Beta- Star- Continuity and Beta- Star- Contra- Continuity

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Abstract—In 2014 Mubarki, Al-Rshudi, and Al-Juhani introduced and studied the notion of a set in general topology called β^*-open set and investigated its fundamental properties and studied the relationships between β^*-open set and other topological sets including β^*-continuity in topological spaces. We introduce and investigate several properties and characterizations of a new class of functions between topological spaces called β^*-open, β^*-closed, β^*-continuous and β^*-irresolute functions in topological spaces. We also introduce slightly β^*-continuous, totally β^*-continuous and almost β^*-continuous functions between topological spaces and establish several characterizations of these new forms of functions. Furthermore, we also introduce and investigate certain ramifications of contra continuous and allied functions, namely, contra $-\beta^*-$continuous, and almost contra $-\beta^*-$continuous functions along with their several properties, characterizations and natural relationships. Moreover, we introduce new types of closed graphs by using β^*-open sets and investigate its properties and characterizations in topological spaces.

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1. INTRODUCTION

In recent literature, we find many topologists have focused their research in the direction of investigating different types of generalized continuity. One of the outcomes of their research leads to the initiation of different orientations of contra continuous functions. In 2014 Mubarki, Al-Rshudi, and Al-Juhani introduced and studied the notion of set in general topology called β^*-open sets and investigated its fundamental properties and studied the relationship between β^*-open set and other topological sets including β^*-continuity in topological spaces. In this paper, we introduce and investigate several properties and characterizations of a new class of maps between topological spaces called β^*-open maps, β^*-open maps, β^*-continuous maps and β^*-irresolute maps. We also introduce slightly β^*-continuous, totally β^*-continuous and almost β^*-continuous maps between topological spaces and establish several characterizations of these new forms of maps. Furthermore, we also introduce and investigate fundamental properties of contra continuous and allied functions, namely, contra $-\beta^*-$continuous, almost $-\beta^*-$continuous, and almost contra $-\beta^*-$continuous functions along with their several properties, characterizations and natural relationships. Moreover, we introduce new types of graphs, called β^*-closed, contra $-\beta^*-$closed and strongly contra $-\beta^*-$closed graphs via β^*-open sets. Several characterizations and properties of such notions are investigated.

Throughout this paper (X,τ) or simply by X we denote topological space on which no separation axioms are assumed unless explicitly stated and $f:(X,\tau)\to(Z,\mu)$ means a mapping f from a topological space X to a topological space Z. If U is a set and x is a point in X then $N(x)$, $Int(U)$, $Cl(U)$ and $U^c = X \setminus U$ denote respectively, the neighbourhood system of x, the interior of U, the closure of U and complement of U.

2. β^*-OPEN SETS IN TOPOLOGICAL SPACES

Definition 2.1. A subset A of a topological space X is called semi $-$ open set if $A\subseteq Cl\left[Int(A)\right]$.

Definition 2.2. A subset A of a topological space X is called $\alpha-$open set if $A\subseteq Int\left[Cl\left(Int(A)\right)\right]$.

Definition 2.3. A subset A of a topological space X is called $\beta-$open set if $A\subseteq Cl\left[Int\left(Cl(A)\right)\right]$.

Definition 2.4. A subset A of a topological space X is called pre$-$open set if $A\subseteq Int\left[Cl(A)\right]$.

Definition 2.5. A subset A of a topological space X is said to be b$-$open set if $A\subseteq Cl\left[Int(A)\right]\cup Int\left[Cl(A)\right]$.

Definition 2.6. Let (X,τ) be a topological space. Then a point $x\in X$ is called the $\delta-$cluster point of $A\subseteq X$ if $A\cap Int\left[Cl(U)\right]\neq\phi$ for every open set U of X containing x. The set of all cluster points of A is called the $\delta-$cluster points of A, denoted by $Cl_\delta(A)$. A subset $A\subseteq X$ is called $\delta-$closed if $A=Cl_\delta(A)$.

Definition 2.7. Let (X,τ) be a topological space and $A\subseteq X$. Then A is called $\delta-$open set if its complement $X-A$ is $\delta-$closed in X. The collection of all $\delta-$open sets in a topological space (X,τ) forms a topology τ_δ on X, weaker than τ and the class of all regular open sets in τ forms an open basis for τ_δ.

Definition 2.8. A subset A of a topological space X is called e$*-$open set if $A\subseteq Cl\left[Int\left(Cl_\delta(A)\right)\right]$.

Definition 2.9. Let (X,τ) be a topological space. Then a subset A of X is said to be $\beta*-$open if $A\subseteq Cl\left[Int\left(Cl(A)\right)\right]\cup Int\left[Cl_\delta(A)\right]$. The family of all $\beta*-$open subsets of a topological space (X,τ) will be as always denoted by $\beta*O(X)$.

Definition 2.10. A subset A of a topological space (X,τ) is said to be a $\beta*-$closed set if $Int\left[Cl\left(Int(A)\right)\right]\cap Cl\left[Int_\delta(A)\right]\subseteq A$.

The family of all $\beta*-$closed subsets of a topological space (X,τ) will be as denoted by $\beta^*C(X)$.

Remark 2.11. The following diagram holds for each a subset A of X.

open set $\rightarrow\alpha-$open set $\rightarrow$ preopen set $\rightarrow$ b$-$open set $\rightarrow\beta-$open set $\rightarrow\beta*-$open set $\rightarrow$ e$*-$open set

Theorem 2.12. Let (X,τ) be a topological space. Then the following assertions hold:

(1) The arbitrary union of $\beta*-$open sets is $\beta*-$open.

(2) The arbitrary intersections of $\beta*-$closed is $\beta*-$closed.

Proof. (1) Let $\{A_i : i\in I\}$ be a family of $\beta*-$open sets. Then $A_i\subseteq Cl\left[Int\left(Cl(A_i)\right)\right]\cup Int\left[Cl_\delta(A_i)\right]$ and therefore immediately it follows that

$$\bigcup_{i\in I}A_i\subseteq\bigcup_{i\in I}\left(Cl\left[Int\left(Cl(A_i)\right)\right]\cup Int\left[Cl_\delta(A_i)\right]\right)\subseteq Cl\left[Int\left(Cl\left(\bigcup_{i\in I}A_i\right)\right)\right]\cup Int\left[Cl_\delta\left(\bigcup_{i\in I}A_i\right)\right],$$

for all $i\in I$. Thus $\bigcup_{i\in I}A_i$ is $\beta*-$open.

(2) It follows from (1).

Remark 2.13. The next example shows that the intersection of any two $\beta*-$open sets is not $\beta*-$open.

Example 2.14. Let $X=\{1,2,3\}$ with topology $\tau=\{\phi,\{1\},\{2\},\{1,2\},X\}$. Then $A=\{1,3\}$ and

$B=\{2,3\}$ are β^*-open sets. But $A\cap B=\{3\}$ is not β^*-open.

Definition 2.15. Let (X,τ) be a topological space. Then:

(1) The union of all β^*-open sets of X contained in A is called the β^*-interior of A and is denoted by $\beta^*-Int(A)$.

(2) The intersection of all β^*-closed sets of X containing A is called the β^*-closure of A and is denoted by $\beta^*-Cl(A)$.

Theorem 2.16. Let A, B be two subsets of a topological space (X,τ). Then the following assertions are true:

(1) $\beta^*-Cl(X)=X$ and $\beta^*-Cl(\phi)=\phi$.

(2) $A\subseteq\beta^*-Cl(A)$.

(3) If $A\subseteq B$, then $\beta^*-Cl(A)\subseteq\beta^*-Cl(B)$.

(4) $x\in\beta^*-Cl(A)$ if and only if for each a β^*-open set U containing x, $U\cap A\neq\phi$.

(5) A is β^*-closed set if and only if $A=\beta^*-Cl(A)$.

(6) $\beta^*-Cl[\beta^*-Cl(A)]=\beta^*-Cl(A)$.

(7) $\beta^*-Cl(A)\cup\beta^*-Cl(B)\subseteq\beta^*-Cl(A\cup B)$.

(8) $\beta^*-Cl(A\cap B)\subseteq\beta^*-Cl(A)\cap\beta^*-Cl(B)$.

Theorem 2.17. Let A, B be two subsets of a topological space (X,τ). Then the following assertions are true:

(1) $\beta^*-Int(X)=X$ and $\beta^*-Int(\phi)=\phi$.

(2) $\beta^*-Int(A)\subseteq A$.

(3) If $A\subseteq B$, then $\beta^*-Int(A)\subseteq\beta^*-Int(B)$.

(4) $x\in\beta^*-Int(A)$ if and only if there exists β^*-open set W such that $x\in W\subseteq A$.

(5) A is β^*-open set if and only if $A=\beta^*-Int(A)$.

(6) $\beta^*-Int[\beta^*-Int(A)]=\beta^*-Int(A)$.

(7) $\beta^*-Int(A\cap B)\subseteq\beta^*-Int(A)\cap\beta^*-Int(B)$.

(8) $\beta^*-Int(A)\cup\beta^*-Int(B)\subseteq\beta^*-Int(A\cup B)$.

3. β^*-CONTINUOUS FUNCTIONS

In this section, we introduce a new type of continuous map called a β^*-continuous map and obtain some of its properties and characterizations.

Definition 3.1. Let (X,τ) and (Y,μ) be two topological spaces. A map $f:(X,\tau)\to(Y,\mu)$ is called an β^*-continuous function if the inverse image of each open set in Y is an β^*-open set in X.

Theorem 3.2. Every continuous function is β^*-continuous.

Proof. Let $f:(X,\tau)\to(Y,\mu)$ be a continuous function and W be an open set in Y. By hypothesis f is continuous. Then $f^{-1}(W)$ is an open set in X. Since $\tau\subseteq\beta^*-O(X,\tau)$. Therefore, $f^{-1}(W)$ is β^*-open in X. Hence f is β^*-continuous.

The converse of the above theorem is not true as shown in the following example.

Example 3.3. Let the set $X=\{a,b,c,d\}$ and let $\tau=\{\phi,\{a\},\{c\},\{a,c\},\{a,b,c\},\{a,c,d\},X\}$ be a topology on X. Let $f:(X,\tau)\to(X,\tau)$ be a function defined by $f(a)=f(b)=f(d)=c, f(c)=a$. We note that $\{a,b,d\}$ is a β^*-open set in X. Then $f^{-1}(\{c\})=\{a,b,d\}$ is a β^*-open set in X. Then clearly f is a $\beta*$–continuous function. Now since $f^{-1}(\{c\})=\{a,b,d\}$ is not an open set in X. Therefore, f is not a continuous map.

Theorem 3.4. Let (X,τ) and (Y,μ) be two topological spaces. Let f be a map from X into Y. Then the following statements are equivalent:

(1) f is a $\beta*$–continuous map;

(2) The inverse image of a closed set in Y is a $\beta*$–closed set in X;

(3) $\beta*\text{–}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Cl(B)\right]$ for every set B in Y;

(4) $f\left\lfloor \beta*\text{–}Cl(A)\right\rfloor\subseteq Cl\left\lfloor f(A)\right\rfloor$ for every set A in X;

(5) $f^{-1}\left[Int(B)\right]\subseteq\beta*\text{–}Int\left[f^{-1}(B)\right]$ for every set B in Y.

Proof. $(1)\Rightarrow(2)$: Let B be a closed set in Y, then $Y\setminus B$ is an open set in Y. Then $f^{-1}(Y\setminus B)=X\setminus f^{-1}(B)$ is a $\beta*$–open set in X; It follows that $f^{-1}(B)$ is a $\beta*$–closed closed subset of X.

$(2)\Rightarrow(3)$: Let B be any subset of Y. Since $Cl(B)$ is closed in Y. then $f^{-1}\left[Cl(B)\right]$ is $\beta*$–closed in X. Therefore, $\beta*\text{–}Cl\left[f^{-1}(B)\right]\subseteq \beta*\text{–}Cl\left[f^{-1}(Cl(B))\right]= f^{-1}\left[Cl(B)\right]$.

$(3)\Rightarrow(4)$: Let A be any subset of X. By (3) we have

$$\beta*\text{–}Cl(A)\subseteq\beta*\text{–}Cl\left[f^{-1}(f(A))\right]\subseteq f^{-1}\left[Cl(f(A))\right].$$

Therefore, $f\left\lfloor\beta*\text{–}Cl(A)\right\rfloor\subseteq Cl\left\lfloor f(A)\right\rfloor$.

$(4)\Rightarrow(3)$: Let B be any subset of Y. Then by hypothesis, we get $f\left[\beta*\text{–}Cl(f^{-1}(B))\right]\subseteq Cl\left[f(f^{-1}(B))\right]\subseteq Cl(B)$. Therefore we obtain $\beta*\text{–}Cl(f^{-1}(B))\subseteq f^{-1}\left[Cl(B)\right]$.

$(3)\Rightarrow(5)$: Let B be any subset of Y. Then by hypothesis, we get $\beta*\text{–}Cl\left(f^{-1}(Z\setminus B)\right)\subseteq f^{-1}\left[Cl(Z\setminus B)\right]$ and hence $X\setminus\left[\beta*\text{–}Int(f^{-1}(B))\right]\subseteq X\setminus f^{-1}(Int(B))$. Therefore we obtain $f^{-1}\left[Int(B)\right]\subseteq\beta*\text{–}Int(f^{-1}(B))$.

$(5)\Rightarrow(1)$: Let B be an open set in Y and $f^{-1}\left[Int(B)\right]\subseteq\beta*\text{–}Int\left[f^{-1}(B)\right]$. Then, $f^{-1}(B)\subseteq\beta*\text{–}Int\left[f^{-1}(B)\right]$. But, $\beta*\text{–}Int\left[f^{-1}(B)\right]\subseteq f^{-1}(B)$. Hence, $f^{-1}(B)=\beta*\text{–}Int\left[f^{-1}(B)\right]$. Therefore, $f^{-1}(B)$ is $\beta*$–open set in X.

Theorem 3.5. Let (X_1,τ_1), (X_2,τ_2) and (X_3,τ_3) be three topological spaces. If a map $f:(X_1,\tau_1)\to(X_2,\tau_2)$ is $\beta*$–continuous and $g:(X_2,\tau_2)\to(X_3,\tau_3)$ is a continuous map, then $gof:(X_1,\tau_1)\to(X_3,\tau_3)$ is $\beta*$–continuous.

Proof. Obvious.

Theorem 3.6. Let (X,τ) and (Y,μ) be two topological spaces. Then $f:(X,\tau)\to(Y,\mu)$ is a $\beta*$–continuous map, if one of the following holds:

(1) $f^{-1}\left[\beta*\text{–}Int(B)\right]\subseteq Int\left[f^{-1}(B)\right]$ for every set B in Y.

(2) $Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[\beta*\text{–}Cl(B)\right]$ for every set B in Y.

(3) $f\left\lfloor Cl(A)\right\rfloor\subseteq\beta*\text{–}Cl\left\lfloor f(A)\right\rfloor$ for every set A in X.

Proof. (1) Let B be any open set of Y. Then $f^{-1}\left[\beta*\text{–}Int(B)\right]\subseteq Int\left[f^{-1}(B)\right]$. We get $f^{-1}(B)\subseteq Int\left[f^{-1}(B)\right]$. Therefore, $f^{-1}(B)$ is an open set. Since very open set is $\beta*$–copen. Hence, f is a $\beta*$–continuous function.

(2) Let B be a closed subset of Y. Then by hypothesis, $Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[\beta*\text{–}Cl(B)\right]$. Since B is closed, $\beta*\text{–}Cl(B)=B$. Thus $Cl\left[f^{-1}(B)\right]\subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is closed in X. So f is a β*–continuous function.

(3) Let B be any open set of Y. Then $f^{-1}(B)$ is a set in X and $f\left[Cl(f^{-1}(B))\right]\subseteq\beta*\text{–}Cl\left[f(f^{-1}(B))\right]$. This implies $f\left[Cl(f^{-1}(B))\right]\subseteq\beta*\text{–}Cl(B)$. This is nothing but condition (2). Hence f is a $\beta*$–continuous map.

4. β^* – OPEN FUNCTIONS AND β^* – CLOSED FUNCTIONS

Definition 4.1. A map $f:(X,\tau)\to(Y,\mu)$ is called β^*–open (resp. β^*–closed) if the image of each open (resp. closed) set in X is β^*–open (resp. β^*–closed) in (Y,μ).

Theorem 4.2. A map $f:(X,\tau)\to(Y,\mu)$ is β^*–open if and only if $f\lfloor Int(A)\rfloor\subseteq \beta^*\text{–}Int\lfloor f(A)\rfloor$ for each set A in X.

Proof. Suppose that f is a β^*–open map. Since $Int(A)\subseteq A$, then $f\lfloor Int(A)\rfloor\subseteq f(A)$. By hypothesis, $f\lfloor Int(A)\rfloor$ is a β^*–open set and $\beta^*\text{–}Int\lfloor f(A)\rfloor$ is the largest β^*–open set contained in $f(A)$. Hence $f\lfloor Int(A)\rfloor\subseteq \beta^*\text{–}Int\lfloor f(A)\rfloor$.

Conversely, suppose A is an open set in X. Then, $f\lfloor Int(A)\rfloor\subseteq\beta^*\text{–}Int\lfloor f(A)\rfloor$. Since $Int(A)=A$, then $f(A)\subseteq Y-Int\lfloor f(A)\rfloor$. Therefore, $f(A)$ is a β^**–open* set in (Y,μ) and f is a β^*–open function.

Theorem 4.3. A function $f:(X,\tau)\to(Y,\mu)$ is β^*–closed if and only if $\beta^*\text{–}Cl\lfloor f(A)\rfloor\subseteq f\lfloor Cl(A)\rfloor$ for each set A in X.

Proof. Suppose f is a β^*–closed function. Since for each set A in X. $Cl(A)$ is closed set in X, then $f\lfloor Cl(A)\rfloor$ is a β^*–closed set in Y. Also, since $f(A)\subseteq f\lfloor Cl(A)\rfloor$, then $\beta^*\text{–}Cl\lfloor f(A)\rfloor\subseteq f\lfloor Cl(A)\rfloor$.

Conversely, Let A be a closed set in X. Since $\beta^*\text{–}Cl\lfloor f(A)\rfloor$ is the smallest β^*–closed set containing $f(A)$, then $f(A)\subseteq\beta^*\text{–}Cl\lfloor f(A)\rfloor\subseteq f\lfloor Cl(A)\rfloor=f(A)$.

Thus, $f(A)=\beta^*\text{–}Cl\lfloor f(A)\rfloor$. Hence, $f(A)$ is a β^*–closed set in Y. Therefore, f is a β^*–closed function.

Theorem 4.4. Suppose that (X_1,τ_1), (X_2,τ_2) and (X_3,τ_3) are any three topological spaces. Suppose also that $f:(X_1,\tau_1)\to(X_2,\tau_2)$ and $g:(X_2,\tau_2)\to(X_3,\tau_3)$ are two functions. Then,

(1) if gof is β^*–open and f is continuous surjective, then g is a β^*–open function.

(2) if gof is open and g is β^*–continuous injective, then f is a β^*–open function.

Proof. (1) Let V be an open set in X_2. Then, $f^{-1}(V)$ is an open set in X_1. Since gof is a β^*–open map, then $(gof)\left[f^{-1}(V)\right]=\quad g\left[f\left(f^{-1}(V)\right)\right]=g(V)$ (because f is surjective) is a β^*–open set in X_3. Therefore, g is a β^*–open function.

(2) Let U be an open set in X_1. Then, $g\lfloor f(U)\rfloor$ is an open set in X_3. Therefore, $g^{-1}\left[g\left(f(U)\right)\right]=f(U)$ (because g is injective) is a β^*–open set in X_2. Hence, f is a β^*–open map.

Theorem 4.5. Let (X,τ) and (Y,μ) be two topological spaces and $f:(X,\tau)\to(Y,\mu)$ be a bijective function. Then the following statements are equivalent:

(1) f is a β^*–open function;

(2) f is a β^*–closed function;

(3) f^{-1} is a β^*–continuous function.

Proof. $(1)\Rightarrow(2)$: Suppose A is a closed set in X. Then $X \mathbf{B} A$ is an open set in X and by (1), $f(X \mathbf{B} A)$ is a β^*–open set in Y. Since f is bijective, then $f(X \mathbf{B} A)=Y \mathbf{B} f(A)$. Hence, $f(A)$ is a β^*–closed set in Y. Therefore, f is a β^*–closed function.

$(2)\Rightarrow(3)$: Let f be a β^*–closed function and A be closed set in X. Since f is bijective, then $\left(f^{-1}\right)^{-1}(A)=f(A)$ which is a β^*–closed set in Y. Therefore, by Theorem 3.4, f^{-1} is a β^*–continuous function.

$(3)\Rightarrow(1)$: Let A be an open set in X. Since f^{-1} is a β^*–continuous function, then $\left(f^{-1}\right)^{-1}(A)=f(A)$ is a β^*–open set in Y. Hence, f is a β^*–open function.

5. β* – IRRESOLUTE FUNCTIONS

In this section, we introduce a new type of function called β^*–irresolute function and obtain some of its properties and characterizations.

Definition 5.1. A function $f:(X,\tau)\rightarrow(Y,\mu)$ is called a β^*–irresolute function if the inverse image of each β^*–open set in Y is a β^*–open set in X.

Theorem 5.2. Every β^*–irresolute function is a β^*–continuous function.

Proof. Straightforward.

Theorem 5.3. Let (X,τ) and (Y,μ) be two topological spaces. Let f be a function from X into Y. Then, the following statements are equivalent:

(1) f is a β^*–irresolute function;

(2) The inverse image of each β^*–closed set in Y is a β^*–closed set in X;

(3) $\beta^*\text{–}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[\beta^*\text{–}Cl(B)\right]$ for every set B in Y;

(4) $f\left\lfloor\beta^*\text{–}Cl(A)\right\rfloor\subseteq\beta^*\text{–}Cl\left\lfloor f(A)\right\rfloor$ for every set A in X;

(5) $f^{-1}\left[\beta^*\text{–}Int(B)\right]\subseteq\beta^*\text{–}Int\left[f^{-1}(B)\right]$ for every B in Y.

Proof. $(1)\Rightarrow(2)$: Let B be a β^*–closed set in Y. Then $Y\setminus B$ is a β^*–open set in Y. Hence $f^{-1}(Y\setminus B)=X\setminus f^{-1}(B)$ is a β^*–open set in X. It follows that $f^{-1}(B)$ is a β^*–closed subset of X.

$(2)\Rightarrow(3)$: Let B be any subset of Y. Since $\beta^*\text{–}Cl(B)$ is a β^*–closed set in Y, then $f^{-1}\left[\beta^*\text{–}Cl(B)\right]$ is a β^*–closed set in X. Thus $\beta^*\text{–}Cl\left[f^{-1}(B)\right]\subseteq\beta^*\text{–}Cl\left[f^{-1}\left(\beta^*\text{–}Cl(B)\right)\right]$ $=f^{-1}\left[\beta^*\text{–}Cl(B)\right]$.

$(3)\Rightarrow(4)$: Let A be any subset of X. By (3), we have $\beta^*\text{–}Cl(A)\subseteq\beta^*\text{–}Cl\left[f^{-1}\left(f(A)\right)\right]\subseteq$ $f^{-1}\left[\beta^*\text{–}Cl\left(f(A)\right)\right]$. Therefore $f\left\lfloor\beta^*\text{–}Cl(A)\right\rfloor\subseteq\beta^*\text{–}Cl\left\lfloor f(A)\right\rfloor$.

$(4)\Rightarrow(5)$: Let B be any subset of Y. By (4), $f\left[\beta^*\text{–}Cl\left(X\setminus f^{-1}(B)\right)\right]\subseteq\beta^*\text{–}Cl\left[f\left(X\setminus f^{-1}(B)\right)\right]$ and $f\left\lfloor X\setminus\left(\beta^*\text{–}Int\left(f^{-1}(B)\right)\right)\right\rfloor\subseteq\beta^*\text{–}Cl(Y\setminus B)=$ $Y\setminus\left\lfloor\beta^*\text{–}Int(B)\right\rfloor$. Therefore we have $X\setminus\left[\beta^*\text{–}Int\left(f^{-1}(B)\right)\right]\subseteq f^{-1}\left[Y\setminus\left(\beta^*\text{–}Int(B)\right)\right]$ and hence $f^{-1}\left[\beta^*\text{–}Int(B)\right]\subseteq\beta^*\text{–}Int\left[f^{-1}(B)\right]$.

$(5)\Rightarrow(1)$: Let B be a β^*–open set in Y and $f^{-1}\left[\beta^*\text{–}Int(B)\right]\subseteq\beta^*\text{–}Int\left[f^{-1}(B)\right]$. Then $f^{-1}(B)\subseteq\ \beta^*\text{–}Int\left[f^{-1}(B)\right]$ But, $\beta^*\text{–}Int\left[f^{-1}(B)\right]\subseteq f^{-1}(B)$. Hence $f^{-1}(B)=\beta^*\text{–}Int\left[f^{-1}(B)\right]$. Therefore $f^{-1}(B)$ is β^*–open in X. Thus f is a β^*–irresolute function.

Theorem 5.4. Let $f:(X_1,\tau_1)\rightarrow(X_2,\tau_2)$ and $g:(X_2,\tau_2)\rightarrow(X_3,\tau_3)$ be β^*–irresolute maps. Then $gof:(X_1,\tau_1)\rightarrow(X_3,\tau_3)$ is β^*–irresolute.

Proof. Obvious.

6. TOTALLY β* – CONTINUOUS FUNCTIONS

In this section, the notion of totally β^*–continuous function is introduced as well as its characterizations are investigated.

Definition 6.1. Let (X,τ) be a topological space. A subset A of X is called $\beta*$–clopen if A is both $\beta*$–open and $\beta*$–closed set in X.

Definition 6.2. Let (X,τ) and (Y,μ) be two topological spaces. A function $f:(X,\tau)\to(Y,\mu)$ is called a totally $\beta*$–continuous function if the inverse image of each open set in Y is $\beta*$–clopen in X.

Definition 6.3. A topological space (X,τ) is called $\beta*$–connected if it is not the union of two non-empty disjoint $\beta*$–open sets.

Theorem 6.4. A topological space (X,τ) is $\beta*$–connected if and only if X and ϕ are the only $\beta*$–clopen subsets of X.

Proof. Obvious.

Theorem 6.5. Let (X,τ) be a topological space. If $f:(X,\tau)\to(Y,\mu)$ is a totally $\beta*$–continuous surjection and (X,τ) is $\beta*$–connected, then (Y,μ) is an indiscrete space.

Proof. Suppose that (Y,μ) is not an indiscrete space and let V be a proper non-empty open subset of (Y,μ). Since f is a totally $\beta*$–continuous function, then $f^{-1}(V)$ is a proper non-empty $\beta*$–clopen subset of X. Therefore $X = f^{-1}(V)\mathrm{U}\left[X \mathbf{B}\ f^{-1}(V)\right]$ and X is a union of two non-empty disjoint $\beta*$–open sets, which is a contradiction. Therefore (Y,μ) must be an indiscrete space.

Theorem 6.6. A topological space (X,τ) is $\beta*$–connected if and only if every totally $\beta*$–continuous function from (X,τ) into any T_0–space (Y,μ) is a constant map.

Proof. $\Rightarrow$: Suppose that $f:(X,\tau)\to(Y,\mu)$ is a totally $\beta*$–continuous function, where (Y,μ) is a T_0–space. Assume that f is not constant and $x,y\in X$ such that $f(x)\neq f(y)$. Since (Y,μ) is T_0, and $f(x)$ and $f(y)$ are distinct points in Y, then there is an open set V in (Y,μ) containing only one of the points $f(x)$ and $f(y)$. We take the case $f(x)\in V$ and $f(y)\notin V$. The proof of the other case is similar. Since f is a totally $\beta*$–continuous function, $f^{-1}(V)$ is a $\beta*$–clopen subset of X and $x\in f^{-1}(V)$, but $y\notin f^{-1}(V)$. Since $X = f^{-1}(V)\mathrm{U}\left[X \mathbf{B}\ f^{-1}(V)\right]$, X is a union of two non-empty disjoint $\beta*$–open subsets of X. Thus (X,τ) is not $\beta*$–connected, which is a contradiction.

$\Leftarrow$: Suppose that (X,τ) is not a $\beta*$–connected space. Then there is a proper non-empty $\beta*$–clopen subset A of X. Let $Y=\{a,b\}$ and $\mu=\{Y,\phi,\{a\},\{b\}\}$, define $f:(X,\tau)\to(Y,\mu)$ by $f(x)=a$ for each $x\in A$ and $f(x)=b$ for $x\in X\ \mathbf{B}\ A$. Clearly f is not constant and totally $\beta*$–continuous where Y is T_0, and thus we have a contradiction.

Definition 6.7. A topological space (X,τ) is said to be:

(i) $\beta*$–T_1 if for each pair of distinct points x and y of X, there exist $\beta*$–open sets U and V containing x and y, respectively such that $y\notin U$ and $x\notin V$.

(ii) $\beta*$–T_2 if for each pair of distinct points x and y in X, there exist disjoint $\beta*$–open sets U and V in X such that $x\in U$ and $y\in V$.

Theorem 6.8. Let $f:(X,\tau)\to(Y,\mu)$ be totally $\beta*$–continuous and Y be a T_1–space. If A is a non-empty $\beta*$–connected subset of X, then $f(A)$ is a singleton.

Proof. Suppose that $f(A)$ is not a singleton. Let $f(x_1)=y_1\in f(A)$ and $f(x_2)=y_2\in f(A)$, such that $y_1\neq y_2$, where $x_1,x_2\in A$ and $x_1\neq x_2$. Since $y_1,y_2\in Y$ such that $y_1\neq y_2$ and Z is a T_1–space, then there exists an open set G in Y

(say) containing y_1 but not y_2. Since f is totally β^*–continuous Then $f^{-1}(G)$ is a β^*–clopen set containing x_1 but not x_2. Now $X = f^{-1}(G) \cup [X \setminus f^{-1}(G)]$. Hence X is the union of two disjoint non-empty β^*–open subsets. Let $A_1 = A \cap f^{-1}(G)$ and $A_2 = A \cap (X - f^{-1}(G))$. Clearly $x_1 \in A_1$ and $x_2 \in A_2$. We observe that A_1 and A_2 are two disjoint nonempty β^*–open subsets of A such that $A = A_1 \cup A_2$. This implies that A is not β^*–connected, which is a contradiction. Thus $f(A)$ is a singleton.

Theorem 6.9. Let (X, τ) and (Y, μ) be two topological spaces. Let $f:(X,\tau)\to(Y,\mu)$ be a totally β^*–continuous injection. If Y is T_0, then (X,τ) is β^*–T_2.

Proof. Let $x, y \in X$ with $x \neq y$. Since f is injection, $f(x) \neq f(y)$. Since Y is T_0, there exists an open subset V of Y containing $f(x)$ but not $f(y)$, or containing $f(y)$ but not $f(x)$. Thus for the first case we have, $x \in f^{-1}(V)$ and $y \notin f^{-1}(V)$. Since f is totally β^*–continuous and V is an open subset of Y, $f^{-1}(V)$ and $X \setminus f^{-1}(V)$ are disjoint β^*–clopen subsets of X containing x and y, respectively. The second case is proved in the same way. Thus X is β^*–T_2.

7. SLIGHTLY β^* – CONTINUOUS FUNCTIONS

In this section, the notion of slightly β^*–continuous function is introduced and characterizations and some relationships of slightly β^*–continuous functions and basic properties of slightly β^*–continuous functions are investigated and obtained.

Definition 7.1. Let (X, τ) and (Y, μ) be two topological spaces. Then a function $f:(X,\tau)\to(Y,\mu)$ is called a slightly β^*–continuous function at a point $x \in X$ if for each clopen subset V in Y containing $f(x)$, there exists a β^*–open subset U in X containing x such that $f(U) \subseteq V$. The function f is said to be slightly β^*–continuous if it has this property at each point of X.

Theorem 7.2. Let (X, τ) and (Y, μ) be two topological spaces. The following statements are equivalent for a function $f:(X,\tau)\to(Y,\mu)$:

(1) f is slightly β^*–continuous;

(2) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is β^*–open;

(3) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is β^*–closed;

(4) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is β^*–clopen;

Proof. $(1)\Rightarrow(2)$: Let V be a clopen subset of Y and let $x \in f^{-1}(V)$. Since f is slightly β^*–continuous, by (1), there exists a β^*–open set U_x in X containing x such that $f(U_x) \subseteq V$; hence $U_x \subseteq f^{-1}(V)$. We obtain that $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$. Thus $f^{-1}(V)$ is β^*–open.

$(2)\Rightarrow(3)$: Let V be a clopen subset of Y. Then $Y \setminus V$ is clopen. By (2), $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is Y–open. Thus $f^{-1}(V)$ is Y–closed.

$(3)\Rightarrow(4)$: It can be shown easily.

$(4)\Rightarrow(1)$: Let $x \in X$ and V be a clopen subset of Z with $f(x) \in V$. Let $U = f^{-1}(V)$. By assumption U is β^*–clopen and so β^*–open. Also $x \in U$ and $f(U) \subseteq V$.

Theorem 7.3. Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be topological spaces. Let $f:(X_1,\tau_1)\to(X_2,\tau_2)$ and

$g:(X_2,\tau_2)\to(X_3,\tau_3)$ be functions. Then, the following properties hold:

(1) If f is β^*–irresolute and g is slightly β^*–continuous, then gof is slightly β^*–continuous.

(2) If f is slightly β^*–continuous and g is continuous, then gof is slightly β^*–continuous.

Proof. (1) Let V be any clopen set in Y. Since g is slightly β^*–continuous, $g^{-1}(V)$ is β^*–open. Since f is β^*–irresolute, $f^{-1}\left[g^{-1}(V)\right]=(gof)^{-1}(V)$ is β^*–open. Therefore, gof is slightly β^*–continuous.

(2) Let V be any clopen set in Y. By the continuity of g, $g^{-1}(V)$ is clopen. Since f is slightly β^*–continuous, so $f^{-1}\left[g^{-1}(V)\right]=(gof)^{-1}(V)$ is β^*–open. Therefore, gof is slightly β^*–continuous.

Corrolary 7.4. Let (X_1,τ_1), (X_2,τ_2) and (X_3,τ_3) be topological spaces. If $f:(X_1,\tau_1)\to(X_2,\tau_2)$ is a β^*–irresolute function and $g:(X_2,\tau_2)\to(X_3,\tau_3)$ is a β^*–continuous function. then gof is slightly β^*–continuous.

Theorem 7.5. Let (X_1,τ_1), (X_2,τ_2) and (X_3,τ_3) be topological spaces. Let $f:(X_1,\tau_1)\to(X_2,\tau_2)$ be a β^*–irresolute, β^*–open surjection and $g:(X_2,\tau_2)\to(X_3,\tau_3)$ be a function. Then g is slightly β^*–continuous if and only if gof is slightly β^*–continuous.

Proof. $\Rightarrow$: Let g be slightly β^*–continuous. Then by Theorem 7.3, gof is slightly β^*–continuous.

$\Leftarrow$: Let gof be slightly β^*–continuous and V be clopen set in Y. Then $(gof)^{-1}(V)$ is β^*–open. Since f is a β^*–open surjection, then $f\left[(gof)^{-1}(V)\right]=g^{-1}(V)$ is β^*–open in Y. This shows that g is slightly β^*–continuous.

Theorem 7.6. Let (X,τ) and (Y,μ) be two topological spaces. Suppose that a function $f:(X,\tau)\to(Z,\mu)$ is a slightly β^*–continuous function and (X,τ) is β^*–connected. Then Y is connected.

Proof. Suppose that Y is a disconnected space. Then there exist non-empty disjoint open sets U and V such that $Y=U\cup V$. Therefore, U and V are clopen sets in Y. Since f is slightly β^*–continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are β^*–open in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint and $X=f^{-1}(U)\cup f^{-1}(V)$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Therefore, X is not β^*–connected. This is a contradiction and hence Y is connected.

Corrolary 7.7. The inverse image of a disconnected space under a slightly β^*–continuous surjection is β^*–disconnected.

Definition 7.8. A topological space (X,τ) is said to be

(1) locally indiscrete if every open set of X is closed in X,

(2) $0-$dimentional if its topology has a base consisting of clopen sets.

Theorem 7.9. Let (X,τ) be a topological space. If $f:(X,\tau)\to(Y,\mu)$ is a slightly β^*–continuous function and Y is locally indiscrete, then f is β^*–continuous.

Proof. Let V be any open set of Y. Since Y is locally indiscrete, V is clopen and hence $f^{-1}(V)$ is β^*–open in X. Therefore, f is β^*–continuous.

Theorem 7.10. Let (X,τ) be a topological space. If $f:(X,\tau)\to(Y,\mu)$ is a slightly

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$\beta*-$continuous function and Y is $0-$dimentional, then f is $\beta*-$continuous.

Proof. Let $x \in X$ and $V \subseteq Y$ be any open set containing $f(x)$. Since Y is $0-$dimentional, there exists a clopen set U containing $f(x)$ such that $U \subseteq V$. But f is slightly $\beta*-$continuous, then there exists a $\beta*-$open set G of X containing x such that $f(x) \in f(G) \subseteq U \subseteq V$. Hence f is $\beta*-$continuous.

Theorem 7.11. Let (X,τ) be a topological space. Let $f:(X,\tau) \to (Y,\mu)$ be a slightly $\beta*-$continuous injection and Y is $0-$dimentional. If Y is T_1, (resp. T_2), then X is $\beta*-T_1$, ($resp.\ \beta*-T_2$).

Proof. We prove only the second statement, the proof of the first being analogous. Let Y be T_2. Since f is injective, for any pair of distinct points $x,y \in X$, $f(x) \neq f(y)$. Since Y is T_2, there exist open sets V_1, V_2 in Y such that $f(x) \in V_1$, $f(y) \in V_2$ and $V_1 \cap V_2 = \phi$. Since Y is $0-$dimentional, there exist clopen sets U_1, U_2 in Y such that $f(x) \in U_1 \subseteq V_1$ and $f(y) \in U_2 \subseteq V_2$. Consequently $x \in f^{-1}(U_1) \subseteq f^{-1}(V_1)$, $y \in f^{-1}(U_2) \subseteq f^{-1}(V_2)$ and $f^{-1}(U_1) \cap f^{-1}(U_2) = \phi$. Since f is slightly $\beta*-$continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\beta*-$open sets and this implies that X is $\beta*-T_2$.

Definition 7.12. A topological space (X,τ) is said to be:

(1) clopen T_1 if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

(2) clopen T_2 (clopen Hausdorff or ultra-Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 7.13. Let (X,τ) be a topological space. Let $f:(X,\tau) \to (Y,\mu)$ be a slightly $\beta*-$continuous injection and (Y,μ) be clopen T_1, then X is $\beta*-T_1$.

Proof. Suppose that Y is clopen T_1. For any distinct points x and y in X, there exist clopen sets V and W such that $f(x) \in V$, $f(y) \notin V$ and $f(y) \in W$, $f(x) \notin W$. Since f is slightly $\beta*-$continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\beta*-$open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$ and $y \in f^{-1}(W)$, $x \notin f^{-1}(W)$. This shows that X is $\beta*-T_1$.

Theorem 7.14. Let (X,τ) be a topological space. Let $f:(X,\tau) \to (Y,\mu)$ be a slightly $\beta*-$continuous injection and Y is clopen T_2, then X is $\beta*-T_2$.

Proof. For any pair of distinct points x and y in X, there exist disjoint clopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is slightly $\beta*-$continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\beta*-$open subsets of X containing x and y, respectively. Therefore $f^{-1}(U) \cap f^{-1}(V) = \phi$ because $U \cap V = \phi$. This shows that X is $\beta*-T_2$.

Definition 7.15. A topological space (X,τ) is said to be mildly compact (resp. mildly Lindelof) if every clopen cover of X has a finite (resp. countable) sub cover.

Definition 7.16. A topological space (X,τ) is called $\beta*-$compact (resp. $\beta*-$Lindelof) if every $\beta*-$open cover of X has a finite (resp. countable) sub cover.

Theorem 7.17. Let (X,τ) be a topological space. Let $f:(X,\tau) \to (Y,\mu)$ be a slightly $\beta*-$continuous surjection, then the following statements hold:

(1) if (X,τ) is β^*–compact, then Y is mildly compact.

(2) if (X,τ) is β^*–Lindelof, then Y is mildly Lindelof.

Proof. We prove (1), the proof of (2) being entirely analogous. Let $\{V_\alpha : \alpha \in \Delta\}$ be a clopen cover of Y. Since f is slightly β^*–continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a β^*–open cover of X. Since X is β^*–compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Thus we have $Y = \cup\{V_\alpha : \alpha \in \Delta_0\}$ which means that Y is mildly compact.

Definition 7.18. A topological space (X,τ) is called β^*–closed compact (resp. β^*–closed Lindelof) if every cover of X by β^*–closed sets has a finite (resp. countable) sub cover.

Theorem 7.19. Let (X,τ) be a topological space. Let $f:(X,\tau)\to(Y,\mu)$ be a slightly β^*–continuous surjection. Then the following statements hold:

(1) if (X,τ) is β^*–closed compact, then Y is mildly compact.

(2) if (X,τ) is β^*–closed Lindelof, then Y is mildly Lindelof.

Proof. It can be obtained similarly as Theorem 7.17.

8. ALMOST β^* – CONTINUOUS FUNCTIONS

Definition 8.1. Let (X,τ) be a topological space. Then a subset A of X is said to be regular open (respectively, regular closed) if $A = Int\lfloor Cl(A)\rfloor$ $(resp.\ A = Cl[Int(A)])$.

Let $x \in X$. Then by $O(X,x)$ we denote the set of all open sets that contains x. Furthermore, by $\beta^* - O(X,x)$ $(resp.\ RO(X,x))$, we denote the set of all β^*–open (resp. regular open) sets that contain x.

Definition 8.2. A map $f:(X,\tau)\to(Y,\mu)$ is called almost continuous at $x \in X$ if for each $V \in RO(Y, f(x))$. there exists $U \in O(X,x)$ such that $f(U) \subseteq V$. If f is almost continuous at every point of X, then it is called almost continuous.

Equivalently, A map $f:(X,\tau)\to(Y,\mu)$ is called almost continuous if $f^{-1}(V)$ is open set in X for every regular open set V of Y.

Definition 8.3. A map $f:(X,\tau)\to(Y,\mu)$ is called almost β^*–continuous at $x \in X$ if for each $V \in RO(Y, f(x))$, there exists $U \in \beta^*–O(X,x)$ such that $f(U) \subseteq V$. If f is almost Y – continuous at every point of X, then it is called almost β^*–continuous.

Definition 8.4. Let (X,τ) be a topological space. Then a subset A of X is said to be δ – open if for each $x \in A$, there exists a regular open set U such that $x \in U \subseteq A$. The complement of a δ – open set is said to be δ – closed. The intersection of all δ – closed sets containing A is called the δ – closure of A and it is denoted by $\delta - Cl(A)$.

The next three results characterize almost β^*–continuous functions.

Definition 8.5. Let $f:(X,\tau)\to(Y,\mu)$ be a function. Then the following statements are equivalent:

(1) f is almost β^*–continuous.

(2) $f^{-1}(V)$ is β^*–closed in X, for every regular closed set V of Y.

(3) $f^{-1}[Cl(Int(V))]$ is β^*–closed in X, for every closed set V of Y.

(4) $f^{-1}[Int(Cl(V))]$ is β^*–open in X, for every open set V of Y.

Proof. $(1)\Rightarrow(2)$: Let V be regular closed set in Y. Then $Y \setminus V$ is regular open set in Y. Since f is almost β^*–continuous, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is β^*–open in X. Hence $f^{-1}(V)$ is β^*–closed in X.

$(2)\Rightarrow(3)$: Let V be closed set in Y. Then $V = Cl\lfloor Int(V)\rfloor$ is regular closed set in Y. Then by hypothesis, $f^{-1}\left[Cl(Int(V))\right]$ is β^*–closed in X.

$(3)\Rightarrow(4)$: Let V be open set in Y. Then $V = Int\lfloor Cl(V)\rfloor$ is regular open set in Y. Then $Y \setminus Int\lfloor Cl(V)\rfloor$ is regular closed set in Y. Then by hypothesis, $f^{-1}\left[Y \setminus Int(Cl(V))\right] = X \setminus f^{-1}\left[Int(Cl(V))\right]$ is β^*–closed in X. Hence $f^{-1}\left[Int(Cl(V))\right]$ is β^*–open in X.

$(4)\Rightarrow(1)$: Let V be regular open set in Y. Then $V = Int\lfloor Cl(V)\rfloor$ is regular open set and every regular open set is open set in Y. Then by hypothesis, $f^{-1}\left[Int(Cl(V))\right] = f^{-1}(V)$ is β^*–open in X. Hence f is almost β^*–continuous.

Theorem 8.6. Let $f:(X,\tau)\to(Y,\mu)$ be a map, then the following statements are equivalent:

(a) f is almost β^*–continuous.

(b) for each $x\in X$ and each open set V containing $f(x)$, there exists β^*–open set U containing x such that $f(U)\subseteq Int\lfloor Cl(V)\rfloor$;

(c) $f^{-1}(F)$ is β^*–closed in X for every regular closed set F in Y;

(d) $f^{-1}(V)$ is β^*–open in X for every regular open set V in Y.

Proof. The proof is obvious and thus omitted

Theorem 8.7. Let $f:(X,\tau)\to(Y,\mu)$ be a function, then the following statements are equivalent:

(a) f is almost β^*–continuous;

(b) $f\lfloor \beta^*\text{–}Cl(A)\rfloor \subseteq \delta\text{–}Cl\lfloor f(A)\rfloor$ for every subset A of X;

(c) $\beta^*\text{–}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[\delta\text{–}Cl(B)\right]$ for every subset B of Y;

(d) $f^{-1}(F)$ is β^*–closed in X for every δ–closed set F of Y;

(e) $f^{-1}(V)$ is β^*–open in X for every δ–open set V of Y.

Proof. $(a)\Rightarrow(b)$: Let A be a subset of X. Since $\delta\text{–}Cl\lfloor f(A)\rfloor$ is δ–closed in Y, it may be denoted by $\bigcap\{F_\alpha : \alpha\in\Delta\}$, where each F_α is regular closed set in Y such that $f(A)\subset F_\alpha$. The set $f^{-1}(F_\alpha)$ is β^*–closed (Theorem 8.5) and contains A. We also have $f^{-1}\left[\delta\text{–}Cl(f(A))\right] = \bigcap\left\{f^{-1}(F_\alpha):\alpha\in\Delta\right\}$. Now we note that $A\subseteq f^{-1}(F_\alpha)$, for each $\alpha\in\Delta$. Since each $f^{-1}(F_\alpha)$ is β^*–closed. Thus $\beta^*\text{–}Cl(A)\subseteq f^{-1}(F_\alpha)$, for each $\alpha\in\Delta$. So $\beta^*\text{–}Cl(A)\subseteq \bigcap\left\{f^{-1}(F_\alpha):\alpha\in\Delta\right\} = f^{-1}\left[\delta\text{–}Cl(f(A))\right]$. Therefore we obtain $f\lfloor\beta^*\text{–}Cl(A)\rfloor\subseteq\delta\text{–}Cl\lfloor f(A)\rfloor$.

$(b)\Rightarrow(c)$: Let B be a subset of Y. We have $f\left[\beta^*\text{–}Cl(f^{-1}(B))\right]\subseteq\delta\text{–}Cl\left[f(f^{-1}(B))\right]\subseteq\delta\text{–}Cl(B)$ and hence $\beta^*\text{–}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[\delta\text{–}Cl(B)\right]$.

$(c)\Rightarrow(d)$: Let F be any δ–closed set of Y. We have $\beta^*\text{–}Cl\left[f^{-1}(F)\right]\subseteq f^{-1}\left[\delta\text{–}Cl(F)\right] = f^{-1}(F)$ and hence $f^{-1}(F)$ is β^*–closed. in X.

$(d)\Rightarrow(e)$: Let V be any δ–open set of Y. Then $Y\setminus V$ is δ–closed. We have $f^{-1}(Y\setminus V) = X\setminus f^{-1}(V)$ is β^*–closed in X. Hence $f^{-1}(V)$ is β^*–open in X.

$(e)\Rightarrow(a)$: Let V be any regular open set in Y. Since V is δ–open in Y, we have $f^{-1}(V)$ is

β^*–open in X and hence by Theorem 8.6, f is almost β^*–continuous.

The following equivalent definition of almost β^*–continuity follows immediately from Theorem 8.6.

Definition 8.8. A map $f:(X,\tau)\to(Y,\mu)$ is called almost β^*–continuous if $f^{-1}(V)$ is β^*–open set in X for every regular open set V of Y.

Theorem 8.9. Every β^*–continuous function is almost β^*–continuous function.

Proof. Let $f:(X,\tau)\to(Y,\mu)$ be a β^*–continuous function. Let V be a regular open set in Y. Then V is open set in Y, since every regular open set is open set. Since f is β^*–continuous function, $f^{-1}(V)$ is β^*–open in X. Therefore f is almost β^*–continuous function.

Theorem 8.10. Every β^*–irresolute function is almost β^*–continuous function.

Proof. Let $f:(X,\tau)\to(Y,\mu)$ be a β^*–continuous function. Let V be regular open set in Y. Then V is β^*–open set in Y, since every regular open set is open set and every open set is β^*–open set. Since f is β^*–irresolute function, then $f^{-1}(V)$ is β^*–open in X. Therefore f is almost β^*–continuous function.

Theorem 8.11. Every almost continuous function is almost β^*–continuous function.

Proof. Let $f:(X,\tau)\to(Y,\mu)$ be an almost continuous function. Let V be regular open set in Y. Since f is almost continuous function, then $f^{-1}(V)$ is open in (X,τ), implies $f^{-1}(V)$ is β^*–open in (X,τ). Therefore f is almost β^*–continuous function.

In fact, we have the following implications:

continuity $\Rightarrow$ β^**–continuity* $\Rightarrow$ *almost* β^**–continuity*

Theorem 8.12. If $f:(X_1,\tau_1)\to(X_2,\tau_2)$ is β^*–irresolute and $g:(X_2,\tau_2)\to(X_3,\tau_3)$ is almost β^*–continuous, then $gof:(X_1,\tau_1)\to(X_3,\tau_3)$ is almost β^*–continuous.

Proof. Let V be regular open set in X_3. Since g is almost β^*–continuous, then $g^{-1}(V)$ is β^*–open set in X_2. Since f is β^*–irresolute, then $f^{-1}\left[g^{-1}(V)\right]$ is β^*–open in X_1. Hence gof is almost β^*–continuous.

Theorem 8.13. Let $f:(X,\tau)\to(Y,\mu)$ be a function and $g:(X,\tau)\to(X\times Y,\tau\times\mu)$ be the graph function defined by $g(x)=(x,f(x))$ for every $x\in X$. If g is almost β^*–continuous, then f is almost Υ–continuous.

Proof. Let $x\in X$ and $V\in RO(Y,f(x))$. Then $g(x)=(x,f(x))\in X\times V$. Observe that $X\times V\in RO(X\times Y,\tau\times\mu)$. If g is almost β^*–continuous, then there exists $U\in\beta^*\text{–}O(X,x)$ such that $g(U)\subseteq X\times V$. It follows that $f(U)\subseteq V$, hence f is almost β^*–continuous.

9. CONTRA – β^* – CONTINUOUS FUNCTIONS

Definition 9.1. A function $f:(X,\tau)\to(Y,\sigma)$ is called contra – β^*–continuous if $f^{-1}(V)$ is β^*–closed in X for every open set V of Y.

Definition 9.2. Let (X,τ) be topological space and $A\subseteq X$. Then the intersection of all open sets of X containing A is called kernel of A and is denoted by $\mathrm{Ker}(A)$.

Lemma 9.3. The following properties hold for subsets A and B of a topological space (X,τ).

(a) $x\in Ker(A)$ if and only if $A\cap F\neq\phi$ for any closed set F of X containing x.

(b) $\mathrm{A} \subseteq \mathrm{Ker}(\mathrm{A})$ and $\mathrm{A} = \mathrm{Ker}(\mathrm{A})$ if A is open in X.

(c) If $\mathrm{A} \subseteq \mathrm{B}$, then $\mathrm{Ker}(\mathrm{A}) \subseteq \mathrm{Ker}(\mathrm{B})$.

Lemma 9.4. The following properties hold for a subset A of a topological space (X, τ):

(i) $\beta^*\text{-}Int(A) = X \mathbf{B} \left[\beta^*\text{-}Cl(X \mathbf{B} A)\right]$;

(ii) $x \in \beta^*\text{-}Cl(A)$ if and only if $\mathrm{A} \cap \mathrm{U} \neq \phi$ for each $x \in \beta^*\text{-}O(X, x)$;

(iii) A is β^*–open if and only if $\mathrm{A} = \beta^*\text{-}\mathrm{Int}(\mathrm{A})$;

(iv) A is β^*–closed if and only if $\mathrm{A} = \beta^*\text{-}\mathrm{Cl}(\mathrm{A})$.

Theorem 9.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following conditions are equivalent:

(a) f is contra – β^*–continuous;

(b) for each $x \in X$ and each closed subset F of Y containing $f(x)$, there exists $U \in \beta^*\text{-}O(X, x)$ such that $f(U) \subseteq F$;

(c) for each closed subset F of Y, $f^{-1}(F)$ is β^*–open in X;

(d) $f\left[\beta^*\text{-}Cl(A)\right] \subseteq Ker\left[f(A)\right]$ for every subset A of X;

(e) $\beta^*\text{-}Cl\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Ker(B)\right]$ for every subset B of Y.

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and F be any closed set of Y containing $f(x)$. Using (a), we have $f^{-1}(Y \mathbf{B} F) = X \mathbf{B} f^{-1}(F)$ is β^*–closed in X and so $f^{-1}(F)$ is β^*–open in X. Taking $U = f^{-1}(F)$, we get $x \in U$ and $f(U) \subseteq \mathrm{F}$.

$(b) \Rightarrow (c)$: Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists a β^*–open subset U_x containing x such that $f(U_x) \subseteq \mathrm{F}$. Therefore, we obtain $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$, which is β^*–open in X.

$(c) \Rightarrow (a)$: Let V be any open set of Y. Then since $(Y \mathbf{B} V)$ is closed in Y, by (c) $f^{-1}(Y \mathbf{B} V) = \mathrm{X}\mathbf{B}\ f^{-1}(V)$ is β^*–open in X. Therefore, $f^{-1}(V)$ is β^*–closed in X.

$(c) \Rightarrow (d)$: Let A be any subset of X. Suppose that $y \notin Ker\left[f(A)\right]$. Then by Lemma 9.3, there exists a closed set F of Y containing y such that $f(A) \cap F = \phi$. This implies that $A \cap f^{-1}(F) = \phi$ and so $\beta^*\text{-}Cl(A) \cap f^{-1}(F) = \phi$. Therefore, we obtain $f\left[\beta^*\text{-}Cl(A)\right] \cap F = \phi$ and $y \notin f\left[\beta^*\text{-}Cl(A)\right]$. Hence, $f\left[\beta^*\text{-}Cl(A)\right] \subseteq Ker\left[f(A)\right]$.

$(d) \Rightarrow (e)$: Let B be any subset of Y. Using (d) and Lemma 3.3 we have $f\left[\beta^*\text{-}Cl\left(f^{-1}(B)\right)\right] \subseteq Ker\left[f\left(f^{-1}(B)\right)\right] \subseteq Ker(B)$. Thus it follows that $\beta^*\text{-}Cl\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Ker(B)\right]$.

$(e) \Rightarrow (a)$: Let V be an open subset of Y. Then from Lemma 9.3 and (e) we have $\beta^*\text{-}Cl\left[f^{-1}(V)\right] \subseteq f^{-1}\left[Ker(V)\right] = f^{-1}(V)$ and hence $\beta^*\text{-}Cl\left[f^{-1}(V)\right] = f^{-1}(V)$. This shows that $f^{-1}(V)$ is β^**–closed* in X.

The following lemma can be verified easily.

Lemma 9.6. A function $f : (X, \tau) \to (Y, \sigma)$ is β^**–continuous* if and only if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists $U \in \beta^*\text{-}O(X, x)$ such that $f(U) \subseteq V$.

Theorem 9.7. Suppose that a function (2) is *contra* – β^**–continuous* and Y is regular. Then f is β^**–continuous.*

Proof. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set G in Y containing $f(x)$ such that $Cl(G) \subseteq V$. Again, since f is *contra* – β^**–continuous*, so by Theorem 9.5, there exists $U \in \beta^*\text{-}O(X, x)$ such that $f(U) \subseteq Cl(G)$. Then $f(U) \subseteq Cl(G) \subseteq V$. Hence f is $\beta^* -$*continuous.*

Definition 9.8. A function $f:(X,\tau)\to(Y,\sigma)$ is called $almost-\beta^*-continuous$ if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq\beta^*-Int[Cl(V)]$.

$Almost-\beta^*-continuous$ function can be equivalently defined as in the following proposition.

Proposition 9.9. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:

(a) f is $almost-\beta^*-continuous$.

(b) For each $x\in X$ and each regular open set V of Y containing $f(x)$, there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq V$.

(c) $f^{-1}(V)$ is β^*-open in X for every regular open set V of Y.

Definition 9.10. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $pre-\beta^*-open$ if image of each β^*-open set of X is a β^*-open set of Y.

Definition 9.11. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $\beta^*-irresolute$ if preimage of a β^*-open subset of Y is a β^*-open subset of X.

Theorem 9.12. Suppose that a function $f:(X,\tau)\to(Y,\sigma)$ is $pre-\beta^*-open$ and $contra-\beta^*-continuous$. Then f is $almost-\beta^*-continuous$.

Proof. Let $x\in X$ and V be an open set containing $f(x)$. Since f is $contra-\beta^*-continuous$, then by Theorem 9.5, there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq Cl(V)$. Again, since f is $pre-\beta^*-open$, $f(U)$ is β^*-open in Y. Therefore, $f(U)=\beta^*-Int\lfloor f(U)\rfloor$ and hence $f(U)\subseteq\beta^*-Int[Cl(f(U))]\subseteq\beta^*-Int[Cl(V)]$. So f is $almost-\beta^*-continuous$.

Theorem 9.13. Let $\{(X_\lambda,\tau_\lambda):\lambda\in\Lambda\}$ be any family of topological spaces. If a function $f:X\longrightarrow\prod_{\lambda\in\Lambda}X_\lambda$ is $contra-\beta^*-continuous$, then $\pi_\lambda of:X\longrightarrow X_\lambda$ is $contra-\beta^*-continuous$, for each $\lambda\in\Lambda$, where π_λ is the projection of $\prod_{\lambda\in\Lambda}X_\lambda$ onto X_λ.

Proof. For a fixed $\lambda\in\Lambda$, let V_λ be any open subset of X_λ. Since π_λ is continuous, $\pi_\lambda^{-1}(V_\lambda)$ is open in $\prod_{\lambda\in\Lambda}X_\lambda$. Since f is $contra-\beta^*-continuous$,

$f^{-1}[\pi_\lambda^{-1}(V_\lambda)]=(\pi_\lambda of)^{-1}(V_\lambda)$ is β^*-closed in X. Therefore, $\pi_\lambda of$ is $contra-\beta^*-continuous$ for each $\lambda\in\Lambda$,

Definition 9.14. Let (X,τ) be a topological space. Then the $\beta^*-frontier$ of a subset A of X, denoted by $\beta^*-Fr(A)$, is defined as $\beta^*-Fr(A)=\lfloor\beta^*-Cl(A)\rfloor\cap\lfloor\beta^*-Cl(X\setminus A)\rfloor$ $=[\beta^*-Cl(A)]\setminus[\beta^*-Int(A)]$.

Theorem 9.15. The set of all points x of X at which $f:(X,\tau)\to(Y,\sigma)$ is not $contra-\beta^*-continuous$ is identical with the union of $\beta^*-frontier$ of the inverse images of closed sets of Y containing $f(x)$.

Proof. ***Necessity:*** Let f be not $contra-\beta^*-continuous$ at a point $x\in X$. Then by Theorem 9.5, there exists a closed set F of Y containing $f(x)$ such that $f(U)\cap(Y\setminus F)\neq\phi$ for every $U\in\beta^*-O(X,x)$, which implies that $U\cap f^{-1}(Y\setminus F)\neq\phi$. Therefore, $x\in\beta^*-Cl[f^{-1}(Y\setminus F)]=\beta^*-Cl[X\setminus f^{-1}(F)]$. Again, since $x\in f^{-1}(F)$, we get $x\in\beta^*-Cl[f^{-1}(F)]$ and so it follows that $x\in\beta^*-Fr[f^{-1}(F)]$.

Sufficiency: Suppose that $x\in(\beta^*-Fr[f^{-1}(F)])$ for some closed set F of Y containing $f(x)$ and f is $contra-\beta^*-continuous$ at x. Then there exists $U \in \beta^* - O(X, x)$ such that $f(U) \subseteq F$. Therefore $x \in U \subseteq f^{-1}(F)$ and hence it follows that $x \in \beta^* - Int[f^{-1}(F)] \subseteq X$ **B** $(\beta^* - Fr[f^{-1}(F)])$. But this is a contradiction. So f is not $contra - \beta^* - continuous$ at x.

Definition 9.16. A function $f:(X, \tau) \to (Y, \sigma)$ is called almost $weakly - \beta^* - continuous$ if, for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists $U \in \beta^* - O(X, x)$ such that $f(U) \subseteq Cl(V)$.

Theorem 9.17. Suppose that a function $f:(X, \tau) \to (Y, \sigma)$ is $contra - \beta^* - continuous$. Then f is almost $weakly - \beta^* - continuous$.

Proof. For any open set V of Y, $Cl(V)$ is closed in Y. Since f is $contra - \beta^* - continuous$, $f^{-1}[Cl(V)]$ is $\beta^* - open$ set in X. We take $U = f^{-1}[Cl(V)]$, then $f(U) \subseteq Cl(V)$. Hence f is almost $weakly - \beta^* - continuous$,

Theorem 9.18. Let $f:(X, \tau) \to (Y, \sigma)$ and $g:(Y, \sigma) \to (Z, \mu)$ be any two functions. Then the following properties hold:

(i) If f is $contra - \beta^* - continuous$ function and g is a continuous function, then gof is $contra - \beta^* - continuous$.

(ii) If f is $\beta^* - irresolute$ and g is $contra - \beta^* - continuous$, then gof is $contra - \beta^* - continuous$.

Proof. (i) For $x \in X$, let W be any closed set of Z containing $(gof)(x)$. Since g is continuous, $V = g^{-1}(W)$ is closed in Y. Also, since f is $contra - \beta^* - continuous$, there exists $U \in \beta^* - O(X, x)$ such that $f(U) \subseteq V$. Therefore $(gof)(U) \subseteq g\lfloor f(U) \rfloor \subseteq g(V) \subseteq W$ and so it implies that $(gof)(U) \subseteq W$. Hence, gof is $contra - \beta^* - continuous$.

(ii) For x ∈ X, let W be any closed set of Z containing $(gof)(x)$. Since g is $contra - \beta^* - continuous$, there exists $V \in \beta^* - O(Y, f(x))$ such that $g(V) \subseteq W$. Again, since f is $\beta^* - irresolute$, there exists $U \in \beta^* - O(X, x)$ such that $f(U) \subseteq V$. This shows that $(gof)(U) \subseteq W$. Hence, gof is $contra - \beta^* - continuous$.

Theorem 9.19. Let $f:(X, \tau) \to (Y, \sigma)$ be surjective $\beta^* - irresolute$ and $pre - \beta^* - open$ function and $g:(Y, \sigma) \to (Z, \eta)$ be any function. Then $gof:(X, \tau) \to (Z, \eta)$ is $contra - \beta^* - continuous$ if and only if g is $contra - \beta^* - continuous$.

Proof. The "if" part is easy to prove. To prove "only if" part, let $gof:(X, \tau) \to (Z, \eta)$ be $contra - \beta^* - continuous$ and let F be a closed subset of Z. Then $(gof)^{-1}(F)$ is a $\beta^* - open$ subset of X i.e. $f^{-1}[g^{-1}(F)]$ is $\beta^* - open$ in X. Since f is $pre - \beta^* - open$, $f[f^{-1}(g^{-1}(F))]$ is a $\beta^* - open$ subset of Y and so $g^{-1}(F)$ is $\beta^* - open$ in Y. Hence, g is $contra - \beta^* - continuous$.

Definition 9.20. A topological space (X, τ) is said to be $\beta^* - normal$ if each pair of $non - empty$ disjoint closed sets can be separated by disjoint $\beta^* - open$ sets.

Definition 9.21. A topological space (X, τ) is said to be ultranormal if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 9.22. Suppose that $f:(X, \tau) \to (Y, \sigma)$ is a $contra - \beta^* - continuous$, closed injection and Y is ultranormal. Then X is $\beta^* - normal$.

Proof. Let A and B be disjoint closed subsets of X. Since f is closed injection, $f(A)$ and $f(B)$ are disjoint closed subsets of Y. Again, since Y is ultranormal, $f(A)$ and $f(B)$ are separated by disjoint clopen sets P and Q (say) respectively. Therefore, $f(A)\subseteq P$ and $f(B)\subseteq Q$ i.e., $A\subseteq f^{-1}(P)$ and $B\subseteq f^{-1}(Q)$, where $f^{-1}(P)$ and $f^{-1}(Q)$ are disjoint β^*-open sets of X (since f is $contra-\beta^*-continuous$). This shows that X is $\beta^*-normal$.

Definition 9.23. A topological space (X,τ) is called $\beta^*-connected$ provided that X is not the union of two disjoint nonempty β^*-open sets of X.

Theorem 9.24. Suppose that $f:(X,\tau)\to(Y,\sigma)$ is $contra-\beta^*-continuous$ surjection, where X is $\beta^*-connected$ and Y is any topological space, then Y is not a discrete space.

Proof. If possible, suppose that Y is a discrete space. Let P be a proper nonempty open and closed subset of Y. Then $f^{-1}(P)$ is a proper nonempty β^*-open and $\beta^*-closed$ subset of X, which contradicts to the fact that X is $\beta^*-connected$ Hence the theorem follows.

Theorem 9.25. Suppose that $f:(X,\tau)\to(Y,\sigma)$ is $contra-\beta^*-continuous$ surjection and X is $\beta^*-connected$. Then Y is connected.

Proof. If possible, suppose that Y is not connected. Then there exist nonempty disjoint open sets P and Q such that $Y=P\bigcup Q$. So P and Q are clopen sets of Y. Since f is $contra-\beta^*-continuous$ function, $f^{-1}(P)$ and $f^{-1}(Q)$ are β^*-open sets of X. Also $f^{-1}(P)$ and $f^{-1}(Q)$ are nonempty disjoint β^*-open sets of X and $X=f^{-1}(P)\bigcup f^{-1}(Q)$, which contradicts to the fact that X is $\beta^*-connected$. Hence Y is connected.

Theorem 9.26. A topological space (X,τ) is $\beta^*-connected$ if and only if every $contra-\beta^*-continuous$ function from X into any $T_1-space$ (Y,σ) is constant.

Proof. Let X be $\beta^*-connected$. Now, since Y is a $T_1-space$, $\Omega=\{f^{-1}(y):y\in Y\}$ is disjoint β^*-open partition of X. If $|\Omega|\geq 2$ (where $|\Omega|$ denotes the cardinality of Ω), then X is the union of two nonempty disjoint β^*-open sets. Since X is $\beta^*-connected$, we get $|\Omega|=1$. Hence, f is constant.

Conversely, suppose that X is not $\beta^*-connected$ and every $contra-\beta^*-continuous$ function from X into any $T_1-space$ Y is constant. Since X is not $\beta^*-connected$, there exists a non-empty proper β^*-open as well as $\beta^*-closed$ set V (say) in X. We consider the space $Y=\{0,1\}$ with the discrete topology σ. The function $f:(X,\tau)\to(Y,\sigma)$ defined by $f(V)=\{0\}$ and $f(X \mathbf{B} V)=\{1\}$ is obviously $contra-\beta^*-continuous$ and which is $non-constant$. This leads to a contradiction. Hence X is $\beta^*-connected$.

Definition 9.27. A topological space (X,τ) is said to be β^*-T_2 if for each pair of distinct points x, y in X there exist $U\in\beta^*-O(X,x)$ and $V\in\beta^*-O(X,y)$ such that $U\bigcap V=\phi$.

Theorem 9.28. Let (X,τ) and (Y,σ) be two topological spaces and suppose that for each pair of distinct points x and y in X there exists a function $f:(X,\tau)\to(Y,\sigma)$ such that $f(x)\neq f(y)$ where Y is an Urysohn space and f is $contra-\beta^*-continuous$ function at x and y. Then X is β^*-T_2.

Proof. Let $x, y \in X$ and $x \neq y$. Then by assumption, there exists a function $f:(X,\tau)\to(Y,\sigma)$, such that $f(x)\neq f(y)$ where Y is Urysohn and f is $contra-\beta^*-continuous$ at x and y. Now, since Y is Urysohn, there exist open sets U and V of Y containing $f(x)$ and $f(y)$ respectively, such that $Cl(U)\cap Cl(V)=\phi$. Also, f being $contra-\beta^*-continuous$ at x and y there exist β^*-open sets P and Q containing x and y respectively such that $f(P)\subseteq Cl(U)$ and $f(Q)\subseteq Cl(V)$. Then $f(P)\cap f(Q)=\phi$ and so $P\cap Q=\phi$. Therefore, X is β^*-T_2.

Corrolary 9.29. If $f:(X,\tau)\to(Y,\sigma)$ is $contra-\beta^*-continuous$ injection where Y is an Urysohn space, then X is β^*-T_2.

Corollary 9.30. If f is $contra-\beta^*-continuous$ injection of a topological space (X,τ) into an ultra Hausdorff space (Y,σ), then X is β^*-T_2.

Proof. Let $x, y \in X$ where $x\neq y$. Then, since f is an injection and Y is ultra Hausdorff, $f(x)\neq f(y)$ and there exist disjoint closed sets U and V containing $f(x)$ and $f(y)$ respectively. Again, since f is $contra-\beta^*-continuous$, $f^{-1}(U)\in\beta^*-O(X,x)$ and $f^{-1}(V)\in\beta^*-O(X,y)$ with $f^{-1}(U)\cap f^{-1}(V)=\phi$. This shows that X is β^*-T_2.

10. ALMOST CONTRA – β^* – CONTINUOUS FUNCTIONS

Definition 10.1. A function $f:(X,\tau)\to(Y,\sigma)$ is called almost $contra-\beta^*-continuous$ if $f^{-1}(V)$ is $\beta^*-closed$ for every regular open set V of Y.

Theorem 10.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:

(a) f is almost $contra-\beta^*-continuous$;

(b) $f^{-1}(F)$ is β^*-open in X for every regular closed set F of Y;

(c) for each $x\in X$ and each regular open set F of Y containing $f(x)$, there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq F$.

(d) for each $x\in X$ and each regular open set V of Y non-containing $f(x)$, there exists a $\beta^*-closed$ set K of X non-containing x such that $f^{-1}(V)\subseteq K$.

Proof. $(a)\Leftrightarrow(b)$: Let F be any regular closed set of Y. Then $(Y\mathbf{B}F)$ is regular open and therefore $f^{-1}(Y\mathbf{B}F)=X\mathbf{B}f^{-1}(F)\in\beta^*-C(X)$. Hence, $f^{-1}(F)\in\beta^*-O(X)$. The converse part is obvious.

$(b)\Rightarrow(c)$: Let F be any regular closed set of Y containing $f(x)$. Then $f^{-1}(F)\in\beta^*-O(X)$ and $x\in f^{-1}(F)$. Taking $U=f^{-1}(F)$ we get $f(U)\subseteq F$.

$(c)\Rightarrow(b)$: Let F be any regular closed set of Y and $x\in f^{-1}(F)$. Then, there exists $U_x\in\beta^*-O(X,x)$ such that $f(U_x)\subseteq F$ and so $U_x\subseteq f^{-1}(F)$. Also, we have $f^{-1}(F)=\bigcup_{x\in f^{-1}(F)}U_x$. Hence $f^{-1}(F)\in\beta^*-O(X)$.

$(c)\Rightarrow(d)$: Let *V* be any regular open set of *Y* non-containing *f*(*x*). Then $(Y\mathbf{B}V)$ is regular closed set of *Y* containing *f*(*x*). Hence by (c), there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq(Y\mathbf{B}V)$. Hence, we obtain $U\subseteq f^{-1}(Y\mathbf{B}V)\subseteq X\mathbf{B}f^{-1}(V)$ and so $f^{-1}(V)\subseteq(X\mathbf{B}U)$. Now, since $U\in\beta^*-O(X)$, $(X\mathbf{B}U)$ is $\beta^*-closed$ set of X not containing *x*. The converse part is obvious.

Theorem 10.3. Let $f:(X,\tau)\to(Y,\sigma)$ be almost $contra-\beta^*-continuous$. Then f is almost $weakly-\beta^*-continuous$.

Proof. For $x\in X$, let H be any open set of Y containing $f(x)$. Then $Cl(H)$ is a regular closed set of Y containing $f(x)$. Then by Theorem 10.2, there exists $G\in\beta^*-O(X,x)$ such that $f(G)\subseteq Cl(H)$. So f is almost $weakly-\beta^*-continuous$.

Theorem 10.4. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $contra-\beta^*-continuous$ injection and Y is weakly Hausdorff. Then X is β^*-T_1.

Proof. Since Y is weakly Hausdorff, for distinct points x, y of Y, there exist regular closed sets U and V such that $f(x)\in U$, $f(y)\notin U$ and $f(y)\in V$, $f(x)\notin V$. Now, f being almost $contra-\beta^*-continuous$, $f^{-1}(U)$ and $f^{-1}(V)$ are β^*-open subsets of X such that $x\in f^{-1}(U)$, $y\notin f^{-1}(U)$ and $y\in f^{-1}(V)$, $x\notin f^{-1}(V)$. This shows that X is β^*-T_1.

Corollary 10.5. If $f:(X,\tau)\to(Y,\sigma)$ is a $contra-\beta^*-continuous$ injection and Y is weakly Hausdorff, then X is β^*-T_1.

Theorem 10.6. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $contra-\beta^*-continuous$ surjection and X be $\beta^*-connected$. Then Y is connected.

Proof. If possible, suppose that Y is not connected. Then there exist disjoint non-empty open sets U and V of Y such that $Y=U\cup V$. Since U and V are clopen sets in Y, they are regular open sets of Y. Again, since f is almost $contra-\beta^*-continuous$ surjection, $f^{-1}(U)$ and $f^{-1}(V)$ are β^*-open sets of X and $X=f^{-1}(U)\cup f^{-1}(V)$. This shows that X is not $\beta^*-connected$. But this is a contradiction. Hence Y is connected.

Definition 10.7. A topological space (X,τ) is said to be $\beta^*-compact$ if every β^*-open cover of X has a finite subcover.

Definition 10.8. A topological space (X,τ) is said to be countably $\beta^*-compact$ if every countable cover of X by β^*-open sets has a finite subcover.

Definition 10.9. A topological space (X,τ) is said to be $\beta^*-Lindelof$ if every β^*-open cover of X has a countable subcover.

Theorem 10.10. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $contra-\beta^*-continuous$ surjection. Then the following statements hold:

(a) If X is $\beta^*-compact$, then Y is $S-closed$.

(b) If X is $\beta^*-Lindelof$, then Y is $S-Lindelof$.

(c) If X is $countably\ \beta^*-compact$, then Y is $countably\ S-closed$.

Proof. (a): Let $\{V_\alpha:\alpha\in I\}$ be any regular closed cover of Y. Since f is almost $contra-\beta^*-continuous$ then $\{f^{-1}(V_\alpha):\alpha\in I\}$ is a β^*-open cover of X. Again, since X is $\beta^*-compact$, there exist a finite subset I_0 of I such that $X=\cup\{f^{-1}(V_\alpha):\alpha\in I_0\}$ and hence $Y=\{V_\alpha:\alpha\in I_0\}$. Therefore, Y is $S-closed$.

The proofs of (b) and (c) are being similar to (a): omitted.

Definition 10.11. A topological space (X,τ) is said to be $\beta^*-closed$ compact if every $\beta^*-closed$ cover of X has a finite subcover.

Definition 10.12. A topological space (X,τ) is said to be countably $\beta^*-closed$ if every countable cover of X by $\beta^*-closed$ sets has a finite subcover.

Definition 10.12. A topological space (X,τ) is said to be $\beta^*-closed\ Lindelof$ if every $\beta^*-closed$ cover of X has a countable subcover.

Theorem 10.14. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $contra-\beta^*-continuous$ surjection. Then the following statements hold:

(a) If X is $\beta^* - closed$ compact, then Y is nearly compact.

(b) If X is $\beta^* - closed\ Lindelof$, then Y is *nearly Lindelof*.

(c) If X is *countably* $\beta^* - closed\ compact$, then Y is *nearly countable compact*.

Proof. (a): Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost $contra - \beta^* - continuous$, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a $\beta^* - closed$ cover of X. Again, since X is $\beta^* - closed\ compact$, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ and hence $Y = \{V_\alpha : \alpha \in I_0\}$. Therefore, Y is *nearly compact*.

The proofs of (b) and (c) are being similar to (a): omitted.

11. CLOSED GRAPHS VIA β^* – OPEN SETS

Definition 11.1. Let $f:(X,\tau) \to (Y,\sigma)$ be a function. Then the graph $G(f) = \{(x, f(x)) : x \in X\}$ of f is said to be $\beta^* - closed$ (resp. $contra - \beta^* - closed$) if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist a $U \in \beta^* - O(X, x)$ and an open set (resp. a closed set) V in Y containing y such that $(U \times V) \cap G(f) = \phi$.

Lemma 11.2. A graph $G(f)$ of a function $f:(X,\tau) \to (Y,\sigma)$ is $\beta^* - closed$ (resp. $contra$ $\beta^* - closed$) in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \beta^* - O(X, x)$ and an open set (resp. a closed set) V in Y containing y such that $f(U) \cap V = \phi$.

Proof. We shall prove that $f(U) \cap V = \phi \Leftrightarrow (U \times V) \cap G(f) = \phi$. Let $(U \times V) \cap G(f) \neq \phi$. Then there exists $(x, y) \in (U \times V)$ and $(x, y) \in G(f)$. This implies that $x \in U$, $y \in V$ and $y = f(x) \in V$. Therefore, $f(U) \cap V \neq \phi$. Hence the result follows.

Theorem 11.3. Suppose that $f:(X,\tau) \to (Y,\sigma)$ is *contra* $\beta^* - continuous$ and Y is Urysohn. Then $G(f)$ is *contra* $\beta^* - closed$ in $X \times Y$.

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since Y is Urysohn, there exist open sets V and W in Y such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \phi$. Now, since f is contra $\beta^* - continuous$, there exists a $U \in \beta^* - O(X, x)$ such that $f(U) \subseteq Cl(V)$ which implies that $f(U) \cap Cl(W) = \phi$. Hence by Lemma 11.2, $G(f)$ is contra $\beta^* - closed$ in $X \times Y$.

Theorem 11.4. Let $f:(X,\tau) \to (Y,\sigma)$ be a function and $g: X \to X \times Y$ be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $\beta^* - continuous$, then f is contra $\beta^* - continuous$.

Proof. Let G be an open set in Y, then $X \times G$ is an open set in $X \times Y$. Since g is contra $\beta^* - continuous$, it implies that $f^{-1}(G) = g^{-1}(X \times G)$ is a $\beta^* - closed$ set of X. Therefore, f is contra $\beta^* - continuous$.

Theorem 11.5. Let $f:(X,\tau) \to (Y,\sigma)$ have a contra $\beta^* - closed$ graph. If f is injective, then X is $\beta^* - T_1$.

Proof. Let x_1 and x_2 be any two distinct points of X. Then, we have $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Then, there exists a $\beta^* - open$ set U in X containing x_1 and $F \in C(Y, f(x_2))$ such that $f(U) \cap F = \phi$. Hence $U \cap f^{-1}(F) = \phi$. Therefore, we have $x_2 \notin U$. This implies that X is $\beta^* - T_1$.

Definition 11.6. The graph $G(f)$ of a function $f:(X,\tau)\to(Y,\sigma)$ is said to be strongly contra $\beta^*-closed$ if for each $(x,y)\in(X\times Y)\setminus G(f)$, there exist $U\in\beta^*-O(X,x)$ and regular closed set V in Y containing y such that $(U\times V)\cap G(f)=\phi$.

Lemma 11.7. The graph $G(f)$ of a function $f:(X,\tau)\to(Y,\sigma)$ is strongly contra $\beta^*-closed$ in $X\times Y$ if and only if for each $(x,y)\in(X\times Y)\setminus G(f)$, there exist $U\in\beta^*-O(X,x)$ and regular closed set V in Y containing y such that $f(U)\cap V=\phi$.

Theorem 11.8. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $weakly-\beta^*-continuous$ and Y is Urysohn. Then $G(f)$ is strongly contra $\beta^*-closed$ in $X\times Y$.

Proof. Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$ and since Y is Urysohn, there exist open sets G,H in Y such that $f(x)\in G$, $y\in H$ and $Cl(G)\cap Cl(H)=\phi$. Now, since f is almost $weakly-\beta^*-continuous$, there exists $U\in\beta^*-O(X,x)$ such that $f(U)\subseteq Cl(G)$. This implies that $f(U)\cap Cl(H)=f(U)\cap Cl[Int(H)]=\phi$, where $Cl\lfloor Int(H)\rfloor$ is regular closed in Y. Hence by above Lemma 11.7, $G(f)$ is strongly contra $\beta^*-closed$ in $X\times Y$.

Theorem 11.9. Let $f:(X,\tau)\to(Y,\sigma)$ be an almost $\beta^*-continuous$ and Y is T_2. Then $G(f)$ is strongly contra $\beta^*-closed$ in $X\times Y$.

Proof. Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$ and since Y is T_2, there exist open sets G and H containing y and $f(x)$, respectively, such that $G\cap H=\phi$; which is equivalent to $Cl(G)\cap Int[Cl(H)]=\phi$. Again, since f is almost $\beta^*-continuous$ and $Int\lfloor Cl(H)\rfloor$ is regular open, so there exists $W\in\beta^*-O(X,x)$ such that $f(W)\subseteq Int[Cl(H)]$. This implies that $f(W)\cap Cl(G)=\phi$ and by Lemma 11.7, $G(f)$ is strongly contra $\beta^*-closed$ in $X\times Y$.

Definition 11.10. A filter base $\Im$ on a topological space (X,τ) is said to be $\beta^*-convergent$ to a point x in X if for any $U\in\beta^*-O(X,\tau)$ containing x, there exists an $F\in\Im$ such that $F\subseteq U$.

Theorem 5.11. Prove that every function $\psi:(X,\tau)\to(Y,\sigma)$, where (Y,σ) is compact with $\beta^*-closed$ graph is $\beta^*-continuous$.

Proof. Let ψ be not $\beta^*-continuous$ at $x\in X$. Then there exists an open set S in Y containing $\psi(x)$ such that $\psi(T)\not\subset S$ for every $T\in\beta^*-O(X,x)$. It is obvious to verify that $\wp=\{T\subseteq X:T\in\beta^*-O(X,x)\}$ is a filterbase on X that $\beta^*-converges$ to x. Now we shall show that $\Upsilon_\wp=\{\psi(T)\cap(Y\setminus S):T\in\beta^*-O(X,x)\}$ is a filterbase on Y. Here for every $T\in\beta^*-O(X,x)$, $\psi(T)\not\subset S$, i.e. $\psi(T)\cap(Y\setminus S)\neq\phi$. So $\phi\notin\Upsilon_\wp$. Let $G,H\in\Upsilon_\wp$. Then there are $T_1,T_2\in\wp$ such that $G=\psi(T_1)\cap(Y\setminus S)$ and $H=\psi(T_2)\cap(Y\setminus S)$. Since $\wp$ is a filterbase, there exists a $T_3\in\wp$ such that $T_3\subseteq T_1\cap T_2$ and so $W=\psi(T_3)\cap(Y\setminus S)\in\Upsilon_\wp$ with $W\subseteq G\cap H$. It is clear that $G\in\Upsilon_\wp$ and $G\subseteq H$ imply $H\in\Upsilon_\wp$. Hence $\Upsilon_\wp$ is a filterbase on Y. Since $Y\setminus S$ is closed in compact space Y, S is itself compact. So, $\Upsilon_\wp$ must adheres at some point $y\in Y\setminus S$. Here $y\neq\psi(x)$ ensures that $(x,y)\notin G(\psi)$. Thus Lemma 11.2 gives us a $U\in\beta^*-O(X,x)$ and an open set V in Y containing y such that $\psi(U)\cap V=\phi$, *i.e.*

$\left[\psi(U)\cap(Y \setminus S)\right]\cap V=\phi.$ But this is a contradiction.

Theorem 11.12. Suppose that an open surjection $\psi:(X,\tau)\to(Y,\sigma)$ possesses a $\beta^*-closed$ graph. Then Y is T_2.

Proof. Let $p_1, p_2 \in Y$ with $p_1 \neq p_2$. Since ψ is a surjection, there exists an $x_1 \in X$ such that $\psi(x_1)=p_1$ and $\psi(x_1)\neq p_2$. Therefore $(x_1, p_2)\notin G(\psi)$ and so by Lemma 11.2, there exist $U_1 \in \beta^*-O(X,x_1)$ and open set V_1 in Y containing p_2 such that $\psi(U_1)\cap V_1=\phi$. Since ψ is β^*-open, $\psi(U_1)$ and V_1 are disjoint open sets containing p_1 and p_2 respectively. So Y is T_2.

Corollary 11.13. If a function $\psi:(X,\tau)\to(Y,\sigma)$ is a surjection and possesses a $\beta^*-closed$ graph, then Y is T_1.

12. CONCLUSIONS

The author introduced β^*-continuous, β^*-open, β^*-closed, β^*-irresolute, totally β^*-continuous, slightly β^*-continuous, almost β^*-continuous, contra$-\beta^*-$continuous and almost contra$-\beta^*-$continuous functions in topological space and investigated several properties and characterizations of these functions. He also presented closed graphs via β^*-open sets.

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