Slightly Continuous Functions in Topological Spaces

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Abstract— In this paper slightly $\omega-$continuity is introduced and studied. Furthermore, basic properties and presentation theorems of slightly $\omega-$continuous functions are investigated and relationships between slightly $\omega-$continuous functions and graphs are studied and investigated.

Keywords— Clopen, $\omega-$open, $\omega-$continuity, slightly continuity, slightly $\omega-$continuity.

I. INTRODUCTION AND PRELIMINARIES

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X,τ), $Cl(A)$, $Int(A)$ and $X-A$ denote the closure of A, the interior of A and the complement of A in X, respectively. Recently, as generalization of closed sets, the notion of $\omega-$closed sets was introduced and studied by Hdeib [13]. A point $x\in X$ is called a condensation point of A if for each $U\in\tau$ with $x\in U$, the set $U \cap A$ is uncountable. A subset A is said to be $\omega-$closed Hdeib [13] if it contains all its condensation points. The complement of an $\omega-$closed set is said to be an $\omega-$open set. It is well known that a subset W of a space (X,τ) is $\omega-$open if and only if for each $x\in W$, there exists $U\in\tau$ such that $x\in U$ and $U\setminus W$ is countable. The family of all $\omega-$open subsets of a topological space (X,τ) forms a topology on X which is finer than τ. The set of all $\omega-$open sets of (X,τ) is denoted by $\omega O(X,\tau)$.

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The set of all $\omega-$open sets of (X,τ) containing a point $x\in X$ is denoted by $\omega O(X,x)$. The intersection of all $\omega-$closed sets containing S is called the $\omega-$closure of S and is denoted by $\omega Cl(S)$. The $\omega-$interior of S is denoted by the union of all $\omega-$open sets contained in S and is denoted by $\omega Int(S)$. The complement of a $\omega-$open set is said to be $\omega-$closed. The intersection of all $\omega-$closed sets of *X* containing *A* is called the $\omega-$closure of *A* and is denoted by $\omega Cl(S)$. The union of all $\omega-$open sets of *X* contained in *A* is called $\omega-$interior of *A* and is denoted by $\omega Int(S)$. The family of all $\omega-$open, $\omega-$closed, clopen, $\omega-$clopen sets of *X* is denoted by $\omega O(X,\tau)$, $\omega Cl(X,\tau)$, $CO(X,\tau)$, $\omega CO(X,\tau)$. Functions and of course continuous functions are stated among the most important and most researched points in the whole of the Mathematical Sciences. Many different forms of continuous functions have been introduced over the years. Various interesting problems arise when one considers continuity. Its importance is significant in various areas of mathematics and related sciences. The aim of this paper is to introduce and study a new weaker form of continuity called slightly $\omega-$continuity. Moreover, basic properties and preservation theorems of slightly $\omega-$continuous functions are investigated and relationships between slightly $\omega-$continuous functions and graphs are investigated. In Section 2, the notion of slightly $\omega-$continuous functions is introduced and characterizations and some relationships of $\omega-$continuous functions and basic properties of slightly $\omega-$continuous functions are investigated and obtained. The relationships between slightly $\omega-$continuity and connectedness are investigated. In Section 3 and in Section 4, the relationships between slightly $\omega-$continuity and compactness and the relationships between slightly $\omega-$continuity and separation axioms and graphs are obtained.

II. SLIGHTLY $\omega-$CONTINUOUS FUNCTIONS

In this section, the notion of slightly $\omega-$continuous functions is introduced and characterizations and some relationships of $\omega-$continuous functions and basic properties of slightly $\omega-$continuous functions are investigated and obtained.

DEFINITION 2.1. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is called slightly $\omega-$continuous at a point $x\in X$ if for each clopen subset V in Y containing $f(x)$, there exists an $\omega-$open subset U in X containing x such that $f(U)\subseteq V.$

DEFINITION 2.2. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is called slightly $\omega-$continuous if it is slightly $\omega-$continuous at each point of X.

THEOREM 2.3. Let (X,τ) and (Y,σ) be topological spaces and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function. Then the following statements are equivalent.

(1) f is slightly $\omega-$continuous;

(2) for every clopen set $V\subseteq Y,$ $f^{-1}(V)$ is $\omega-$open;

(3) for every clopen set $V\subseteq Y,$ $f^{-1}(V)$ is $\omega-$closed;

(4) for every clopen set $V\subseteq Y,$ $f^{-1}(V)$ is $\omega-$clopen.

PROOF. $(1)\Rightarrow(2):$ Let V be a clopen subset of Y and let $x\in f^{-1}(V).$ Since $f(x)\in V,$ by (1), there exists an $\omega-$open set U_x such that $x\in U_x$ and $U_x\subseteq f^{-1}(V).$ We obtain $f^{-1}(V)=\mathrm{U}\{U_x : x\in f^{-1}(V)\}.$ Thus, $f^{-1}(V)$ is $\omega-$open.

$(2)\Rightarrow(3):$ Let V be a clopen subset of Y. Then, $Y\setminus V$ is clopen. By (2), $f^{-1}(Y\setminus V)=X\setminus f^{-1}(V)$ is an $\omega-$open set in X. Thus, $f^{-1}(V)$ is $\omega-$closed.

$(3)\Rightarrow(4):$ Let V be a clopen subset of Y. Then, by (3), $f^{-1}(V)$ is and $\omega-$closed set in $X.$ Note that $Y\setminus V$ is also clopen in $Y.$ Hence by (3), it follows that $f^{-1}(Y\setminus V)=X\setminus f^{-1}(V)$ is an $\omega-$closed set in $X.$ Thus, $f^{-1}(V)$ is $\omega-$clopen in $X.$

$(4)\Rightarrow(1):$ Let V be a clopen subset in Y containing $f(x).$ By (4), $f^{-1}(V)$ is $\omega-$clopen in $X.$ Take $U=f^{-1}(V).$ Then, $f(U)\subseteq V.$ Hence, f is slightly $\omega-$continuous.

THEOREM. 2.4. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function and $\Sigma=\{U_i : i\in I\}$ be a cover of X such that $U_i\in\omega O(X,\tau)$ for each $i\in I.$ If $f\,|\,U_i$ is slightly $\omega-$continuous for each $i\in I,$ then f is a slightly $\omega-$continuous function.

PROOF. Suppose that V be any clopen set of $Y.$ Since $f\,|\,U_i$ is slightly $\omega-$continuous for each $i\in I,$ it follows that $(f\,|\,U_i)^{-1}(V)\in\omega O(U_i,\tau\,|\,U_i).$ We have $f^{-1}(V)=\mathrm{U}\{f^{-1}(V)\mathrm{I}\ U_i : i\in I\}$ $=\mathrm{U}\left\{(f\,|\,U_i)^{-1}(V) : i\in I\right\}.$ We obtain $(f)^{-1}(V)\in\omega O(X,\tau)$ which means that f is slightly $\omega-$continuous.

THEOREM 2.5. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function and $x\in X.$ If there exists $U\in\omega O(X,\tau)$ such that $x\in U$ and the restriction of f to U is a slightly $\omega-$continuous function at $x,$ then f is slightly $\omega-$continuous at $x.$

PROOF. Suppose that $CO(Y,\sigma)$ containing $f(x).$ Since $f\,|\,U$ is slightly $\omega-$continuous at $x,$ there exists $V\in\omega O(U,\tau\,|\,U)$ containing x such that $f(V)=(f\,|\,U)(V)\subseteq F.$ Since $U\in\omega O(X,\tau)$ containing x, it follows that $V\in\omega O(X,\tau)$ containing x. This shows clearly that f is slightly $\omega-$continuous at $x.$

THEOREM 2.6. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function and let Let $g:X\longrightarrow X\times Y$ be the graph function of f, defined by $g(x)=(x,f(x))$ for every $x\in X.$ Then g is slightly $\omega-$continuous if and only if f is slightly $\omega-$continuous.

PROOF. Let $V\in CO(Y,\sigma),$ then $X\times V\in CO(X\times Y).$ Since g is slightly $\omega-$continuous, then $f^{-1}(V)=g^{-1}(X\times V)\in\omega O(X,\tau).$ Thus, f is slightly $\omega-$continuous.

Conversely, let $x\in X$ and let W be a closed subset of $X\times Y$ containing $g(x).$ Then $U\mathrm{I}\ (\{x\}\times Y)$ is clopen in $\{x\}\times Y$ containing $g(x).$ Also $\{x\}\times Y$ is homeomorphic to $Y.$ Hence $\{y\in Y:(x,y)\in W\}$ is a clopen subset of $Y.$ Since f is slightly $\omega-$continuous, $\mathrm{U}\{f^{-1}(y):(x,y)\in W\}$ is an $\omega-$open subset of $X.$

Also $x \in \cup\left\{ f^{-1}(y) : (x,y) \in W \right\} \subseteq g^{-1}(W)$. Hence $g^{-1}(W)$ is $\omega-$open. Then g is slightly $\omega-$continuous.

DEFINITION 2.7. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is called $\omega-$irresolute if and only if for every $\omega-$open subset G of Y, $f^{-1}(G)$ is $\omega-$-open in X.

DEFINITION 2,8. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is called $\omega-$open if for every $\omega-$open subset A of X, $f(A)$ is $\omega-$open in Y.

DEFINITION 2.9. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is said to be $\omega-$continuous if $f^{-1}(V)$ is $\omega-$***open*** set in X for each open set V of Y.

DEFINITION 2.10. [14]. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly continuous if $f^{-1}(V)$ is an open set in X for each clopen set V of Y.

THEOREM. 2.11. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ and $g:(Y,\sigma) \longrightarrow (Z,\delta)$ be functions. Then, the following properties hold:

(1). If f is $\omega-$irresolute and g is slightly $\omega-$continuous, then $gof:(X,\tau) \longrightarrow (Z,\delta)$ is slightly $\omega-$continuous.

(2). If f is $\omega-$irresolute and g is $\omega-$continuous, then $gof:(X,\tau) \longrightarrow (Z,\delta)$ is slightly $\omega-$continuous.

(3). If f is $\omega-$irresolute and g is slightly continuous, then $gof:(X,\tau) \longrightarrow (Z,\delta)$ is slightly $\omega-$continuous.

PROOF. (1) Let V be any clopen set in Z. Since g is slightly $\omega-$continuous, $g^{-1}(V)$ is $\omega-$open, Since f is $\omega-$irresolute, $f^{-1}\left[g^{-1}(V)\right] = (gof)^{-1}(V)$ is $\omega-$open. Therefore gof is slightly $\omega-$continuous.

(2) Let V be any clopen set in Z. Since g is $\omega-$continuous, $g^{-1}(V)$ is $\omega-$open. Since f is $\omega-$irresolute, $f^{-1}\left[g^{-1}(V)\right] = (gof)^{-1}(V)$ is $\omega-$open. Therefore, gof is slightly $\omega-$continuous.

(3). Let V be any clopen sets in Z. Since g is slightly continuous, $g^{-1}(V)$ is open in Y. Since every open set is $\omega-$open, $g^{-1}(V)$ is $\omega-$open in Y. Since f is $\omega-$irresolute, $f^{-1}\left[g^{-1}(V)\right] = (gof)^{-1}(V)$ is $\omega-$open. Therefore, gof is slightly $\omega-$continuous.

THEOREM 2.12. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ and $g:(Y,\sigma) \longrightarrow (Z,\delta)$ be functions. If f is $\omega-$open and surjective and $gof: X \longrightarrow Z$ is slightly $\omega-$continuous, then g is slightly $\omega-$continuous.

PROOF. Let V be any clopen set in Z. Since gof is slightly $\omega-$continuous, gof is slightly $\omega-$continuous, $(gof)^{-1}(V) = f^{-1}\left[g^{-1}(V)\right]$ is $\omega-$open. Since f is $\omega-$open, then $f\left(f^{-1}\left[g^{-1}(V)\right]\right) = g^{-1}(V)$ is $\omega-$open. Hence, g is slightly $\omega-$continuous.

Combining the previous two theorems, we obtain the following result.

THEOREM 2.13. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be surjective and $\omega-$ open and $g:(Y,\sigma) \longrightarrow (Z,\delta)$ be a function. Then $gof:(X,\tau) \longrightarrow (Z,\delta)$ is slightly $\omega-$continuous if and only if g is slightly $\omega-$continuous.

DEFINITION. 2.14. Let (X,τ) be a topological space. Let $x \in X$ and Λ be any filter base in X $\omega-$converging to x if for any $U \in \omega O(X,\tau)$ containing x, there exists a $B \in \Lambda$ such that $B \subseteq U$.

DEFINITION 2.15. Let (X,τ) be a topological space. A filter base Λ is said to be co – convergent to a point $x \in X$ if for any $U \in CO(X,\tau)$ containing x, there exists a $B \in \Lambda$ such that $B \subseteq U$.

THEOREM 2.16. If a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly $\omega-$continuous, then for each point $x \in X$ and each filter base Λ in X $\omega-$converging to x, the filter base $f(\Lambda)$ is co – convergent to $f(x)$.

PROOF. Let $x \in X$ and Λ be any filter base in X $\omega-$converging to x. Since f is slightly $\omega-$**continuous**, then for any $V \in CO(Y,\sigma)$ containing $f(x)$, there exists a $U \in \omega O(X,\tau)$ containing x such that $f(U) \subseteq V$. Since Λ is $\omega-$converging to x, there exists a $B \in \Lambda$ such that $B \subseteq U$. This means that $f(B) \subseteq V$ and therefore the filter base $f(\Lambda)$ is co-convergent to $f(x)$.

DEFINITION 2.17. A topological space (X,τ) is called $\omega-$connected provided that X is not the union of two disjoint nonempty $\omega-$open sets.

THEOREM 2.18. If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous surjective function and X is $\omega-$connected space, then Y is connected space.

PROOF. Suppose that Y is not connected space. Then there exists nonempty disjoint open sets U and V such that $Y=U\cup V.$ Therefore, U and V are clopen sets in $Y.$ Since f is slightly $\omega-continuous,$ then $f^{-1}(U)$ and $f^{-1}(V)$ are $\omega-$closed and $\omega-$open in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint and $X=f^{-1}(U)\cup f^{-1}(V).$ This shows that X is not $\omega-$connected. This is a contradiction. Hence, Y is connected.

III. COVERING PROPERTIES

In this section, the relationship between slightly $\omega-$continuous functions and compactness are investigated.

DEFINITION 3.1. A topological space (X,τ) is said to be mildly compact if for every clopen cover of X has a finite subcover.

DEFINITION 3.2. A topological space (X,τ) is said to be $\omega-$compact if for every $\omega-$open cover of X has a finite subcover.

DEFINITION 3.3. A subset A of a space X is said to be mildly compact (respectively $\omega-compact$) relative to X if every cover of A by clopen (resp. $\omega-$open) sets of X has a finite subcover.

DEFINITION. 3.4. A subset A of a space X is said to be mildly compact (respectively $\omega-$compact) if the subspace A is mildly compact (resp. $\omega-$compact).

THEOREM. 3.5 If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous and K is $\omega-$compact relative to X, then $f(K)$ is mildly compact in $Y.$

PROOF. Let $\{H_\alpha:\alpha\in I\}$ be any cover of $f(K)$ by clopen sets of the subspace $f(K).$ For each $\alpha\in I,$ there exists a clopen set K_α of Y such that $H_\alpha=K_\alpha\cap f(K).$ For each $x\in K,$ there exists $\alpha_x\in I,$ such that $f(x)\in K_{\alpha_x}$ and there exists $U_x\in\omega O(X,\tau)$ containing x such that $f(U_x)\subseteq K_{\alpha_x}.$ Since the family $\{U_x:x\in K\}$ is a cover of K by $\omega-$open sets of K, there exists a finite subset K_0 of K such that $K\subseteq\{U_x:x\in K_0\}.$ Therefore, we obtain $f(K)\subseteq\cup\{f(U_x):x\in K_0\}$ which is a subset of $\cup\{K_{\alpha_x}:x\in K_0\}.$ Thus $f(K)=\cup\{H_{\alpha_x}:x\in K_0\}$ and hence $f(K)$ is mildly compact.

COROLLARY 3.6. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous and X is $\omega-$compact, then Y is mildly compact.

DEFINITION 3.7. A topological space (X,τ) said to be mildly countably compact if every clopen countable cover of X has a finite subcover.

DEFINITION 3.8. A topological space (X,τ) is said to be mildly Lindelof if every cover of X by clopen sets has a countable subcover.

DEFINITION 3.9. A topological space (X,τ) said to be countably $\omega-$compact if every $\omega-open$ countable cover of X has a finite subcover.

DEFINITION 3.10. A topological space (X,τ) said to be $\omega-$Lindelof if every $\omega-$open cover of X has a countable subcover.

DEFINITION 3.11. A topological space (X,τ) said to be $\omega-$closed – compact if every $\omega-$closed cover of X has a finite subcover.

DEFINITION 3.12. A topological space (X,τ) said to be countably $\omega-$closed – compact if every countable cover of X by $\omega-$closed sets has a finite subcover.

DEFINITION 3.13. A topological space (X,τ) said to be $\omega-$closed – Lindelof if every cover of X by $\omega-$closed sets has a countable subcover.

THEOREM 3.14. Let (X,τ) be $\omega-$Lindelof and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega-continuous$ surjection. Then Y is mildly Lindelof.

PROOF. Let $\gamma=\{V_\alpha:\alpha\in I\}$ be any clopen cover of Y. Since f is slightly $\omega-$continuous, then $\lambda=f^{-1}(\gamma)=\{f^{-1}(V_\alpha):\alpha\in I\}$ is an $\omega-$open cover of X. Since X is $\omega-$Lindelof, there exists a countable subset I_0 of I such that $X=\cup\{f^{-1}(V_\alpha):\alpha\in I_0\}.$ Thus, we have $Y=\cup\{V_\alpha:\alpha\in I_0\}$ and Y is mildly Lindelof.

THEOREM 3.15. Let (X,τ) be countably $\omega-$compact and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega-$continuous surjection. Then Y is mildly countably compact.

PROOF. Let $\gamma=\{V_\alpha:\alpha\in I\}$ be a countable open cover of X. Since f is slightly $\omega-$continuous, then $\lambda=f^{-1}(\gamma)=\{f^{-1}(V_\alpha):\alpha\in I\}$ is an $\omega-$open cover of X. Since X is countably $\omega-$compact, there exists a finite subset I_0 of I such that $X=\bigcup\{f^{-1}(V_\alpha):\alpha\in I_0\}$. Thus, we have $Y=\bigcup\{V_\alpha:\alpha\in I_0\}$ and Y is mildly countably compact.

The proofs of the next three theorems can be obtained similarly as the previous two theorems.

THEOREM 3.16. Let (X,τ) be $\omega-$closed – compact and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega-$continuous surjection. Then Y is mildly compact.

THEOREM 3.17. Let (X,τ) be $\omega-$closed – Lindelof and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega-$continuous surjection. Then Y is mildly Lindelof.

THEOREM 3.18. Let (X,τ) be countably $\omega-$closed – compact and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega-$continuous surjection. Then Y is mildly countably compact.

IV. SEPARATION AXIOMS

In this section, the relationship between slightly $\omega-$continuous functions and separation axioms are investigated.

DEFINITION 4.1. A topological space (X,τ) said to be $\omega-T_1$ if for each pair of distinct points x and y of X, there exist $\omega-$open sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

DEFINITION 4.2. A topological space (X,τ) said to be $\omega-T_2$ $(\omega-\textit{Hausdorff})$ if for each pair of distinct points x and y in X, there exist disjoint $\omega-$open sets U and V in X such that $x\in U$ and $y\in V$.

DEFINITION 4.3. A topological space (X,τ) said to be clopen T_1 if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

DEFINITION 4.4. A topological space (X,τ) said to be clopen T_2 ***(clopen Hausdorff or ultra Hausdorff)*** if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x\in U$ and $y\in V$.

PROPOSITION 4.5. A topological space (X,τ) is $\omega-T_1$ if and only if the singletons are $\omega-$***closed*** sets.

PROPOSITION 4.6. A topological space (X,τ) is $\omega-T_2$ $(\omega-\textit{Hausdorff})$ if and only if the intersection of all $\omega-$closed $\omega-$neighbourhoods of each point of X is reduced to that point.

THEOREM 4.7. If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous injection and Y is clopen T_1, then X is $\omega-T_1$.

PROOF. Suppose that Y is T_1. For any distinct points x and y in X, there exist $V,W\in CO(Y,\sigma)$ such that $f(x)\in V$, $f(y)\notin V$, $f(x)\notin W$ and $f(y)\in W$. Since f is slightly $\omega-$continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\omega-$open subsets of X such that $x\in f^{-1}(V)$, $y\notin f^{-1}(V)$, $x\notin f^{-1}(W)$ and $y\notin f^{-1}(W)$. This shows that X is $\omega-T_1$.

THEOREM 4.8. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is a slightly $\omega-$continuous injection and Y is clopen T_2, then X is $\omega-T_2$.

PROOF. For any pair of distinct points x and y in X, there exist disjoint clopen sets U and V in Y such that $f(x)\in U$ and $f(y)\in V$. Since f is slightly $\omega-$continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\omega-$open in X containing x and y respectively. Therefore $f^{-1}(U)\cap f^{-1}(V)=\phi$ because $U\cap V=\phi$. This shows that X is $\omega-T_2$.

THEOREM 4.9. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is a slightly continuous function and $g:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous function and Y is clopen Hausdorff, then $E=\{x\in X: f(x)=g(x)\}$ is $\omega-$closed in X.

PROOF. If $x\in X\setminus E$, then it follows that $f(x)\neq g(x)$. Since Y is clopen Hausdorff, there exist $f(x)\in V\in CO(Y,\sigma)$ and $g(x)\in W\in CO(Y,\sigma)$

such that $V \cap W = \phi.$ Since f is slightly continuous and g is slightly $\omega-$continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is $\omega-$open in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W).$ Set $O = f^{-1}(V) \cap g^{-1}(W).$ Since $\tau \subseteq \omega O(X,\tau)$ and so O is $\omega-$open. Therefore $f(O) \cap g(O) = \phi$ and it follows that $x \notin \omega Cl(E).$ This shows that E is $\omega-$closed in $X.$

DEFINITION 4.10. A topological space (X,τ) is called clopen regular (respectively $\omega-$regular) if for each clopen (respectively $\omega-$closed) set F and each point $x \notin F,$ there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V.$

DEFINITION 4.11. A topological space (X,τ) is said to be clopen normal (respectively $\omega-$normal) if for every pair of disjoint clopen (respectively $\omega-$closed) subsets F_1 and F_2 of $X,$ there exist disjoint open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V.$

THEOREM 4.12. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous injective open function from a $\omega-$regular space X onto a space $Y,$ then Y is clopen regular.

THEOREM 4.13. Let F be closed set in Y and be $y \notin F.$ Take $y = f(x).$ Since f is slightly $\omega-$***continuous***, $f^{-1}(F)$ is a $\omega-$closed set. Take $G = f^{-1}(F).$ We have $x \notin G.$ Since X is $\omega-$***regular***, there exist disjoint open sets U and V such that $G \subseteq U$ and $x \in V.$ We obtain that $F = f(G) \subseteq f(U)$ and $y = f(x) \in f(V)$ such that *f*(*U*) and *f*(*V*) are disjoint open sets. This shows that Y is clopen regular.

THEOREM 4.14. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous injective open function from a $\omega-$normal space X onto a space $Y,$ then Y is clopen normal.

PROOF. Let F_1 and F_2 be disjoint clopen subsets of Y. Since f is slightly $\omega-$continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed sets. Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2).$ We have $U \cap V = \phi.$ Since X is $\omega-$normal, there exist disjoint open sets A and B such that $U \subseteq A$ and $V \subseteq B.$ We obtain that $F_1 = f(U) \subseteq f(A)$ and $F_2 = f(V) \subseteq f(B)$ such that $f(A)$ and $f(B)$ are disjoint open sets. Thus, Y is clopen normal.

DEFINITION 4.15. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function. Then the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f).$

DEFINITION 4.16. A graph $G(f) = \{(x, f(x)) : x \in X\}$ of a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is said to be strongly $\omega-$co$-$closed if for each $(x,y) \in (X \times Y)\setminus G(f),$ there exist $U \in \omega CO(X,\tau)$ containing x and $V \in \omega CO(Y,\sigma)$ containing y such that $(U \times V) \cap G(f) = \phi.$

LEMMA 4.17. A graph $G(f) = \{(x, f(x)) : x \in X\}$ of a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is strongly $\omega-$co$-$closed in $X \times Y$ if and only if for each $(x,y) \in (X \times Y)\setminus G(f),$ there exist $U \in \omega CO(X,\tau)$ containing x and $V \in \omega CO(Y,\sigma)$ containing y such that $f(U) \cap V = \phi.$

THEOREM 4.18. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous function and Y is clopen $T_1,$ then $G(f)$ is strongly $\omega-$co$-$closed in $X \times Y.$

PROOF. Let $(x,y) \in (X \times Y)\setminus G(f),$ then $f(x) \neq y$ and there exists a clopen set V of Y such that $f(x) \in V$ and $y \notin V.$ Since f is slightly $\omega-$continuous, then $f^{-1}(V) \in \omega CO(X,\tau)$ containing $x.$ Take $U = f^{-1}(V).$ We have $f(U) \subseteq V.$ Therefore we obtain $f(U) \cap (Y \setminus V) = \phi$ and $Y \setminus V \in CO(Y,\sigma)$ containing $y.$ This shows that $G(f)$ is strongly $\omega-$co$-$closed in $X \times Y.$

COROLLARY 4.19. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega-$continuous function and Y is clopen Hausdorff, then $G(f)$ is strongly $\omega-$co$-$closed in $X \times Y.$

THEOREM 4.20. Suppose that the function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is an injection and has a strongly $\omega-$co$-$closed graph $G(f).$ Then X is $\omega-T_1.$

PROOF. Let x and y be any two distinct points of $X.$ Then, we have $(x, f(y)) \in (X \times Y)\setminus G(f).$ By

Lemma 4.17, there exists an $\omega-$clopen set U of X and $V \in CO(Y,\sigma)$ such that $(x, f(y)) \in U \times Y$ and $f(U) \cap V = \phi.$ Hence $U \cap f^{-1}(V) = \phi$ and $y \notin U.$ This implies that X is $\omega - T_1.$

THEOREM 4.21. Suppose that the function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is a surjection and has a strongly $\omega-$co – closed graph $G(f).$ Then Y is $\omega - T_2.$

PROOF. Let y_1 and y_2 be any distinct points of $Y.$ Since f is surjective. So $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \setminus G(f).$ By Definition 4.16, there exist a $\omega-$clopen set U of X and $V \in CO(Y,\sigma)$ such that $(x, y_2) \in U \times V$ and $(U \times V) \cap G(f) = \phi.$ Then, we have $f(U) \cap V = \phi.$ Since f is $\omega-$open, then $f(U)$ is $\omega-$open set such that $f(x) = y_1 \in f(U).$ This implies that Y is $\omega - T_2.$

DEFINITION 4.22. A topological space (X,τ) is $0-$dimensional if its topology has a base consisting of clopen sets.

THEOREM 4.23. Suppose that $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly $\omega-$continuous and Y is $0-$*dimensional* space, then f is $\omega-$continuous.

PROOF. Let $x \in X$ and V be an open subset of Y containing $f(x).$ Since Y is a $0-$*dimensional* space, there exists a clopen set U containing $f(x)$ such that $U \subseteq V.$ Since f is slightly $\omega-$continuous, then there exists an $\omega-$open subset G in X containing x such that $f(G) \subseteq U \subseteq V.$ Thus, f is $\omega-$continuous.

DEFINITION 4.24. A subset M of a topological space (X,τ) is said to be $\omega-$dense in X if there is no proper $\omega-$closed set C in X such that $M \subseteq C \subseteq X.$

EXAMPLE 4.25. Let $X = R$ with topology $\tau = \{\phi,$ ‴ $,$ ¤ $\}.$ It is easy to see that ‴ \¤ is an $\omega-$dense set in X, but ¤ is not an $\omega-$dense set in X.

PROPOSITION 4.26. A subset M of a topological space (X,τ) is said to be $\omega-$dense in X if for any nonempty $\omega-$open set U in X, $U \cap M \neq \phi.$

THEOREM 4.27. For a surjective function $f:(X,\tau) \longrightarrow (Y,\sigma)$, the following statements are equivalent:

(1). f is slightly $\omega-$continuous.

(2). If C is a clopen subset of Y such that $f^{-1}(C) \neq X$, then there is a proper $\omega-$closed subset D of X such that $f^{-1}(C) \subseteq D.$

(3). If M is an $\omega-$dense subset of X, then $f(M)$ is a dense subset of Y.

PROOF. $(1) \Rightarrow (2):$ Let C be a clopen subset of Y such that $f^{-1}(C) \neq X.$ Then $Y \setminus C$ is a clopen set in Y such that $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C) \neq \phi.$ By (1), there exists an $\omega-$open set V in Y such that $V \neq \phi$ and $V \subseteq f^{-1}(Y \setminus C) = X \setminus f^{-1}(C).$ This shows that $f^{-1}(C) \subseteq (X \setminus V)$ and $X \setminus V = D$ is a proper $\omega-$closed set in X.

$(2) \Rightarrow (3):$ Let M be an $\omega-$dense set in X. Suppose that $f(M)$ is not dense in Y. Then there exists a nonempty proper closed set C in Y such that $f(M) \subseteq C \subset Y.$ Clearly $f^{-1}(C) \neq X.$ Then $U = Y \setminus C$ is a nonempty proper open subset of Y. Since Y is $0-$dimensional. So there exists a nonempty proper clopen set E in Y such that $E \subseteq U = Y \setminus C.$ Then $C \subseteq Y \setminus E = F$ and $F = Y \setminus C$ is a nonempty proper clopen set in Y. Also $f(M) \subseteq F \subseteq Y$ and $f^{-1}(F) \neq \phi.$ By (2), there exists a nonempty proper closed set D such that $M \subseteq f^{-1}(F) \subseteq D \subset X.$ This is a contradiction to the fact that M is $\omega-$dense in X.

$(3) \Rightarrow (1):$ Suppose that f is not a slightly $\omega-$continuous function, then there exits a nonempty proper clopen set U in Y such that $\omega-$interior of $f^{-1}(U)$ is empty, that is $X - f^{-1}(U)$ is $\omega-$*dense* in X, while $f(X - f^{-1}(U)) = Y - U$ is not dense in Y. This is a contradiction.

LEMMA 4.28.[32]. Let A and B be subsets of a topological space (X,τ).

(1). If $A \in \omega O(X,\tau)$ and $B \in \tau,$ then $A \cap B \in \omega O(B,\tau_B).$

(3). If $A \in \omega O(B,\tau_B)$ and $B \in \omega O(X,\tau),$ then $A \in \omega O(X,\tau).$

PROPOSITION 4.29. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be a function and $X = A \cup B,$ where $A, B \in \tau.$ If the restriction functions $f_{|A}:(A,\tau_{|A}) \longrightarrow (Y,\sigma)$ and

$f_{|B}:(B,\tau_{|B})\longrightarrow(Y,\sigma)$ are slightly $\omega-$*continuous*, then f is slightly $\omega-$continuous.

PROOF. Let $x\in X$ and let U be any clopen subset of Y such that $f(x)\in U.$ Now $x\in f^{-1}(U).$ Then $x\in(f_{|A})^{-1}(U)$ or $x\in(f_{|B})^{-1}(U)$ or both $x\in(f_{|A})^{-1}(U)$ and $x\in(f_{|B})^{-1}(U)$. Suppose $x\in(f_{|A})^{-1}(U)$. Since $f_{|A}$ is slightly $\omega-$continuous, there exists an $\omega-$open set V in A such that $x\in V$ and $x\in V\subseteq(f_{|A})^{-1}(U)\subseteq f^{-1}(U).$ Since V is $\omega-$open in A and A is open in X, V is $\omega-$*open* in X. Thus we find that f is slightly $\omega-$continuous. The proof of other cases are similar.

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