.

(ii) This part of the proof is based on the Riccati technique. Suppose, by contradiction, that (2) is nonoscillatory and w is a solution of the associated Riccati equation (4). Then, according to [8, Chap. XI], the solution w can be expressed in the form

$$w(t) = \int_t^\infty c(s)\,ds + \int_t^\infty \frac{w^2(s)}{r(s)}\,ds.$$

Multiplying the last equation by $\int^t r^{-1}(s)\,ds$, we have

$$\left(\int^t r^{-1}(s)\,ds\right) w(t) = \left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty c(s)\,ds\right) + \left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty \frac{w^2(s)}{r(s)}\,ds\right).$$

Suppose first that $\liminf_{t\to\infty}\left(\int^t r^{-1}(s)\,ds\right) w(t) = \lambda$ exists finite. Then, using (9), there exists $\varepsilon > 0$ such that

$$\left(\int^t r^{-1}(s)\,ds\right) w(t) \ge \frac{1}{4} + \varepsilon + \left(\int^t r^{-1}(s)\,ds\right) \times \int_t^\infty \left[\left(\int^s r^{-1}(\tau)\,d\tau\right)^2 w^2(s)\cdot\frac{r^{-1}(s)}{\left(\int^s r^{-1}(\tau)\,d\tau\right)^2}\right] ds$$

for large t, and hence, letting $t \to \infty$ in the last inequality,

$$\lambda \ge \frac{1}{4} + \varepsilon + \lambda^2,$$

i.e., $-\left(\lambda - \frac{1}{2}\right)^2 \ge \varepsilon$, which is a contradiction.

If

$$\liminf_{t\to\infty}\left(\int^t r^{-1}(s)\,ds\right) w(t) = \infty, \tag{10}$$

denote

$$m(t) = \inf_{t\le s}\left(\int_T^s r^{-1}(\tau)\,d\tau\right) w(s).$$

Then m is nondecreasing and using (9) there exists $\varepsilon > 0$ such that

$$\left(\int^t r^{-1}(s)\,ds\right) w(t) \ge \frac{1}{4} + \varepsilon + \left(\int^t r^{-1}(s)\,ds\right) \times \int_t^\infty m^2(t)\cdot\frac{r^{-1}(s)}{\left(\int^s r^{-1}(\tau)\,d\tau\right)^2}\,ds,$$

which means

$$\left(\int^t r^{-1}(s)\,ds\right) w(t) \ge \frac{1}{4} + \varepsilon + m^2(t).$$

Since m is nondecreasing, we have for $s > t$

$$m(s) \ge \frac{1}{4} + \varepsilon + m^2(t),$$

thus,

$$m(t) \ge \frac{1}{4} + \varepsilon + m^2(t),$$

which is a contradiction with (10). □

Now we modify the previous result to the situation when (2) is viewed as a perturbation of the linear Sturm-Liouville differential equation

$$(r(t)x')' + \tilde{c}(t)x = 0. \tag{11}$$

If (11) is nonoscillatory and h is its *principal* solution (see [5]), then the transformation $x = h(t)y$ transforms (2) into the equation

$$\left(r(t)h^2(t)y'\right)' + \left(c(t) - \tilde{c}(t)\right)h^2(t)y = 0,$$

where $\displaystyle\int^\infty r^{-1}(t)h^{-2}(t)\,dt = \infty$.

Applying this transformation, Theorem 1 can be reformulated as follows, we present the result for the part (ii) (see [11]), the part (i) can be reformulated analogicaly.

Theorem 2: Suppose that $\displaystyle\int^\infty \frac{dt}{r(t)h^2(t)} = \infty$ and the integral

$$\int^\infty \left(c(t) - \tilde{c}(t)\right)h^2(t)\,dt$$

is convergent. If

$$\liminf_{t\to\infty}\left(\int^t \frac{1}{r(s)h^2(s)}\,ds\right)\left(\int_t^\infty \left(c(s) - \tilde{c}(s)\right)h^2(s)\,ds\right) > \frac{1}{4},$$

then (2) is oscillatory.

If h is a *nonprincipal* solution (see [5]), it can be shown that (2) is oscillatory provided one of the following conditions holds:

(i) $c(t) \ge \tilde{c}(t)$ for large t and

$$\limsup_{t\to\infty}\left(\int_t^\infty \frac{1}{r(s)h^2(s)}\,ds\right)\left(\int^t (c(s) - \tilde{c}(s))h^2(s)\,ds\right) > 1$$

(ii)

$$\liminf_{t\to\infty}\left(\int_t^\infty \frac{1}{r(s)h^2(s)}\,ds\right)\left(\int^t (c(s) - \tilde{c}(s))h^2(s)\,ds\right) > \frac{1}{4}.$$

B. Oscillation criteria for half-linear equation

Now, we turn out attention to Hille-Nehari type oscillation criteria for the half-linear differential equation (1).

A direct modification of the proof of Theorem 1 shows that the criteria given in that theorem can be extended to (1) as follows, see [5, Sec 3.1.1].

Theorem 3: Suppose that $\int^\infty r^{1-q}(t)\,dt = \infty$ and the integral $\int^\infty c(t)\,dt$ is convergent. Equation (1) is oscillatory provided one of the following conditions holds:

(i) $c(t) \ge 0$ for large t and

$$\limsup_{t\to\infty}\left(\int^t r^{1-q}(s)\,ds\right)^{p-1}\left(\int_t^\infty c(s)\,ds\right) > 1,$$

(ii)

$$\liminf_{t\to\infty}\left(\int^t r^{1-q}(s)\,ds\right)^{p-1}\left(\int_t^\infty c(s)\,ds\right) > \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}.$$

Note that in the modification of the part (ii) of the proof of Theorem 3, one needs to use the fact that

$$|\lambda|^q-\lambda=\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}\ge 0 \quad \text{for every} \quad t\in\mathbb{R}.$$

Similarly as in the linear case, we we will regard (1) as a perturbation of nonoscillatory equation

$$(r(t)\Phi(x'))'+\tilde{c}(t)\Phi(x)=0. \tag{12}$$

The following theorem is proved in [4].

Theorem 4: Let $\int^\infty r^{1-q}(t)\,dt=\infty$,

$$\int^\infty c(t)\,dt \text{ converges} \quad \text{and} \quad \int_t^\infty c(s)\,ds\ge 0 \quad \text{for large } t.$$

Further suppose that equation (12) is nonoscillatory and possesses a positive solution satisfying

(i) The derivative $h'(t)>0$ for large t,

(ii) It holds
$$\int^\infty r(t)\,(h'(t))^p\,dt=\infty,$$

(iii) There exists a finite limit
$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L>0.$$

Denote by

$$G(t)=\int^t\frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}$$

and suppose that the integral

$$\int^\infty (c(t)-\tilde{c}(t))\,h^p(t)\,dt=\lim_{b\to\infty}\int^b (c(t)-\tilde{c}(t))\,h^p(t)\,dt$$

is convergent. If

$$\liminf_{t\to\infty} G(t)\int_t^\infty (c(s)-\tilde{c}(s))\,h^p(s)\,ds>\frac{1}{2q} \tag{13}$$

then equation (1) is oscillatory.

Note that the function h is essentially the so-called *principal solution* of (12), in particular, from (13) follows that

$$G(t)=\int^t\frac{ds}{r(s)h^2(s)|h'(s)|^{p-2}}\longrightarrow\infty \quad \text{as} \quad t\to\infty.$$

If the function h is the *nonprincipal* solution of (12) then we have the following oscillation criterion for (1). The proof of this criterion can be found in [6].

Theorem 5: Let $\tilde{x}$, h be the positive principal and nonprincipal solutions of (nonoscillatory) equation (12), respectively, and suppose that

$$\lim_{t\to\infty} h^p(t)[w_h(t)-\tilde{w}(t)]=\infty,$$

where $w_h=\frac{r\Phi(h')}{\Phi(h)}, \tilde{w}=\frac{r\Phi(\tilde{x}')}{\Phi(\tilde{x})}$. If

$$\liminf_{t\to\infty}\frac{1}{h^p(t)[w_h(t)-\tilde{w}(t)]}\int_T^t[c(s)-\tilde{c}(s)]h^p(s)\,ds>1 \tag{14}$$

for some $T\in\mathbb{R}$ sufficiently large, then (1) is oscillatory. Moreover, if $c(t)\ge\tilde{c}(t)$ for large t, $\liminf$ in (14) can be replaced by $\limsup$.

IV. Nonoscillation criteria

In this part we present Hille-Nehari type nonoscillation criteria for equations (1) and (2), respectively. Let us start with the linear case.

A. *Nonoscillation criteria for Sturm-Liouville equation*

In this subsection we formulate nonoscillation criteria for (2) with $\int^\infty r^{-1}(t)\,dt=\infty$. Similar nonoscillation criteria can be formulated also in the case when $\int^\infty r^{-1}(t)\,dt<\infty$, see [9].

The following theorem is presented and proved in [10].

Theorem 6: Suppose that $\int^\infty r^{-1}(t)\,dt = \infty$ and $\int^\infty c(t)\,dt<\infty$. If

$$\limsup_{t\to\infty}\left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty c(s)\,ds\right)<\frac{1}{4}, \tag{15}$$

$$\liminf_{t\to\infty}\left(\int^t r^{-1}(s)\,ds\right)\left(\int_t^\infty c(s)\,ds\right)>-\frac{3}{4}, \tag{16}$$

then (2) is nonoscillatory.

PROOF. It is well known from the linear Sturmian theory that equation (2) is nonoscillatory provided there exists a differentiable function u which satisfies the Riccati-type inequality

$$u'+c(t)+\frac{u^2}{r(t)}\le 0 \tag{17}$$

for large t (see [8, Chap. XI]).
We will show that the function

$$u(t)=\frac{1}{4\int^t r^{-1}(s)\,ds}+\int_t^\infty c(s)\,ds$$

satisfies (17) for large t. To this end, denote

$$G(t):=\int^t r^{-1}(s)\,ds, \quad C(t):=\int_t^\infty c(s)\,ds,$$

thus,

$$u(t)=\frac{1}{4G(t)}+C(t).$$

Then we have

$$\begin{aligned} u'+c(t)+\frac{u^2}{r(t)} &= -\frac{r^{-1}(t)}{4Gt(t}-C(t)+C(t)\\ &\quad+\frac{(1+4G(t)C(t))^2}{16G^2(t)r(t)}\\ &=\frac{r^{-1}(t)}{4G^2(t)}\left[-1+\frac{(1+4G(t)C(t))^2}{4}\right]. \end{aligned}$$

Now, since (15) and (16) hold, there exists $\delta > 0$ such that

$$-\frac{3+\delta}{4} < G(t)C(t) < \frac{1+\delta}{4} \quad \Longleftrightarrow$$

$$|1+4G(t)C(t)| < 2-\delta \quad \Longleftrightarrow \quad \frac{(1+4G(t)C(t))^2}{4} < 1.$$

Consequently, we have

$$u' + c(t) + \frac{u^2}{r(t)} \le 0,$$

thus, (2) is nonoscillatory. □

Let us now regard equation (2) as a perturbation of another Sturm-Liouville equation of the same form

$$(r(t)x')' + \tilde{c}(t)x = 0,$$

and let h be its solution. The transformation $x = hy$ transforms equation (2) into the equation

$$\left(r(t)h^2(t)y'\right)' + (c(t) - \tilde{c}(t))h^2(t)y = 0. \tag{18}$$

By the substitution

$$v = \frac{r(t)h^2(t)y'}{y},$$

we obtain the Riccati equation associated with (18) in the form

$$v' + (c(t) - \tilde{c}(t))h^2(t) + \frac{v^2}{r(t)h^2(t)} = 0. \tag{19}$$

Considering equation (2) and denoting $u(t)$ solution of the corresponding Riccati type equation

$$u' + c(t) + \frac{u^2}{r(t)} = 0, \tag{20}$$

the relationship between the solution $v(t)$ of (19) and $u(t)$ of (20) reads as follows. If $x = hy$, then $x' = h'y + hy'$ and sustituting x' into $u = \frac{rx'}{x}$ we obtain

$$u(t) = w_h(t) + h^{-2}(t)v(t),$$

where $w_h(t) = \frac{r(t)h'(t)}{h(t)}$. The above mentioned facts we will use in the outline of the proof of the next statement (see [10]). Here, the idea of the proof of Theorem 6 is applied to the transformed equation (18).

Theorem 7: Suppose that $\int^{\infty} r^{-1}(t)h^{-2}(t)\,dt = \infty$ and the integral

$$\int^{\infty} (c(t) - \tilde{c}(t))\, h^2(t)\,dt$$

is convergent. If

$$\limsup_{t\to\infty} \left(\int^t r^{-1}(s)h^{-2}(s)\,ds\right)\left(\int_t^{\infty} (c(s)-\tilde{c}(s))\,h^2(s)\,ds\right) < \frac{1}{4}, \tag{21}$$

$$\liminf_{t\to\infty} \left(\int^t r^{-1}(s)h^{-2}(s)\,ds\right)\left(\int_t^{\infty} (c(s)-\tilde{c}(s))\,h^2(s)\,ds\right) > -\frac{3}{4}, \tag{22}$$

then (2) is nonoscillatory.

OUTLINE OF THE PROOF. Let

$$v(t) = \frac{1}{4\int^t r^{-1}(s)h^{-2}(s)\,ds} + \int_t^{\infty} (c(s)-\tilde{c}(s))\,h^2(s)\,ds$$

Denoting

$$G(t) = \int^t r^{-1}(s)h^{-2}(s)\,ds,$$

$$C(t) = \int_t^{\infty} (c(s)-\tilde{c}(s))\,h^2(s)\,ds,$$

we have

$$v(t) = \frac{1}{4G(t)} + C(t).$$

Similarly as in the proof of the previous theorem, using (21) and (22), we obtain

$$v' + (c(t)-\tilde{c}(t))h^2(t) + \frac{v^2}{r(t)h^2(t)} \le 0.$$

This means that (18) is nonoscillatory and hence (2) is nonoscillatory as well. □

B. Nonoscillation criteria for half-linear equation

Before formulating Hille-Nehari type nonoscillation criteria for (1), let us recall that the basic facts of the half-linear oscillation theory can be found in [5]. In particular, the Riccati type equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q = \frac{p}{p-1} \tag{23}$$

and the associated inequality play the same role as (20) and (17) in the linear theory. Note that since the function Φ is not additive, we have no half-linear analogue of the transformation identity used in the previous section. This is why we need a different approach which is based on the quadratization of some nonlinear terms in the Riccati equation and Picone's identity ([5]).

Now, we present the main results which are formulated and proved in [3].

Theorem 8: Suppose that $\int^{\infty} r^{-1}(t)\,dt = \infty$ and $\int^{\infty} c(t)\,dt < \infty$. If

$$\limsup_{t\to\infty} \left(\int^t r^{1-q}(s)\,ds\right)^{p-1}\left(\int_t^{\infty} c(s)\,ds\right) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

and

$$\liminf_{t\to\infty} \left(\int^t r^{1-q}(s)\,ds\right)^{p-1}\left(\int_t^{\infty} c(s)\,ds\right) > -\frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

then (1) is nonoscillatory.

The next theorem deals with the case $\int^{\infty} r^{1-q}(t)\,dt < \infty$.

Theorem 9: Suppose that $\int^{\infty} r^{-1}(t)\,dt < \infty$. If

$$\limsup_{t\to\infty} \left(\int_t^{\infty} r^{1-q}(s)\,ds\right)^{p-1}\left(\int^t c(s)\,ds\right) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

and

$$\liminf_{t\to\infty} \left(\int_t^{\infty} r^{1-q}(s)\,ds\right)^{p-1}\left(\int_t^{\infty} c(s)\,ds\right) > -\frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

then (1) is nonoscillatory.

The proof of each statement follows the idea introduced in the proof of Theorem 6. We put

$$w = \left(\frac{p-1}{p}\right)^p \left(\int^t r^{1-q}(s)\,ds\right)^{1-p} + \int_t^\infty c(s)\,ds$$

in the proof of Theorem 8 and

$$w = -\left(\frac{p-1}{p}\right)^p \left(\int_t^\infty r^{1-q}(s)\,ds\right)^{1-p} + \int^t c(s)\,ds$$

in the proof of Theorem 9. By a direct computation similar to that in the proof of Theorem 6 one can show that

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q \le 0$$

and this implies, by [5, Theorem 2.2.1], that (1) is nonoscillatory.

Similarly as in linear case, let us now regard equation (1) as a perturbation of another half-linear equation of the same form

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{24}$$

and suppose that h is its solution. Then $w_h = \dfrac{r(t)\Phi(h')}{\Phi(h)}$ solves the Riccati equation

$${w_h}' + \tilde{c}(t) + (p-1)r^{1-q}(t)|w_h|^q = 0.$$

The following theorem is taken from [4]. We present here the main idea of the proof which consists in the quadratization of a certain nonlinear function appearing in the bellow given equation (28).

Theorem 10: Suppose that equation (24) is nonoscillatory and let h be its solution satisfying $h'(t) > 0$ for large t. Further suppose that there exists a finite limit

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) = L > 0$$

and

$$\int^\infty \frac{dt}{r(t)h^2(t)(h'(t))^{p-2}} = \infty.$$

If

$$\limsup_{t\to\infty} \left(\int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}\right)\left(\int_t^\infty (c(s)-\tilde{c}(s))\,h^p(s)\,ds\right) < \frac{1}{2q} \tag{25}$$

and

$$\liminf_{t\to\infty} \left(\int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}\right)\left(\int_t^\infty (c(s)-\tilde{c}(s))\,h^p(s)\,ds\right) > -\frac{3}{2q}, \tag{26}$$

then (1) is nonoscillatory.

PROOF. To prove that (1) is nonoscillatory, it suffices to find a differentiable function w satisfying the inequality

$$w' + \tilde{c}(t) + (p-1)r^{1-q}(t)|w|^q \le 0.$$

We will show that the function $w = w_h + h^{-p}v$ satisfies this inequality for large t. To this end, denote

$$R(t) = \frac{1}{r(t)h^2(t)(h'(t))^{p-2}}.$$

Further, consider the function

$$v(t) = \frac{1}{2q\int^t R^{-1}(s)\,ds} + \int_t^\infty (c(s)-\tilde{c}(s))\,h^p(s)\,ds \tag{27}$$

and denote

$$C(t) = \int_t^\infty (c(s)-\tilde{c}(s))h^p(s)\,ds$$
$$G(t) = \int^t R^{-1}(s)\,ds.$$

Substituting for $v = h^p[w - w_h]$ we obtain (suppressing the argument t)

$$\begin{aligned} v' &= p\Phi(h)h'[w-w_h] + h^p[w' - {w_h}'] \\ &= (\tilde{c}-c)h^p - pr^{1-q}h^p\left[\frac{1}{p}r^q\left(\frac{h'}{h}\right)^p - r^{q-1}\frac{h'}{h}w + \frac{|w|^q}{q}\right] \\ &= (\tilde{c}-c)h^p - pr^{1-q}h^pP\left(\Phi^{-1}(w_h), w\right), \end{aligned}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q}.$$

Thus, we get

$$v' + (c-\tilde{c})h^p + pr^{1-q}h^pP\left(\Phi^{-1}(w_h), w\right) = 0. \tag{28}$$

By the second order degree Taylor formula, we have

$$\begin{aligned} pr^{1-q}h^pP\left(\Phi^{-1}(w_h), w\right) &\sim \frac{q}{2}\frac{1}{rh^2(h')^{p-2}}\left[h^p\left(w-w_h\right)\right]^2 \\ &= \frac{q}{2}\frac{v^2}{R}. \end{aligned}$$

This means that equation (28) can be approximately expressed in the form

$$v' + (c(t)-\tilde{c}(t))h^p(t) + \frac{q}{2}\frac{v^2}{R(t)} = 0.$$

By (27) we obtain

$$\begin{aligned} v' + (c-\tilde{c})h^p + \frac{q}{2}\frac{v^2}{R} &= -\frac{R^{-1}}{2qG^2} - (c-\tilde{c})h^p + (c-\tilde{c})h^p \\ &\quad + \frac{q}{2}\frac{1}{R}\left(\frac{1}{2qG} + C\right)^2 \\ &= \frac{R^{-1}}{2qG^2}\left[-1 + \frac{(1+2qGC)^2}{4}\right]. \end{aligned}$$

According to (25) and (26), there exists $\delta > 0$ such that

$$\frac{-3+\delta}{2q} < G(t)C(t) < \frac{1-\delta}{2q} \quad\Longleftrightarrow\quad |1+2qG(t)C(t)| < 2-\delta$$

for large t, which means that

$$\frac{(1+2qG(t)C(t))^2}{4} < 1.$$

Consequently,

$$v' + (c(t)-\tilde{c}(t))h^p(t) + \frac{q}{2}\frac{v^2}{R(t)} \le 0.$$

Now, substituting $v = h^p(w - w_h)$ in the last inequality, a direct computation gives the inequality

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q \le 0,$$

thus, (1) is nonoscillatory. □

In contrast to the previous theorem, we do not suppose that h is a solution of (24). This criterion is formulated and proved in [7]. The idea of the proof of this statement is the same as in the previous theorem. Observe that if h is a solution of (24) satisfying assumptions of Theorem 10, then the bellow given additional conditions (29) and (30) (with respect to Theorem 10) are satisfied.

Theorem 11: Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$, $\int^{\infty} r^{-1}(t)h^{-2}(t)(h'(t))^{2-p}\,dt < \infty$, and denote

$$G(t) := \int_t^{\infty} \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}.$$

Suppose that

$$\lim_{t\to\infty} G(t)r(t)h(t)\Phi(h'(t)) = \infty \qquad (29)$$

and

$$\lim_{t\to\infty} G^2(t)r(t)h^3(t)(h'(t))^{p-2}\left[(r(t)\Phi(h'(t)))' + \tilde{c}(t)\Phi(h(t))\right] = 0. \qquad (30)$$

If

$$\limsup_{t\to\infty} G(t)\int_T^t [c(s)-\tilde{c}(s)]h^p(s)\,ds < \frac{1}{2q},$$

and

$$\liminf_{t\to\infty} G(t)\int_T^t [c(s)-\tilde{c}(s)]h^p(s)\,ds > -\frac{3}{2q}$$

for some $T \in \mathbb{R}$ sufficiently large, then (1) is nonoscillatory.

V. Conclusion

(Non)oscillation criteria for (1) and (2) presented in this paper were proved using the variational principle and the Riccati technique. Comparing results using these methods, we conlude that oscillation criteria proved using the variational prinpciple contain the oscillation contant which is 4-times larger than the constant in the same criterion proved by the Riccati technique. Based on this observation, we conjecture that this property holds also in other oscillation and nonoscillation criteria for linear and half-linear equations.

Presented results can be used as a theoretical base in the investigation of some physical, biological, and chemical phenomena (for example non-Newtonian fluid theory) which are described by the partial differential equations with the so called p-Laplacian and these PDE's can be reduced under some assumptions to the half-linear ordinary differential equations.

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