The Darboux theory and geodesics in the metric space with torsion

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Abstract— The geometry of Y^n space is generated congruently together by the metric tensor and the torsion tensor. In the presented article has been obtained an analog of the Darboux theory in the Y^n space, also studied the deduction of the equation of the geodesic lines on the hypersurface that embedded in such spaces, showed that in the Y^n space the structure of the curvature tensor has special features and for curvature tensor obtained Ricci - Jacobi identity. We establish that the equations of the geodesics have additional summands, which are caused by the presence of torsion in the space. In Y^n space, the variation of the length of the geodesic lines is proportional to the product of metric and torsion tensors $g_{ij}S^{j}_{pk}$. We have introduced the second fundamental tensor $\pi_{\alpha\beta}$ for the hypersurface Y^{n-1} and established its structure, which is fundamentally different from the case of the Riemannian spaces with zero torsion. Furthermore, the results on the structure of the curvature tensor have been obtained.

Keywords—Darboux theory, metric tensor, torsion tensor, curvature tensor, connection, geodesic equation, Darboux line, hypersurface, tangent bundle.

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I. Introduction

THIS work is dedicated to the theory of the Y^n space, analytically, this space is an n-dimensional differentiable real manifold, at each point of which are given metric and torsion tensors [29-33]. From a geometrical perspective, such space can be defined as a real n-dimensional metric space equipped with a connection on the tangent bundle (that connection can be with torsion); this metric is generated by a given symmetrical covariant tensor, and the torsion of the connection of the space coincides with given torsion tensor [1-5, 29-33].

We believe that introduction Y^n - space is the answer to the next words that were written in the 1928 year by Albert Einstein "Riemannian Geometry has led to a physical description of the gravitational field in the theory of general relativity, but it did not provide concepts that can be attributed to the electromagnetic field [13, 14, 19]. Therefore, theoreticians aim to find natural generalizations or extensions of Riemannian geometry that are richer in concepts, hoping to arrive at a logical construction that unifies all physical field concepts under one single leading point." [6, 7, 21-24]

The main object of this work is the geometric properties of Y^n space, to construct the geometry by means of two tensors - the metric and torsion tensors; obtain the field equations from the variation principle in such spaces.

The rest of the article is organized as follows. In section 2, we collect some properties of the curvature tensor in Y^n space. In section 3, we consider the properties geodesic lines in space with torsion and obtain the necessary and sufficient conditions of a line to be geodesic. Section 4, we study the geometrical structure of hypersurfaces Y^{n-1} in Y^n space and develop the Darboux theory for the hypersurfaces Y^{n-1}. In section 5, we present an exemplar of geodesics in space with the Euclidean metric.

II. The geometric meaning of the curvature tensor in Y^n space

Let us formulate the main theorem of the Y^n space.

Theorem 1. ***Let us assume that the metric g_{ik} and torsion S^{i}_{jk} are given, the metric tensor g_{ik} is symmetric, and the torsion tensor S^{i}_{jk} is asymmetric. If we demand $d(g_{ik}A^iB^k)=\mathbf{0}$ for arbitrary vector A^i and vector B^k, then the connection (a geometric object that defines this connection) Γ^{i}_{jk} is uniquely defined by a formula***

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$$\Gamma^p_{kl} = \frac{1}{2} g^{pi}\left(g_{ik,l} + g_{li,k} - g_{kl,i}\right) + \\ + \frac{1}{2} g^{pi}\left(g_{km}S^m_{li} + g_{lm}S^m_{ki}\right) + \frac{1}{2} S^p_{kl} \quad (1)$$

The proof of this theorem can be found in [29].

Next, we introduce a notation

$$\mathrm{P}^p_{kl} = \frac{1}{2} g^{pi}\left(g_{ik,l} + g_{li,k} - g_{kl,i}\right) \quad (2)$$

and

$$L^p_{kl} \equiv \frac{1}{2} S^p_{kl} + \frac{1}{2} g^{pi}\left(g_{km}S^m_{li} + g_{lm}S^m_{ki}\right) \quad (3)$$

constitutes the tensor with the transformation law $L^i_{jk} = L^{\tilde{i}}_{\tilde{j}\tilde{k}} \frac{\partial x^i}{\partial x^{\tilde{i}}} \frac{\partial x^{\tilde{j}}}{\partial x^j} \frac{\partial x^{\tilde{k}}}{\partial x^k}$ so

$$\Gamma^p_{kl} = \mathrm{P}^p_{kl} + L^p_{kl}. \quad (4)$$

Let us consider in the Y^n space a two-dimensional surface $x^i = x^i(u,v)$, where the rank of Jacobi matrix is

$$rank\begin{bmatrix} \frac{\partial x^1}{\partial u} & . & . & \frac{\partial x^n}{\partial u} \\ \frac{\partial x^1}{\partial v} & . & . & \frac{\partial x^n}{\partial v} \end{bmatrix} = 2.$$

On this surface, we consider a curve $s \in [0, S]$ as $u = u(s), \quad v = v(s)$. So, applying the surface equation $x^i = x^i(u,v)$, we can rewrite the equation of the curve as $x^i = x^i(s), \quad s \in [0,S]$ in the parametric form in Y^n space, where s is the arc length of the curve, this curve links two points $P^i = x^i(0)$ *and* $Q^i = x^i(S)$. Alternatively, more precisely, in Y^n space, we start from a point P^i are moving on a two-dimensional surface along a curve $u = u(s), \quad v = v(s)$ that joins the points $P^i = x^i(0), \quad Q^i = x^i(S)$ on the two-dimensional surface $x^i = x^i(u,v)$.

Therefore, we can think of the curve $u = u(s), \quad v = v(s),$ as a set of curves $x^i = x^i(s,\alpha), \quad s \in [0,S]$ on the surface $u = u(s,\alpha), \quad v = v(s,\alpha),$ so we denote by D displacement $s \to s + ds$ with fixed α as a parameter and we denote by $\tilde{D}$ displacement $\alpha \to \alpha + d\alpha$ with fixed s as a parameter. By definition, using the curvature tensor, we have $\tilde{D}D\xi^i - D\tilde{D}\xi^i = -R^i_{klp}\xi^p \tilde{d}x^k dx^l$, here $\xi^i = \frac{dx^i}{ds}(P)$ and here $D\xi^i = \mathbf{0}$, thus we have an interesting identity $D\tilde{D}\xi^i = R^i_{klp}\xi^p \tilde{d}x^k dx^l$.

The parallel transport of vector ξ^i is given by the formula $d\xi^i = -\Gamma^i_{pl}\xi^p dx^l$,

$\Delta\xi^i = \xi^i(s) - \xi^i(P) = -\int_0^s \Gamma^i_{pl}\xi^p \frac{dx^l}{ds} ds$ and $\Delta\xi^i = -\Gamma^i_{pl}(P)\xi^p(P)\Delta x^l$ here, we denote $\Delta x^l = \int_0^s \frac{dx^l}{ds} ds$.

We write the next equation $\Delta\Gamma^i_{pl} = \Gamma^i_{pl}(P) - \Gamma^i_{pl}(Q) = \left(\frac{\partial}{\partial x^m}\Gamma^i_{pl}(PQ)\right)\Delta x^m$.

A vector of parallel transportation is $\xi^i(s) = \xi^i(P) + \Delta\xi^i(P)$, where $\Delta\xi^i$ is a difference of vector ξ^i during parallel transportation along the curve s. Then, we use equations $u = u(s), \quad v = v(s)$ and obtain $\xi^i = \frac{dx^i}{ds} = \frac{\partial x^i}{\partial u}\frac{du}{ds} + \frac{\partial x^i}{\partial v}\frac{dv}{ds}$.

Let us use an approximation $\xi^i(s) \approx \xi^i(P) - \Gamma^i_{pm}(P)\xi^p(P)\Delta x^m,$ and $\Gamma^i_{pl} = \Gamma^i_{pl}(P) + \frac{\partial\Gamma^i_{pl}}{\partial x^m}(P)\Delta x^m.$

For the difference $\Delta\xi^i$, we have an important formula

$$\Delta\xi^i = -\int_0^s \begin{pmatrix} \left(\Gamma^i_{pl}\xi^p\right)(P) + \frac{\partial\Gamma^i_{pl}}{\partial x^q}(P)\xi^p(P)\Delta x^q - \\ -\left(\Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\xi^p(P)\Delta x^q \end{pmatrix} \frac{dx^l}{ds} ds,$$

and deduce

$$\Delta\xi^i = -\left(\Gamma^i_{pl}\xi^p\right)(P)\Delta x^l - \\ -\left(\frac{\partial\Gamma^i_{pl}}{\partial x^q}(P)\left(\Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\right)(P)\xi^p(P)\int_0^s \Delta x^q \frac{dx^l}{ds} ds.$$

Let us suppose that the curve s forms a closed loop with some area inside, we can rewrite $\Delta\xi^i$ in the form

$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\xi^p(P)\int_0^s \Delta x^q \frac{dx^l}{ds} ds$$

using the contour integration formula, we have

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$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\,\xi^p(P)\times$$
$$\times\oint \Delta x^q\left(\frac{\partial x^l}{\partial u}du + \frac{\partial x^l}{\partial v}dv\right).$$

Since the curve s is closed, we obtain the next equality

$$\frac{\partial}{\partial u}\left(\Delta x^q\frac{\partial x^l}{\partial v}\right) - \frac{\partial}{\partial v}\left(\Delta x^q\frac{\partial x^l}{\partial u}\right) =$$
$$= \frac{\partial x^q}{\partial u}\frac{\partial x^l}{\partial v} - \frac{\partial x^l}{\partial u}\frac{\partial x^q}{\partial v},$$

then we denote by Ω the area inside the loop of the curve s and

$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\,\xi^p(P)$$
$$\iint_\Omega\left(\frac{\partial x^q}{\partial u}\frac{\partial x^l}{\partial v} - \frac{\partial x^l}{\partial u}\frac{\partial x^q}{\partial v}\right)dudv,$$

or

$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq}\right)(P)\,\xi^p(P)$$
$$\left(\frac{\partial x^q}{\partial u}\frac{\partial x^l}{\partial v} - \frac{\partial x^l}{\partial u}\frac{\partial x^q}{\partial v}\right)(P)\iint_\Omega dudv$$

we denote the bivector by $x^{ql} = \left(\frac{\partial x^q}{\partial u}\frac{\partial x^l}{\partial v} - \frac{\partial x^l}{\partial u}\frac{\partial x^q}{\partial v}\right)$,

then we have

$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq}\right)(P)$$
$$\xi^p(P)\,x^{ql}(P)\iint_\Omega dudv$$

since bivector x^{ql} asymmetrical, we obtain

$$\Delta\xi^i = \left(-\frac{\partial\Gamma^i_{pq}}{\partial x^l} + \Gamma^i_{mq}\Gamma^m_{pl}\right)(P)\,\xi^p(P)\,x^{lq}(P)\iint_\Omega dudv$$

and we obtain the result

$$\Delta\xi^i = \begin{pmatrix} -\frac{\partial\Gamma^i_{pl}}{\partial x^q} + \Gamma^i_{ml}\Gamma^m_{pq} + \\ +\frac{\partial\Gamma^i_{pq}}{\partial x^l} - \Gamma^i_{mq}\Gamma^m_{pl} \end{pmatrix}(P)\,\xi^p(P)\,x^{ql}(P)\iint_\Omega dudv$$

and finally, we have obtained the formula $\Delta\xi^i = -R^i_{qlp}\xi^p x^{ql}\iint_\Omega dudv$.

As a result, the vector $\Delta\xi^i = -R^i_{qlp}\xi^p x^{ql}\iint_\Omega dudv$ is a difference between the vector ξ^i before parallel transportation along the curve s and the vector ξ^i after parallel transportation along the curve s, i.e. $\Delta\xi^i$ is the deviation of vector ξ^i during parallel transportation along the curve s.

If we assume that the metric tensor $g_{ik} = \delta_{ik}$ is diagonal, the curvature tensor is equal to $R^p_{ikl} = L^p_{li,k} - L^p_{lk,i} + L^p_{qk}L^q_{li} - L^p_{qi}L^q_{lk}$, where $L^p_{kl} \equiv \frac{1}{2}S^p_{kl} + \frac{1}{2}\delta^{pi}\left(\delta_{km}S^m_{li} + \delta_{lm}S^m_{ki}\right)$ and the difference $\Delta\xi^i$ can be written as

$$\Delta\xi^i = \begin{pmatrix} -L^i_{pq,l} + L^i_{pl,q} - \\ -L^i_{ml}L^m_{pq} + L^i_{mq}L^m_{lp} \end{pmatrix}\xi^p x^{ql}\iint_\Omega dudv$$

so a deviation $\Delta\xi^i$ of the vector ξ^i is proportional to the measure of the surface inside the loop along which the vector was parallel transported

$$\Delta\xi^i = \left(L^i_{pl,q} - L^i_{pq,l} + L^i_{mq}L^m_{lp} - L^i_{ml}L^m_{pq}\right)\xi^p x^{ql}\,mes(\Omega).$$

III. The equation of geodesic lines in space with torsion

Definition 1. ***The line in space is a geodesic line if a tangent vector to this line at any point continues to be tangent at the parallel transport along this line.***

Theorem 2. ***In order not an isotropic line in space, which is generated by the g_{ik} and the S^j_{ik} tensors, was geodetic, it is necessary and sufficient that a variation of the arc δs was equaled:***

$$\delta s = \int_{t_1}^{t_2} g_{ij}S^j_{pk}\frac{dx^i}{dt}dx^p\delta x^k.$$

Consequence. In the case of spaces with the affine connection is known that there is a breach of closure during the transition from the original to the image and vice versa, in the case of an infinitely small contour that is determined up 2 - second-order relative to τ. If we specify the torsion tensor S^k_{ij} at the corresponding point, and if this gap is denoted by Ψ^k, then we can assert that such a gap exists, and extrapolate the square of the length as $|\Psi|^2 = g_{pq}S^p_{ij}A^iB^jS^q_{kl}A^kB^l\tau^4$ where the parallelogram $A^i\tau$ and $B^j\tau$ shrinks to a point at $\tau \to 0$.

Now, we discuss the relationship between a classical Riemannian space, geodetics, and geodesics in the Y^n space.

Let the vector A^i be tangent to any geodesic and A^i is being parallel transported in corresponding with the connection Γ_{kl}^{p} that is defined by (1) and a vector B^i is being parallel transported in corresponding with the connection P_{kl}^{p}.

It was found that in the Y^n space the geodesics coincide with geodesics with connection $\mathrm{P}_{kl}^{p} + M_{kl}^{p}$; here M_{ij}^{k} is an arbitrary symmetric tensor. Since both vectors are tangential, then $B^i = aA^i$, where the coefficient a is a variable parameter and $a \neq \mathbf{0}$. Tensor-vector A^i is given by the formula $dA^k = -\left(\mathrm{P}_{ij}^{k} + M_{ij}^{k}\right) A^i dx^j$, $dB^k = -\mathrm{P}_{ij}^{k} B^i dx^j$.

Then, we have

$$A^k da + adA^k = -\mathrm{P}_{ij}^{k} aA^i dx^j,$$

$$\frac{A^k da}{a} = M_{ij}^{k} A^i dx^j. \quad (5)$$

The tangent vector A^i can be written as $A^i = \dfrac{dx^i}{d\tau}$, where τ is the canonical parameter relative to the connection $\mathrm{P}_{kl}^{p} + M_{kl}^{p}$.

Then, after division by $d\tau$, we obtained $\dfrac{d\ln a}{d\tau} A^k = M_{ij}^{k} A^i A^j$. Since geodesic lines can be drawn through any point and in any direction, then this equality must be true at any point and for any vector A^i, the functional dependence of the point and direction, clearly exists.

The last equality is multiplied by A^l and alternate by k and l, we have

$$\delta_m^l M_{ij}^{k} A^i A^j A^m - \delta_m^k M_{ij}^{l} A^i A^j A^m = \mathbf{0},$$

where we denoted $\delta_m^l = g_{mp} g^{pl}$. This equation must hold identically with respect to all vectors $A^{\mathbf{1}}, \ldots\ldots, A^n$, consequently, after adding a similar summand all the coefficients of the cubic form must vanish. We compute the total coefficient

$$\delta_m^l M_{ij}^{k} - \delta_m^k M_{ij}^{l} + \delta_i^l M_{jm}^{k} - \delta_i^k M_{jm}^{l} +$$
$$+ \delta_j^l M_{mi}^{k} - \delta_j^k M_{mi}^{l} = \mathbf{0}$$

then we contract a tensor with indices l and j. Since $\delta_i^i = n$ we have $M_{ij}^{k} = \dfrac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k M_{jl}^{l} + \delta_j^k M_{il}^{l}\right)$.

All calculations presented above do not consider the specificity of the tensor M_{ij}^{k}, then let $M_{kl}^{p} \equiv \dfrac{\mathbf{1}}{\mathbf{2}} g^{pi}\left(g_{km} S_{li}^{m} + g_{lm} S_{ki}^{m}\right)$, then substitute in the last equation $M_{kl}^{l} \equiv \dfrac{\mathbf{1}}{\mathbf{2}} S_{kl}^{l}$, we have obtained

$$M_{ij}^{k} = \frac{\mathbf{1}}{\mathbf{2}} \frac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k S_{jl}^{l} + \delta_j^k S_{il}^{l}\right).$$

Theorem 3. ***In order to the Riemannian space with a torsion-free connection*** P_{kl}^{p} ***shares geodesic lines with a metric space*** Y^n ***with a connection*** Γ_{ij}^{k} ***with the torsion, where the connection*** Γ_{ij}^{k} ***generated congruently by the metric tensor and torsion tensor, it is necessary and sufficient that the difference*** $\mathrm{P}_{ij}^{k} - \Gamma_{ij}^{k}$ ***equals to the tensor***

$$\frac{\mathbf{1}}{\mathbf{2}} \frac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k S_{jl}^{l} + \delta_j^k S_{il}^{l}\right).$$

Proof. The necessity was derived above.

Now, we will prove the sufficiency. Let us assume that $\mathrm{P}_{ij}^{k} = \Gamma_{ij}^{k} - \dfrac{\mathbf{1}}{\mathbf{2}} \dfrac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k S_{jl}^{l} + \delta_j^k S_{il}^{l}\right)$. Then we again use (5) and have $\dfrac{d\ln a}{d\tau} = S_{il}^{l} A^i$, where the constant $a > \mathbf{0}$ is a scale parameter from $B^i = aA^i$.

Since along the curve the tensor $S_{il}^{l} A^i$ is a definite function of the parameter τ, we will find $\ln a$ after integration with precision to a constant a, but only up to a constant factor. Therefore, the vector is found $B^i = aA^i$ and all geodesic coincide. The theorem is proved. Therefore, for existence the Riemannian space with a connection P_{kl}^{p} that has the same geodesic properties that Y^n space, the identity $\Gamma_{ij}^{k} - \mathrm{P}_{ij}^{k} = \dfrac{\mathbf{1}}{\mathbf{2}} \dfrac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k S_{jl}^{l} + \delta_j^k S_{il}^{l}\right)$ must be true.

IV. The hypersurfaces $Y^{n-\mathbf{1}}$ in Y^n space

4.1. The geometrical structure of hypersurfaces $Y^{n-\mathbf{1}}$ that are embedded in Y^n space

Due to the presence of torsion, in these cases, there is a significant difference from Riemann space. For example, the derivation equations (analog Peterson Codazzi equations) take a more complicated form, in which there are new summands, which are caused by the presence of torsion in the space.

We will study the geometry of hypersurfaces $Y^{n-\mathbf{1}}$ in a metric space with torsion. We are assuming that the hypersurface is defined by a system of equations $x^i = x^i\left(y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}}\right)$, where x^i is a coordinate system in Y^n space, y^α is a coordinate system in $Y^{n-\mathbf{1}}$ subspace and the rank of the matrix $\left[\dfrac{\partial x^i}{\partial y^\alpha}\right]$ equals $n-\mathbf{1}$. The metric tensor of a hypersurface $Y^{n-\mathbf{1}}$ is given by the formula

$$a_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}, \quad (6)$$

Let $G^\alpha_{\beta\gamma}$ be a geometric object and let it is subjected to the law of the transformation from one coordinate system y^α to another hachure coordinate system $y^{\alpha'}$ by the formula

$$G^\alpha_{\beta\gamma} = G^{\alpha'}_{\beta'\gamma'} \frac{\partial y^\alpha}{\partial y^{\alpha'}} \frac{\partial y^{\beta'}}{\partial y^\beta} \frac{\partial y^{\gamma'}}{\partial y^\gamma} + \frac{\partial y^\alpha}{\partial y^{\alpha'}} \frac{\partial^2 y^{\alpha'}}{\partial y^\beta \partial y^\gamma}.$$

We have obtained the next formula for the torsion tensor of hypersurface Y^{n-1}

$$T^\gamma_{\alpha\beta} = a^{\gamma\eta} g_{pq} S^p_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^q}{\partial y^\eta} \quad (7)$$

using tensors $a_{\alpha\beta}$ and $T^\gamma_{\alpha\beta}$, both metric and torsion, we can explore the geometry of the space hypersurface Y^{n-1}.

Let $G^\alpha_{\beta\gamma}$ be a connection of Y^{n-1} and we assume that $G^\alpha_{\beta\gamma}$ is expressed via the metric $a_{\alpha\beta}$ and torsion $T^\gamma_{\alpha\beta}$ similarly to as the connection Γ^k_{ij} can be expressed by means of g_{ij} and S^j_{ik}, we have

$$G^\alpha_{\beta\gamma} = \frac{\mathbf{1}}{\mathbf{2}} \left(a^{\alpha\eta} \begin{pmatrix} a_{\beta\eta,\gamma} + a_{\gamma\eta,\beta} - a_{\beta\gamma,\eta} + \\ + a_{\beta\mu} T^\mu_{\gamma\eta} + a_{\gamma\mu} T^\mu_{\beta\eta} \end{pmatrix} + T^\alpha_{\gamma\beta} \right). \quad (8)$$

Below we use the values of the mixed tensor enumerated two types of indices, while Latin indices refer to the containing space Y^n and responsive to the coordinate transformation x^i, and Greek indices belong to the space hypersurface $Y^{n-\mathbf{1}}$ and responsive to the transformation of coordinate y^α. The index i is not responsive to the coordinate y^α transformation into $Y^{n-\mathbf{1}}$, and the index α does not respond to the coordinate x^i transformation in Y^n.

A further aim of our study is to obtain some analogs of Peterson - Codazzi equations. To do this, consider the system of values $\xi^i_\alpha = \dfrac{\partial x^i}{\partial y^\alpha}$.

At each point of the hypersurface $Y^{n-\mathbf{1}}$, we can build a basis consisting of the vectors $\xi^i_{\mathbf{1}}, \ldots, \xi^i_{n-\mathbf{1}}, \nu^i$, where $\xi^i_{\mathbf{1}}, \ldots, \xi^i_{n-\mathbf{1}}$ linearly independent tangent vectors and ν^i normal vector, defined since the metric and connection agreed.

Next, we act formally, the idea is the same as in the classical case, and we will indicate significant new moments. We compute the derivative of the mixed tensors ξ^i_α such that

$$\xi^i_{\alpha;\gamma} = \xi^i_{\alpha,\gamma} + \Gamma^i_{pq} \xi^p_\alpha \frac{\partial x^q}{\partial y^\gamma} - G^\eta_{\alpha\gamma} \xi^i_\eta.$$

In contrast to the case of torsion-free connection, we have equality

$$\xi^i_{\alpha;\gamma} - \xi^i_{\gamma;\alpha} = S^i_{pq} \xi^p_\alpha \xi^q_\gamma + T^\eta_{\gamma\alpha} \xi^i_\eta,$$

however, we have

$$\xi^i_{\alpha;\gamma} - \xi^i_{\gamma;\alpha} = \left(S^i_{pq} + S^k_{qp} \xi^i_\eta \xi^m_\mu g_{km} a^{\mu\eta} \right) \xi^p_\alpha \xi^q_\gamma = \mathbf{0}.$$

Next, we permute the indices in the equation

$$\mathbf{0} = a_{\alpha\beta;\gamma} = \left(g_{ij} \xi^i_\alpha \xi^j_\beta \right)_{;\gamma} = g_{ij} \xi^i_{\alpha;\gamma} \xi^j_\beta + g_{ij} \xi^i_\alpha \xi^j_{\beta;\gamma},$$

And we obtain $g_{ij} \xi^i_{\alpha;\gamma} \xi^j_\beta = \mathbf{0}$. Hence, we can write decomposition

$$\xi^i_{\beta;\alpha} = \pi_{\alpha\beta} \nu^i. \quad (9)$$

Remark 1. Set $\pi_{\alpha\beta}$ is a tensor, which similar to the second fundamental tensor of hypersurfaces $Y^{n-\mathbf{1}}$, but its structure in this space substantially different from the case of Riemannian spaces with zero torsion. We remark that the equation $S^i_{pq} \xi^p_\beta \xi^q_\alpha = (\xi^i_{\beta;\alpha} - \xi^i_{\alpha;\beta})$ is the simple result of the definition. Therefore, we have equality $\pi_{\alpha\beta} - \pi_{\beta\alpha} = g_{ij} S^i_{pq} \xi^p_\beta \xi^q_\alpha \nu^j$.

Then we have obtained by differentiating $g_{ij} \nu^i \xi^j_\alpha = \mathbf{0}$ at γ

$$g_{ij} \nu^i_{;\gamma} \xi^i_\alpha = -\pi_{\gamma\alpha}. \quad (10)$$

Similarly, by differentiating $g_{ij} \nu^i \nu^j = \mathbf{1}$ at γ, and we obtain

$$\nu^i_{;\gamma} = -a^{\eta\mu} \pi_{\mu\gamma} \xi^i_\eta. \quad (11)$$

Formula (8) and (11) characterize the change of vectors in the small accompanying frame relative to this frame itself.

Remark 2. If we consider the system (9) and (11) from a geometric point of view, then we can formulate the problem for differential equations, where the unknowns are considered the functions $\xi^i_\alpha (y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}})$ $\nu^i (y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}})$ and are given (known) g_{ik}, $a_{\alpha\beta}$, S^i_{jk}, $T^\gamma_{\alpha\beta}$, $\pi_{\alpha\beta}$, as a function of $y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}}$. Then the connection coefficients Γ^i_{jk} as a function of $y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}}$ must be considered as known, and it means that we know exactly how the hypersurface $Y^{n-\mathbf{1}}$ is embedded in the space Y^n, but then ξ^i_α, ν^i we have to consider as the known functions of $y^{\mathbf{1}}, \ldots, y^{n-\mathbf{1}}$ and so the problem makes no sense.

Furthermore, we obtain

$$\begin{aligned} \xi^i_{\beta;\chi;\lambda} - \xi^i_{\beta;\lambda;\chi} &= -R^i_{klp} \xi^k_\lambda \xi^l_\chi \xi^p_\beta + R^\sigma_{\lambda\chi\beta} \xi^i_\sigma + T^\sigma_{\lambda\chi} \xi^i_{\beta;\sigma} = \\ &= \left(\pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi} \right) \nu^i - \\ &- \left(\pi_{\chi\beta} \pi_{\eta\lambda} a^{\eta\sigma} - \pi_{\lambda\beta} \pi_{\eta\chi} a^{\eta\sigma} \right) \xi^i_\sigma \end{aligned}. \quad (12)$$

Equation (12) is multiplying by $g_{ij} \xi^j_\alpha$, we have

$$R_{\alpha\lambda\chi\beta} = R_{iklp}\xi^k_\lambda \xi^l_\chi \xi^p_\beta \xi^i_\alpha - \left(\pi_{\chi\beta}\pi_{\alpha\lambda} - \pi_{\lambda\beta}\pi_{\alpha\chi}\right). \quad (13)$$

Similarly, we derive a formula

$$\nu^i_{;\chi;\lambda} - \nu^i_{;\lambda;\chi} = -R^i_{klp}\xi^k_\lambda \xi^l_\chi \nu^p + T^\sigma_{\lambda\chi}\nu^i_{;\sigma} = \\ = \left(\pi_{\eta\lambda;\chi}a^{\eta\sigma} - \pi_{\eta\chi;\lambda}a^{\eta\sigma}\right)\xi^i_\sigma. \quad (14)$$

We contract (3.8) with $g_{ij}\nu^j$, then $-R_{iklp}\xi^k_\lambda \xi^l_\chi \xi^p_\beta \nu^i + T^\sigma_{\lambda\chi}\pi_{\sigma\beta} = \pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi}$. Formula (14) is multiplying by $g_{ij}\xi^j_\alpha$, we concluded that $-R_{iklp}\xi^k_\lambda \xi^l_\chi \nu^p \xi^i_\alpha + T^\sigma_{\lambda\chi}\pi_{\alpha\sigma} = \pi_{\alpha\lambda;\chi} - \pi_{\alpha\chi;\lambda}$.

Remark 3. If (14) contracts with $g_{ij}\nu^j$, then we obtain identically zero.

Thus, we have two types of formulas. Formula (13) does not contain the torsion tensor explicitly, but it is counted in the tensor $\pi_{\alpha\beta}$.

In the formula (14), the torsion tensor of the hypersurface presented explicitly and in the form of coefficients of $\pi_{\alpha\beta}$, *and appears in the calculation of the covariant derivative.*

Then we denote $\vartheta_{\alpha\beta}$ symmetrical tensor $g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta}$ and we have

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = g_{ij}a^{\eta\mu}\pi_{\mu\alpha}\xi^i_\eta a^{\chi\delta}\pi_{\delta\beta}\xi^j_\chi = \\ = g_{ij}\pi^\eta_\alpha \pi^\chi_\beta \xi^i_\eta \xi^j_\chi = a_{\eta\chi}\pi^\eta_\alpha\pi^\chi_\beta = a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta},$$

or

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = \\ = g_{ij}\left(\nu^i_{,\alpha} + \Gamma^i_{lk}\nu^k\xi^l_\alpha\right)\left(\nu^j_{,\beta} + \Gamma^j_{pq}\nu^q\xi^p_\beta\right) = \\ = g_{ij}\nu^i_{,\alpha}\nu^j_{,\beta} + g_{ij}\nu^j_{,\beta}\Gamma^i_{lk}\nu^k\xi^l_\alpha + \\ + g_{ij}\nu^i_{,\alpha}\Gamma^j_{pq}\nu^q\xi^p_\beta + g_{ij}\Gamma^i_{lk}\Gamma^j_{pq}\nu^k\xi^l_\alpha\nu^q\xi^p_\beta$$

so, we see that the asymmetrical part vanished.

We denote $M = \dfrac{1}{2}a^{\alpha\beta}\pi_{\alpha\beta}$, $\vartheta_{\alpha\beta} = a_{\eta\chi}\pi^\eta_\alpha\pi^\chi_\beta = a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta}$ and $\mathbf{2}Ma^{\eta\chi} = a^{\alpha\beta}\pi_{\alpha\beta}a^{\eta\chi}$ then $a^{\eta\chi}\left(\pi_{\eta\alpha}\pi_{\chi\beta} - a^{\alpha\beta}\pi_{\alpha\beta}\pi_{\eta\chi}\right) = -\dfrac{\pi}{a}a_{\alpha\beta}$ and $a^{\eta\chi}\left(\pi_{\eta\alpha}\pi_{\chi\beta} - a^{\alpha\beta}\pi_{\alpha\beta}\pi_{\eta\chi}\right) + \dfrac{\pi}{a}a_{\alpha\beta} = \mathbf{0}$, next, we can write $\vartheta_{\alpha\beta} - \mathbf{2}M\pi_{\alpha\beta} + \dfrac{\pi}{a}a_{\alpha\beta} = \mathbf{0}$ and we obtain $\vartheta_{\alpha\beta} = \mathbf{2}M\pi_{\alpha\beta} - \dfrac{\pi}{a}a_{\alpha\beta}$.

We calculate

$$\vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} = a^{\eta\chi}\pi_{\eta\alpha;\omega}\pi_{\chi\beta} + \\ + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega;\alpha}\pi_{\chi\beta} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha} = \\ = a^{\eta\chi}\pi_{\chi\beta}\left(\pi_{\eta\alpha;\omega} - \pi_{\eta\omega;\alpha}\right) + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha}$$

and

$$\vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} = a^{\eta\chi}\pi_{\chi\beta}\left(-R_{iklp}\xi^i_\eta\xi^k_\alpha\xi^l_\omega\nu^p + T^\sigma_{\omega\alpha}\pi_{\eta\sigma}\right) + \\ + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha} = \\ = -R_{iklp}\xi^i_\eta\xi^k_\alpha\xi^l_\omega\nu^p a^{\eta\chi}\pi_{\chi\beta} + T^\sigma_{\omega\alpha}\pi_{\eta\sigma}a^{\eta\chi}\pi_{\chi\beta} + \\ + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha}.$$

A tensor $\vartheta_{\alpha\beta}$ ***can be associated with the square of the angle between normal and adjacent normal*** $\vartheta_{\alpha\beta}dy^\alpha dy^\beta = d\varphi^2$.

Therefore, let in space Y^n ***with coordinates*** $x^1,\dots,x^n$ ***given the system of nondegenerate equations*** $x^i = x^i\left(y^1,\dots,y^{n-1}\right),\quad i = 1,\dots,n$ ***so are determined the hypersurface*** Y^{n-1} ***and the metric and torsion of*** Y^{n-1} ***and since the connection of*** Y^{n-1}***. We can consider the hypersurface like*** Y^{n-1} ***space, and so we obtain all the internal (intrinsic) geometry structure of*** Y^{n-1}***, but formulas*** $x^i = x^i\left(y^1,\dots,y^{n-1}\right),\quad i = 1,\dots,n$ ***define more than the internal (intrinsic) geometry structure of*** Y^{n-1}***, they define the external geometry of*** Y^{n-1} ***(embedding) as well. External geometry or "how the hypersurface*** Y^{n-1} ***is embedded" defined by one of the tensors*** $\pi_{\alpha\beta}$ ***or*** $\vartheta_{\alpha\beta}$ ***which determines the position of a hypersurface in*** Y^n ***space. As an example, internal (intrinsic) geometry in*** Y^{n-1}***, we considered geodesic on*** Y^{n-1}***.***

4.2. Geodesic lines on the hypersurface Y^{n-1}

According to the definition, the geodesic on Y^{n-1} is determined by a formula

$$\frac{d^2y^\alpha}{ds^2} = -G^\alpha_{\beta\gamma}\frac{dy^\beta}{ds}\frac{dy^\gamma}{ds}.$$

Let us define a curve by the formula $y^\alpha = y^\alpha(\tau),\quad \tau\in[\tau_1;\tau_2],\quad \alpha = 1,\dots,n-1$.

We calculate the variation of the length of geodesic δS of the curve S

$$\delta\left(a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau}\right) = a_{\alpha\beta}\tilde{D}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau} + \\ + a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau} = 2a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau},$$

where denotes $\tilde{D}$ the absolute differential at the parameter curves of the family at a constant value τ, and D is an absolute differential displacement $d\tau$ curve at a constant parameter of the family, then

$$\delta s=\int_{\tau_1}^{\tau_2} a_{\alpha\beta}\frac{dy^\alpha}{ds}D\delta y^\beta+\int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda=$$
$$=\int_{\tau_1}^{\tau_2} D\left(a_{\alpha\beta}\frac{dy^\alpha}{ds}\delta y^\beta\right)-\int_{\tau_1}^{\tau_2} a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta+$$
$$+\int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda$$

since the ends of the variable curve are fixed

$$\delta s=\int_{\tau_1}^{\tau_2}\left(a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda-a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta\right),$$

suppose the considered curve has a fixed length (analytically $\delta s=0$), then we obtain

$$\delta s=\int_{\tau_1}^{\tau_2}\left(a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda-a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta\right)=0.$$

By the fundamental lemma of the calculus of variations, it follows $a_{\alpha\lambda}T^\lambda_{\gamma\beta}\frac{dy^\alpha}{d\tau}dy^\gamma-a_{\alpha\beta}D\frac{dy^\alpha}{ds}=0.$ The variation of the length of the geodesic is given by the formula $\delta s=\int_{t_1}^{t_2} a_{\alpha\lambda}T^\lambda_{\gamma\beta}\frac{dy^\alpha}{d\tau}dy^\gamma.$ We remark that the geodesics on Y^{n-1} which are determined by connection $G^\alpha_{\beta\gamma}$ do not depend on terms that contain tensor $T^\alpha_{\beta\gamma}$.

Now, we can construct a semi-geodesic coordinate system at any point Y^{n-1}, but we cannot integrate it.

Therefore, similarly to embedding space Y^n, we define the geodesic lines in Y^{n-1} space by formula $\frac{d^2y^\alpha}{ds^2}+G^\alpha_{\beta\gamma}\frac{dy^\beta}{ds}\frac{dy^\gamma}{ds}=\mathbf{0}$, and a variation of the length of the geodetic δs on Y^{n-1} $\delta s=\int_{t_1}^{t_2} a_{\alpha\beta}T^\beta_{\gamma\eta}\frac{dy^\alpha}{dt}dy^\gamma\delta y^\eta$, which depends on the torsion of the hypersurface Y^{n-1} and can be express in terms of torsion in Y^n.

We define the geodesic hypersurface as the hypersurface Y^{n-1} on which any geodesic line in Y^{n-1} is a geodesic line in the embedding space Y^n.

4.3. Properties of the second tensor $\pi_{\alpha\beta}$ of the hypersurface Y^{n-1}

Now we will repeat our reasoning scheme of construction of hypersurface and attempt more completely to understand the structure of embedding space Y^n. In space Y^n with coordinates $x^1,...,x^n$, we have the system of nondegenerate equations $x^i=x^i\left(y^1,...,y^{n-1}\right)$ which is determined by the hypersurface Y^{n-1}, then we calculate the metric and torsion of Y^{n-1} by formulas (6) and (7), and connection by (8). Then we studied some tensors $\xi^i_1,..,\xi^i_{n-1},\nu^i$ and obtained the tensor $\pi_{\alpha\beta}$, which is similar to the second tensor of Riemannian hypersurface but not symmetrical $\pi_{\alpha\beta}-\pi_{\beta\alpha}=g_{ij}S^i_{pq}\xi^p_\beta\xi^q_\alpha\nu^j$.

From the theory of surface in R^3, we know that the covariant derivative of the second tensor of any enough smooth surface is an asymmetrical tensor, on another hand, as we can see from $-R_{iklp}\xi^k_\lambda\xi^l_\chi\xi^p_\beta\nu^i+T^\sigma_{\lambda\chi}\pi_{\sigma\beta}=\pi_{\chi\beta;\lambda}-\pi_{\lambda\beta;\chi}$ the tensor $\pi_{\lambda\beta;\chi}$ is not symmetrical.

The formula $\pi_{\alpha\beta}-\pi_{\beta\alpha}=g_{ij}S^i_{pq}\xi^p_\beta\xi^q_\alpha\nu^j$ shows that external properties of geometry (embedding) of hypersurface can be associated with tensor $\pi_{\alpha\beta}$ and torsion S^i_{pq} of embedding space Y^n is influenced not only tensor $T^\alpha_{\beta\gamma}$ but also $\pi_{\alpha\beta}$ and $\vartheta_{\alpha\beta}$.

We associate with Y^{n-1} a coordinate system in Y^n, which denote by $u^1,...,u^{n-1},u^n$ the rule

$$u^1=y^1,...,u^{n-1}=y^{n-1},u^n=z,$$

with a new metric $\tilde{g}_{ik}$ defined by

$$\tilde{g}_{\alpha\beta}=a_{\alpha\beta},\ \tilde{g}_{n\alpha}=0,\ \tilde{g}_{nn}=1,$$

where z is a geodesic line directed along ν^i, where ν^i is normal to the hypersurface.

Since the rank of the matrix $\left[\frac{\partial x^i}{\partial y^\alpha}\right]$ equal $n-1$, suppose that

$$rank\left[\frac{\partial x^\gamma}{\partial y^\alpha}\right]>0$$

then exist the solution of a system of equations

$$x^1=x^1\left(y^1,...,y^{n-1}\right),$$
$$.....,$$
$$x^{n-1}=x^{n-1}\left(y^1,...,y^{n-1}\right),$$

which we denote by

$$u^1=y^1=y^1(x^1,...,x^{n-1}),$$
$$.....,$$
$$u^{n-1}=y^{n-1}=y^{n-1}(x^1,...,x^{n-1}),$$

and

$$u^n=z=z(x^1,...,x^{n-1},x^n)$$

herewith cometric tensor $\tilde{g}^{ik}$ equals

$$\tilde{g}^{ik}=\xi^i_\alpha \xi^k_\beta a^{\alpha\beta}+\nu^i\nu^k.$$

Remark 4. Whereas all our researches have a local character, we will make some remarks about the Taylor series.

Let us denote by $\Delta\xi^i$ the infinitesimal vector on the hypersurface Y^{n-1}, we can represent it as an infinite sum of terms is a Taylor series and contract this infinitesimal vector with $g_{ij}\nu^j$, so

$$g_{ij}\nu^j\Delta\xi^i=\frac{1}{2}g_{ij}\nu^j\xi^i_{\alpha;\beta}Dy^\alpha Dy^\beta+$$

$$+\frac{1}{6}g_{ij}\nu^j\xi^i_{\alpha;\beta;\gamma}Dy^\alpha Dy^\beta Dy^\gamma+....$$

since $\pi_{\alpha\beta}=g_{ij}\xi^i_{\beta;\alpha}\nu^j$ and $\pi_{\alpha\beta;\gamma}=g_{ij}\nu^j\xi^i_{\beta;\alpha;\gamma}$, and it can be rewritten as

$$g_{ij}\nu^j\Delta\xi^i=\frac{1}{2}\pi_{\alpha\beta}Dy^\alpha Dy^\beta+$$

$$+\frac{1}{6}\pi_{\alpha\beta;\gamma}Dy^\alpha Dy^\beta Dy^\gamma+....$$

where $Dy^\alpha=dy^\alpha+G^\alpha_{\beta\gamma}y^\beta dy^\gamma$, and $G^\alpha_{\beta\gamma}$ defined by (8).

Definition 2. ***If***

$$\varphi^{\alpha\beta}\omega_{\alpha\beta\gamma....}=0$$

for all γ..., then tensor $\varphi_{\alpha\beta}$ is called apolar with (or to) tensor $\omega_{\alpha\beta\gamma....}$.

We present the tensor $\pi_{\alpha\beta}$ in the form of a sum of two tensors symmetrical $\pi_{(\alpha\beta)}=\frac{1}{2}g_{ij}\nu^j(\xi^i_{\beta;\alpha}+\xi^i_{\alpha;\beta})$ and asymmetrical tensor $\pi_{[\alpha\beta]}=\frac{1}{2}g_{ij}\nu^j(\xi^i_{\beta;\alpha}-\xi^i_{\alpha;\beta})$ with properties $2\pi_{[\alpha\beta]}=g_{ij}S^i_{pq}\xi^p_\beta\xi^q_\alpha\nu^j$ and $\pi_{[\alpha\beta]}=\frac{1}{2}g_{ij}S^i_{pq}\xi^p_\beta\xi^q_\alpha\nu^j$.

The quantity

$$K=\frac{\pi_{(\alpha\beta)}dy^\alpha dy^\beta}{g_{\alpha\beta}dy^\alpha dy^\beta}=\frac{\frac{1}{2}g_{ij}\nu^j(\xi^i_{\beta;\alpha}+\xi^i_{\alpha;\beta})dy^\alpha dy^\beta}{g_{\alpha\beta}dy^\alpha dy^\beta}$$

is an analog of principal curvature at a given point of a surface and the direction which determinates by a vector $\left(\pi_{(\alpha\beta)}-Kg_{\alpha\beta}\right)dy^\beta=0$ is called the principal direction of the hypersurface. Using this definition one can classify the points on a hypersurface, we will not do that.

We denote $a^{\alpha\beta}\pi_{\alpha\beta;\gamma}=(n-1)F_\gamma$ and $\pi^{(\alpha\beta)}\pi_{(\alpha\beta);\gamma}=H_\gamma$, here the tensor $\pi^{(\alpha\beta)}$ is constructed from minors of the tensor $\pi_{(\alpha\beta)}$ multiplied by C. It is easy to see $a^{\alpha\beta}\pi_{\alpha\beta;\gamma}=(n-1)F_\gamma=a^{\alpha\beta}\pi_{(\alpha\beta);\gamma}$, but the connection $G^\gamma_{\alpha\beta}$ is not symmetrical, so $\pi_{\alpha\beta;\gamma}$ is not symmetrical at γ. By applying the equality $a_{\alpha\beta}a^{\alpha\beta}=n-1$, we have two equalities $a^{\alpha\beta}(\pi_{\alpha\beta;\gamma}-a_{\alpha\beta}F_\gamma)=0$ and $\pi^{(\alpha\beta)}(\pi_{(\alpha\beta);\gamma}-\frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma)=0.$

Therefore, we obtained two tensors $\pi_{\alpha\beta;\gamma}-a_{\alpha\beta}F_\gamma$ and $\pi_{(\alpha\beta);\gamma}-\frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma$, which are apolar with tensors $a_{\alpha\beta}$ and $\pi_{(\alpha\beta)}$ correspondingly.

The tensor $\pi_{(\alpha\beta);\gamma}-\frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma$, which is apolar with tensor $\pi_{(\alpha\beta)}$, can be symmetrized (in case of space R^n that tensor called Darboux's tensor) and written in the form

$$\theta_{\alpha\beta\gamma}=\pi_{(\alpha\beta);\gamma}-\frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma+\pi_{(\beta\gamma)}H_\alpha+\pi_{(\gamma\alpha)}H_\beta)$$

it is a thrice covariant symmetric tensor of the third order, defined on the hypersurface.

From the definition of Darboux's tensor, we can conclude that the vector on hypersurface ϖ_γ equals to zero

$$\varpi_\gamma=\pi^{(\alpha\beta)}\theta_{\alpha\beta\gamma}=$$

$$=\pi^{(\alpha\beta)}\begin{pmatrix}\pi_{(\alpha\beta);\gamma}-\frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma+\\+\pi_{(\beta\gamma)}H_\alpha+\pi_{(\gamma\alpha)}H_\beta)\end{pmatrix}=0.$$

The tensor $\theta_{\alpha\beta\gamma}$ is associated with the cubic form

$$\theta_{\alpha\beta\gamma}dy^\alpha dy^\beta dy^\gamma=\pi_{(\alpha\beta);\gamma}dy^\alpha dy^\beta dy^\gamma-$$

$$-(\pi_{(\alpha\beta)}H_\gamma+\pi_{(\beta\gamma)}H_\alpha+\pi_{(\gamma\alpha)}H_\beta)\frac{dy^\alpha dy^\beta dy^\gamma}{C(n+1)}=0,$$

and

$$\pi_{(\alpha\beta);\gamma}dy^\alpha dy^\beta dy^\gamma=\frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma+\pi_{(\beta\gamma)}H_\alpha+\pi_{(\gamma\alpha)}H$$

it is easy to see that here symmetry is not essential, we can rewrite

$$\pi_{\alpha\beta;\gamma}dy^\alpha dy^\beta dy^\gamma=\frac{3}{C(n+1)}\pi_{\alpha\beta}dy^\alpha dy^\beta H_\gamma dy^\gamma$$

,

and correlate this equation with a curve on the hypersurface, we obtain the equation

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$$\pi_{(\alpha\beta);\gamma}\frac{dy^{\alpha}}{ds}\frac{dy^{\beta}}{ds}\frac{dy^{\gamma}}{ds}- -(\pi_{(\alpha\beta)}H_{\gamma}+\pi_{(\beta\gamma)}H_{\alpha}+ +\pi_{(\gamma\alpha)}H_{\beta})\frac{1}{C(n+1)}\frac{dy^{\alpha}}{ds}\frac{dy^{\beta}}{ds}\frac{dy^{\gamma}}{ds}=0$$

which determines some directions on the hypersurface.

In general, this equation is too complicated for studying, however, in an important case, when $n=3$ the situation is simplified and we obtained the analog Darboux's theory. Let $n=3$, so we have

$$0=\sum_{\alpha,\beta,\gamma=1,2}\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=$$

$$\sum_{\alpha,\beta,\gamma=1,2}\left(\begin{array}{l}\pi_{(\alpha\beta);\gamma}- \\ -\dfrac{(\pi_{(\alpha\beta)}H_{\gamma}+\pi_{(\beta\gamma)}H_{\alpha}+\pi_{(\gamma\alpha)}H_{\beta})}{4C}\end{array}\right)du^{\alpha}du^{\beta}du^{\gamma}$$

it is a cubic equation with respect to $\frac{du^2}{du^1}$. The fraction $\frac{du^2}{du^1}$ is called Darboux's direction. Two-dimensional hypersurfaces on which $\sum_{\alpha,\beta,\gamma=1,2}\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=0$ is called Darboux's surface and for $\alpha,\beta,\gamma=1,2$ we have

$$\pi_{\alpha\beta;\gamma}du^{\alpha}du^{\beta}du^{\gamma}=\frac{3}{4C}\pi_{\alpha\beta}du^{\alpha}du^{\beta}H_{\gamma}du^{\gamma}.$$

4.4. Two-dimensional Darboux's theory

Definitions. *The value*

$$K=\frac{\pi_{(\alpha\beta)}du^{\alpha}du^{\beta}}{g_{\alpha\beta}du^{\alpha}du^{\beta}}=\frac{\frac{1}{2}g_{ij}\nu^{j}(\xi^{i}_{\beta;\alpha}+\xi^{i}_{\alpha;\beta})du^{\alpha}du^{\beta}}{g_{\alpha\beta}du^{\alpha}du^{\beta}}$$

*is called a **principal curvature** at a given point of a surface, and the direction, which is determined by a vector*

$$\left(\pi_{(\alpha\beta)}-Kg_{\alpha\beta}\right)du^{\beta}=0$$

*called the **principal direction** of the hypersurface.*

From equality $a_{\alpha\beta}a^{\alpha\beta}=2$, we are obtaining two equations

$$a^{\alpha\beta}(\pi_{\alpha\beta;\gamma}-a_{\alpha\beta}F_{\gamma})=0$$

and

$$\pi^{\alpha\beta}\left(\pi_{\alpha\beta;\gamma}-\frac{1}{2K}\pi_{\alpha\beta}H_{\gamma}\right)=0.$$

Therefore, we have two tensors

$$\pi_{\alpha\beta;\gamma}-a_{\alpha\beta}F_{\gamma}$$

and

$$\pi_{\alpha\beta;\gamma}-\frac{1}{2K}\pi_{\alpha\beta}H_{\gamma},$$

which are apolar with tensors $a_{\alpha\beta}$ and $\pi_{\alpha\beta}$ correspondingly.

Darboux's tensor is

$$\theta_{\alpha\beta\gamma}=\pi_{\alpha\beta;\gamma}-\frac{1}{4K}(\pi_{\alpha\beta}H_{\gamma}+\pi_{\beta\gamma}H_{\alpha}+\pi_{\gamma\alpha}H_{\beta}),$$

it is a thrice covariant symmetric tensor of the third order, defined on the surface.

The tensor ϖ_{γ} on hypersurface equals to zero

$$\varpi_{\gamma}=\pi^{\alpha\beta}\theta_{\alpha\beta\gamma}=$$

$$=\pi^{\alpha\beta}\left(\pi_{\alpha\beta;\gamma}-\frac{1}{4K}(\pi_{\alpha\beta}H_{\gamma}+\pi_{\beta\gamma}H_{\alpha}+\pi_{\gamma\alpha}H_{\beta})\right)=0.$$

Thus, we can construct the cubic form that associated with the tensor $\theta_{\alpha\beta\gamma}$, in the form

$$\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=$$

$$=\pi_{\alpha\beta;\gamma}du^{\alpha}du^{\beta}du^{\gamma}-\pi_{\alpha\beta}H_{\gamma}\frac{3}{4K}du^{\alpha}du^{\beta}du^{\gamma}=0,$$

and we have

$$\pi_{\alpha\beta;\gamma}du^{\alpha}du^{\beta}du^{\gamma}=\frac{3}{4K}\pi_{\alpha\beta}H_{\gamma}du^{\alpha}du^{\beta}du^{\gamma}.$$

Correlating this equation with a curve on the hypersurface, we obtain

$$\pi_{\alpha\beta;\gamma}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds}-\pi_{\alpha\beta}H_{\gamma}\frac{3}{4K}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds}=0$$

which determine some directions on the surface.

Thus, we obtained Darboux's theory

$$\sum_{\alpha,\beta,\gamma=1,2}\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=$$

$$=\sum_{\alpha,\beta,\gamma=1,2}\left(\pi_{\alpha\beta;\gamma}-\frac{3}{4K}\pi_{\alpha\beta}H_{\gamma}\right)du^{\alpha}du^{\beta}du^{\gamma}=0$$

it is a cubic equation with respect to $\frac{du^2}{du^1}$. The fraction $\frac{du^2}{du^1}$ is called Darboux's direction. Two-dimensional hypersurfaces on which

$$\sum_{\alpha,\beta,\gamma=1,2}\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=0$$

is called Darboux's surface, for which we have obtained

$$\pi_{\alpha\beta;\gamma}du^{\alpha}du^{\beta}du^{\gamma}=\frac{3}{4K}\pi_{\alpha\beta}H_{\gamma}du^{\alpha}du^{\beta}du^{\gamma}.$$

Thus, the lines defined by equation

$$\theta_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}=0$$

are called Darboux's lines. If we divide this equation on du^1 this equation can be considered as the third-degree equation relative to $\frac{du^2}{du^1}$, and as the third-degree equation, in general, it has three solutions and at least one of which is real.

Without proving, let us formulate the next theorem concerning Darboux's lines.

Theorem (*about Darboux's lines***).**

A surface is a ruled surface (scroll) if and only if all three Darboux's directions and so Darboux's lines coincide in one direction, and this Darboux's direction defines the generator line (Darboux's line coincides with the generator line).

If the surface is not a ruled surface (scroll) with negative curvature, then there is only one Darboux's direction and only one Darboux's line family.

On any surface with positive curvature, all three Darboux's directions always real.

On any second-degree surface, Darboux's tensor equals zero.

The directions on the surface defined by the next equation

$$\pi_{\alpha\beta;\gamma} du^\alpha du^\beta du^\gamma = 0$$

are called Codazzi's directions and lines defined by these directions called Codazzi's lines.

From the equation

$$\pi_{\alpha\beta;\gamma} du^\alpha du^\beta du^\gamma = \frac{3}{4K} \pi_{\alpha\beta} H_\gamma du^\alpha du^\beta du^\gamma ,$$

we can deduce that on Darboux's surfaces Codazzi's lines consist of the lines on which the curvature is a constant $K = const$ and the asymptotic lines.

V. An exemplar of geodesics on the hypersurface Y^{n-1} in the space with a flat metric.

We assume that a hypersurface Y^3 is embedded in the four-dimensional space Y^4 with the Euclidean metric $g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and let the torsion tensor be given as

$$S_{ik}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S_{ik}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{ik}^3 = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S_{ik}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (15)$$

The connection can be obtained as follows

$$\Gamma_{ik}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Gamma_{ik}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_{ik}^3 = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Gamma_{ik}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (16)$$

Next, we denote the hypersurface coordinate system as u^1, u^2, u^3 and let the hypersurface be given as

$$x^1 = u^1,$$
$$x^2 = u^2,$$
$$x^3 = u^3,$$
$$x^4 = 0;$$

and for hypersurface, we have

$$T_{ik}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{pmatrix}, T_{ik}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_{ik}^3 = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \ (17)$$

The connection of hypersurface can be obtained as

$$G_{ik}^1 = \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, G_{ik}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_{ik}^3 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ (18)$$

and we have the following system of differential equations

$$\begin{cases} \dfrac{d^2u^1}{ds^2} - a\dfrac{du^2}{ds}\dfrac{du^3}{ds} = 0 \\ \dfrac{d^2u^2}{ds^2} = 0 \\ \dfrac{d^2u^3}{ds^2} + a\dfrac{du^1}{ds}\dfrac{du^2}{ds} = 0. \end{cases} \qquad (19)$$

This system defines geodesics, has two solutions, general and particular. Its general solution is

$$u^1 = -\frac{k_1}{k_2 a}\cos\left(k_2 as\right) - \frac{k_3}{k_2 a}\sin\left(k_2 as\right) + k_4$$
$$u^2 = k_2 s + k_5$$
$$u^3 = \frac{k_1}{k_2 a}\sin\left(k_2 as\right) - \frac{k_3}{k_2 a}\cos\left(k_2 as\right) + k_6$$
(20)

where $k_1, k_2, k_3, k_4, k_5, k_6$ are independent parameters.

The particular solution is

$$u^1 = M_1 s + M_2$$
$$u^2 = M_3 \qquad (21)$$
$$u^3 = M_4 s + M_5$$

here M_1, M_2, M_3, M_4, M_5 are arbitrary parameters.

VI. Conclusion

In this paper, we establish an analog Darboux's theory for the hypersurfaces in the spaces with metric and torsion. To achieve this goal, we have studied the structural special features of the curvature tensor and obtained Ricci - Jacobi identity for the curvature tensor in a case when the torsion tensor is nonzero.

If and only if all three Darboux's directions, which define Darboux's lines, coincide in one direction, a hypersurface is a scroll or ruled surface. The Darboux's line coincides with the generator line of the scroll and the Darboux's direction defines the generator line.

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