On Matching Problem in Linear Hypergraphs

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Abstract: - In this paper, an attempt is made to deal with the aspect of matching in linear hypergraph and obtained in the process, various matching parameters with characterizations. Characterization of these parameters is investigated with illustrative examples leading to striking results!

Key-Words: - Linear hypergraphs, matching, hypergraphs.

I. INTRODUCTION

Hypergraph terminology is taken from [1] and [2]. Linear hypergraph is a generalization of simple graph. The linear hypergraph is defined as follows:

The couple H=(X, E) is called **linear hypergraph**, where $X = \{x_1, x_2, x_3, ..., x_n\}$ is a finite set and $E = \{E_1, E_2, E_3, ..., E_m\}$ is a family of subsets of X such that

(1) $E_i \neq \phi$ and $|E_i| \leq 1$ i = 1, 2, 3,…,m

(2) $\bigcup_{i=1}^{m} E_i = X$

(3) $|E_i \cap E_j| \leq 1$ for $i \neq j$.

The elements of X are called vertices and the elements of E are called edges of H. Two vertices in a linear hypergraph are adjacent if they have on the same edge. The number of the elements in X is called the order of hypergraph H.

A **Matching** [1] for a hypergraph H=(X, E) is a family of pair wise disjoint edges.

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II. MATCHING IN LINEAR HYPERGRAPH

Two distinct edges E_1 and E_2 in a linear hypergraph are said to be disjoint or adjacent according as $|E_1 \cap E_2| = 0$ or $|E_1 \cap E_2| = 1$.

Definition 2.1 A vertex v of a linear hypergraph H = (V, X) is said to be **saturated**, if there exists a matching M such that $v \in E$ for some $E \in M$.

Definition 2.2 An edge E of a linear hypergraph H = (V, X) is said to be **saturated** by a matching M, if $E \in M$.

Definition 2.3 A matching that saturates all vertices of a linear hypergraph is called a **perfect matching**.

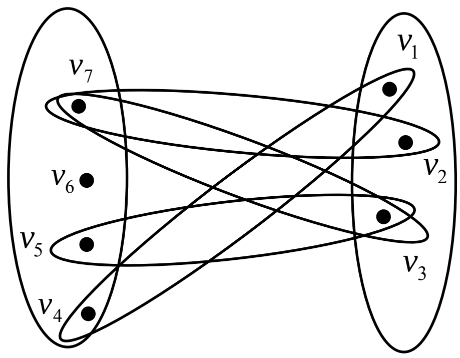
Clearly, a perfect matching and a 1-factor of a linear hypergraph are one and the same thing.

Definition 2.4 A matching M saturating the maximum number of edges of a linear hypergraph H is called a **maximum matching**.

A set consisting of $\beta_1(H)$ independent edges in a linear hypergraph H is a maximum matching of H where, $\beta_1(H)$ is the edge independence number of H.

An observation is in order now regarding the fact that in a graph, every perfect matching is maximum one, but in linear hypergraph, every perfect matching may not be maximum one and this has been illustrated in Fig.- 2.1.

Fig.- 2.1



In the hypergraph shown in Fig.- 2.1, the matching $M = \{(v_1, v_2, v_3), (v_4, v_5, v_6, v_7)\}$ is a perfect matching which is not a maximum matching. In this hypergraph $\{(v_1, v_4), (v_2, v_7), (v_3, v_5)\}$ is the maximum matching.

We now illustrate some examples of matching and perfect matching.

Example 2.1 There is a collection of groups of workers $\{W_1, W_2, ..., W_q\}$ such that in each group there exist at least two workers and between any two groups there is at most one worker in common. The issue of determining the maximum number of groups to be involved in a certain number of works so that no worker is attached to more than two works at a time or the number of groups of workers to be involved in certain number of works so that all workers are involved in the groups with the obvious condition that no worker can do more than two works at a time reflects the example of a maximum matching or a perfect matching.

Example 2.2 There are p cities (or stations) and q passenger train routes (a route may be used to imply the passage of a train touching all the stations on its way or those stations on its way at which it has stoppages) through these cities such that between any two routes there exists at most one city in common and in any route there exist more than or equal to two cities. Such a network is clearly capable of describing various types of matching of hypergraphs.

C. Berge [5] gives a characterization of maximum matching in graph and hypergraph.

Definition 2.5 A collection $\{E_1, E_2, E_3, ..., E_n\}$ of edges of a linear hypergraph $H = (V, X)$ is said to be a **chain** if (i) $|E_i \cap E_{i+1}| = 1$ and (ii) $|E_i \cap E_j| = 0$, for every $i \neq j, j \neq i+1$, i.e. no edges are adjacent except the consecutive ones.

The length of a chain is odd (respectively, even) if the number of edges involved in it is odd (respectively, even).

Definition 2.6 An **M-alternating chain** is a chain with respect to a matching M that alternates edges between those in M and those not in M. An M-alternating chain is ***M*-augmented** with respect to a matching M, if both the starting and ending edges are M-unsaturated edges.

Clearly, the length of an M-augmenting chain relative to a matching M is always odd. It is always possible to obtain an s-path from a chain.

Proposition 2.1

Let $M_1 \Delta M_2 = (M_1 - M_2) \cup (M_2 - M_1)$ be the symmetric difference of two matching of linear hypergraph H and H_1 be the ***sub linear hypergraph*** of H induced by $M_1 \Delta M_2$. Then the components of H either contain an s-cycle or an s-path with edges (or partial edges) alternately in M_1 and M_2 such that the starting and the ending vertices of this s-cycle or s-path are unsaturated in M_1 or M_2

Proposition 2.2 A matching M in a linear hypergraph $H = (V, X)$ is a maximum matching if and only if H contains no M-augmenting chain.

Proposition 2.3

Let M be a matching and C be a vertex cover of a linear hypergraph H. Then, $|C| \geq |M|$.

Proof:

The set C covers every edges of the linear hypergraph H whereas the set M contains only the disjoint edges of H. Therefore, we have $|C| \geq |M|$. ■

III. MATCHING PARAMETERS OF LINEAR HYPERGRAPH

For any vertex v in a linear hypergraph $H = (V, X)$, the set $N_a(v) = \{x \in V \mid x \text{ is adjacent to } v\}$ is **neighbour set** of the vertex v. In graphs two maximum matching have the same number of vertices. But the same is not true in case of linear hypergraphs. However, in this connection, we have the following definition.

Definition 3.1 A maximum matching M of a linear hypergraph H is said to be **maximal vertex-saturated,** if it saturates maximum number of vertices of H among all the maximum matching in H. We denote it by M_{mvs} and the number of vertices saturated by M_{mvs} is called the power of M_{mvs}, denoted by $p(M_{mvs})$.

This is illustrated by the example given below (Fig.-4.2). The edges of the hypergraph shown in the Fig.-3.1 are $E_1 = \{v_1, v_2, v_3\}$, $E_2 = \{v_3, v_6, v_7\}$, $E_3 = \{v_2, v_4\}$, $E_4 = \{v_4, v_5\}$ and $E_5 = \{v_5, v_6\}$. The matching $M_1 = \{E_1, E_5\}$ and $M_2 = \{E_3, E_5\}$ are both maximum matching. But the number of vertices saturated by M_1 is more than those saturated by M_2 .

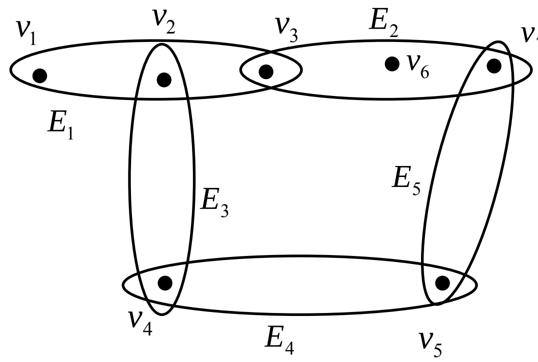


Fig.-3.1

The three immediate results in this connection are produced below.

Proposition 3.1

In a linear hypergraph $H = (V, X)$, let M be a maximal vertex-saturated matching. If $p(M_{mvs}) = |V|$ then the matching M is the perfect matching of H.

Proof:

The proof is trivial since for a maximal vertex-saturated matching M with $p(M_{mvs}) = |V|$ the matching M saturates all the vertices of H and therefore it is clearly a perfect matching in H. ■

Proposition 3.2

Let M be a maximum matching in a linear hypergraph H. Then M is a maximal vertex-saturated matching if and only if

$$\sum_{v\in M} |N_a(v)| > \sum_{v\in M'} |N_a(v)|$$

for any maximum matching M' other than M in H.

Proof:

Suppose M is a maximal vertex-saturated matching and M' is any other maximum matching of the linear hypergraph H. Then there is at least one M' – unsaturated vertex in H which is saturated by M. So we have,

$$\sum_{v\in M} |N_a(v)| > \sum_{v\in M'} |N_a(v)|.$$

Conversely, suppose

$$\sum_{v\in M} |N_a(v)| > \sum_{v\in M'} |N_a(v)|$$

for any two maximum matching M and M' in H.

We are to show that M is M_{mvs}. Let us assume the contrary i.e., let M be not M_{mvs}. But the number of vertices saturated by M and M' is same. Thus, for ***sub linear hypergraphs*** induced by the vertices saturated by M and M', we have

$$\sum_{v\in M} |N_a(v)| = \sum_{v\in M'} |N_a(v)|$$

which contradicts the assumption. Hence follows the result.■

Proposition 3.3

Let M be a maximum matching in a linear hypergraph H. Then M is maximal vertex-saturated matching if and only if

$$\sum_{v\in M} |N_{ca}(v)| > \sum_{v\in M'} |N_{ca}(v)|$$

for any maximum matching M' other than M in H.

Proof:

Trivial.■

While studying the maximum matching and maximal vertex saturated matching for linear hypergraphs we made an attempt to characterize them for **(*p, q*) complete linear hypergraphs**, though success did not prevail on us. To express in particular terms, the problem remains open to find the maximal vertex-saturated matching on a **(*p, q*) complete linear hypergraph *H*** and its power $p(M_{mvs})$.

Similar results can be obtained in connection with the concept of perfect matching of linear hypergraphs. We observe that, contrary to the cases of graphs, there are examples of perfect matching **(which are clearly 1-factors** [3]**)** of linear hypergraphs having distinct number of edges. To confirm our assertion, we require the following definition of minimal edge saturated matching for linear hypergraphs.

Definition 3.2 A perfect matching M of a linear hypergraph H is said to be **minimal edge-saturated**, if it saturates the minimum number of edges of H among all perfect matching of H and it is denoted by M_{mes}. The number of edges saturated by M_{mes} is called the power of M_{mes} which we denote by $p(M_{mes})$.

We demonstrate the situation with the help of an example as shown in Fig.-3.2 below where, we have two perfect matching $M = \{\{v_1, v_2, v_3, v_4\}, \{v_5, v_6, v_7, v_8\}\}$ and $M' = \{\{v_1, v_8\}, \{v_2, v_7\}, \{v_3, v_6\}, \{v_4, v_5\}\}$ out of which the matching M is minimal edge-saturated.

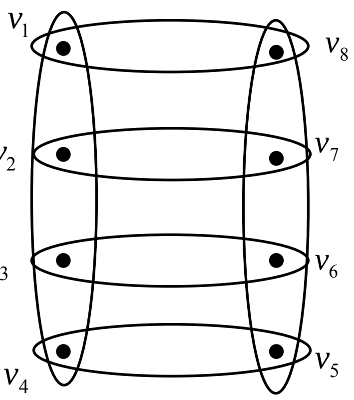


Fig.-3.2

Proposition 3.4

In a linear hypergraph H, a perfect matching M is minimal edge-saturated if and only if, $\alpha_1(H) = |M| = P(M_{mes})$ where $\alpha_1(H)$ denotes the **edge covering number** of H.

Proof:

Suppose M is a minimal edge-saturated matching of a linear hypergraph H. Then it is a perfect matching of H covering all of its vertices. Also it contains the least number of edges covering all vertices of H. Consequently, $\alpha_1(H) = |M| = P(M_{mes})$.

Conversely, suppose M is a perfect matching of H such that, $\alpha_1(H) = |M| = P(M_{mes})$. Thus, M is a perfect matching saturating minimum number of edges of H. Also, by definition of minimal edge-saturated matching, it is clear that M is minimal edge-saturated in H. ■

In case of an ordinary graph a maximum matching saturates largest number of its vertices. However, the same is not always true for linear hypergraphs. In other words, a matching in a linear hypergraph may saturate largest number of vertices though it may not be maximum one. Therefore, it is not out of context to formalize this situation in the form of a definition which may help characterization of distinguishing aspects of linear hypergraphs. We like to name such a matching as an **optimum matching**.

Definition 3.3 A matching M of a linear hypergraph H is called an **optimum matching** if it has the smallest number of edges saturating the largest number of vertices of H.

In the light of this definition, it follows that a minimal edge saturated matching of a hypergraph is always an optimum matching.

The Fig.-3.3 shows a linear hypergraph in which the matching $M' = \{E_2, E_3, E_5\}$ is a maximum matching while the matching $M = \{E_1, E_5\}$ is not a maximum one. However, the number of vertices saturated by M is more than the number of vertices saturated by M'. Thus, the matching M is an optimum matching.

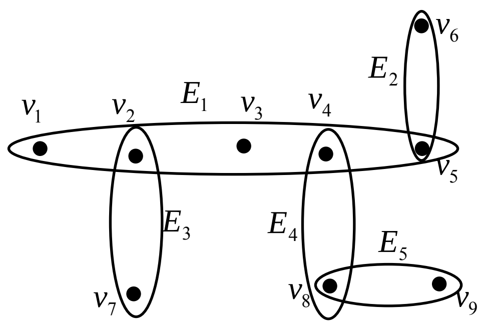


Fig.-3.3

Application: The Example 2.2 mentioned above hints at scope for applications in networking problems particularly, in railway networks of a country. Perhaps, we may design a rail network to have maximum number of mutually disjoint routes in which we can provide trains to reach maximum number of cities (stations) running at the same time (corresponding to a maximal vertex saturated matching) or a rail network to have a minimum number of routes reaching maximum number of stations (corresponding to an optimum matching of a linear hypergraph).

We establish the following property of an optimum matching in linear hypergraph.

Proposition 3.5

Let M_{op} and M be an optimum matching and maximum matching of a linear hypergraph H respectively. Then M_{op} is an optimum matching of H if and only if

$|N_a(M_{op})| \geq |N_a(M)|$,

where, $N_a(M_{op}) = \{\cup N_a(v) \mid v \in E, E \in M_{op}\}$

and $N_a(M) = \{\cup N_a(v) \mid v \in E, E \in M\}$

Proof:

Let M_{op} be an optimum matching of H. Then there is at least one vertex of H which is not saturated by M. So, we have,

$|N_a(M_{op})| > |N_a(M)|$.

If the maximum matching M is also a perfect matching of H, we have

$|N_a(M_{op})| = |N_a(M)|$.

Hence, $|N_a(M_{op})| \geq |N_a(M)|$.

Conversely, let M_{op} and M be a matching and maximum matching of a linear hypergraph H respectively such that $|N_a(M_{op})| \geq |N_a(M)|$. Then from the definition of optimum matching it is clear that M_{op} is an optimum matching of G. ■

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