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Delta – Open Sets And Delta – Continuous Functions

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Abstract—In 1968 Velicko [30] introduced the concepts of $\delta-closure$ and $\delta-interior$ operations. We introduce and study properties of $\delta-derived$, $\delta-border$, $\delta-frontier$ and $\delta-exterior$ of a set using the concept of $\delta-open$ sets. We also introduce some new classes of topological spaces in terms of the concept of $\delta-D-sets$ and investigate some of their fundamental properties. Moreover, we investigate and study some further properties of the well-known notions of $\delta-closure$ and $\delta-interior$ of a set in a topological space. We also introduce $\delta-R_0$ space and study its characteristics. We also introduce $\delta-R_0$ space and study its characteristics. We introduce $\delta-irresolute$, $\delta-closed$, $pre-\delta-open$ and $pre-\delta-closed$ mappings and investigate properties and characterizations of these new types of mappings and also explore further properties of the well-known notions of $\delta-continuous$ and $\delta-open$ mappings.

Keywords—Pure Mathematics, Topology

I. INTRODUCTION AND PRELIMINARIES

Velicko $[30]$ introduced the notion of $\delta-closure,$ and $\delta-interior$ operations. Throughout this paper, (X,τ) (simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., interior) of S will be denoted by $Cl(S)$, $(resp., Int(S))$. A subset S of X is called a semi-open set $[21]$ if $S \subseteq Cl[Int(S)]$. The complement of a semi-open set is called a semi-closed set. The intersection of all semi-closed sets containing A is called the semi-closure of A and is denoted by $sCl(A)$. The family of all semi-open sets in a topological space (X,τ) will be denoted by $SO(X,\tau)$. A subset $M(X)$ of a space X called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [19] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous function. A point $x \in X$ is called the $\delta-cluster$ point of $A \subseteq X$ if $A \cap Int[Cl(U)] \neq \phi$ for every open set U of X containing x. The set of all $\delta-cluster$ points of A is called the $\delta-closure$ of A, denoted by $Cl_\delta(A)$. A subset $A \subseteq X$ is called $\delta-closed$ if $A = Cl_\delta(A)$. The complement of a $\delta-closed$ set is called $\delta-open$. The collection of all $\delta-open$ sets in a topological space (X,τ) forms a topology τ_δ on X, called the semi-generalization topology of τ, weaker than τ and the class of all regular open sets in τ forms an open basis for (X,τ_δ). In this paper, we introduce and study properties of $\delta-derived$, $\delta-border$, $\delta-frontier$ and $\delta-exterior$ of a set using the concept of $\delta-open$ and study also other properties of the well-known notions of $\delta-closure$ and $\delta-interior$. The notion of $\theta-open$ subsets, $\theta-closed$ subsets and $\theta-closure$ were introduced by Velicko [30] for the purpose of studying the important class of

$H-closed$ spaces in terms of arbitrary filterbases. A point $x \in X$ is called a $\theta-adherent$ point of A [7], if $A \cap Cl(V) \neq \phi$ for every open set V containing x. The set of all $\theta-adherent$ points of A is called the $\theta-closure$ of A and is denoted by $Cl_\theta(A)$. A subset A of X is called $\theta-closed$ if $A = Cl_\theta(A)$. Dontchev and Maki $[[7]$, Lemma 3.9$]$ have shown that if A and B are subsets of a space (X,τ), then $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$ and $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$. Note also that the $\theta-closure$ of a given set need not be a $\theta-closed$ set. But it is always closed. The complement of a $\theta-closed$ set is called a $\theta-open$ set. The $\theta-\text{int}erior$ of set A in X, written $Int_\theta(A)$, consists of those points x of A such that for some open set U containing x, $Cl(U) \subseteq A$. A set A is $\theta-open$ if and only if $A = Int_\theta(A)$, or equivalently, $X-A$ is $\theta-closed$. The collection of all $\theta-open$ sets in a topological space (X,τ) forms a topology τ_θ on X, weaker than τ. We observe that for any topological space (X,τ), the relation $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ always holds. We also have $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A)$, for any subset A of X.

2. BASIC PROPERTIES AND APPLICATIONS OF DELTA OPEN SETS

Definition 1.1. Let A be a subset of a topological space (X,τ). A point $x \in A$ is said to be a $\delta-limit$ point of A if for each $\delta-open$ set U containing x, $U \cap (A-\{x\}) \neq \phi$. The set of all $\delta-limit$ points of A is called the $\delta-derived$ set of A and is denoted by $D_\delta(A)$.

Theorem 1.2. For subsets A, B of a space X, the following statements hold:

(1) $D(A) \subseteq D_\delta(A)$, where $D(A)$ is the derived set of A;

(2) If $A \subseteq B$, then $D_\delta(A) \subseteq D_\delta(B)$;

(3) $D_\delta(A) \cup D_\delta(B) = D_\delta(A \cup B)$ and $D_\delta(A \cap B) \subseteq D_\delta(A) \cap D_\delta(B)$;

(4) $[D_\delta(D_\delta(A)) - A] \subseteq D_\delta(A)$;

(5) $D_\delta[A \cup D_\delta(A)] \subseteq [A \cup D_\delta(A)]$.

Proof. (1) It suffices to observe that every $\delta-open$ set is an open set.

(2) Obvious.

(3) $D_\delta(A) \cup D_\delta(B) = D_\delta(A \cup B)$ is a modification of the standard proof for D, where open sets are replaced by $\delta-open$ sets. $D_\delta(A \cap B) \subseteq D_\delta(A) \cap D_\delta(B)$ follows by (2).

(4) If $x \in [D_\delta(D_\delta(A)) - A]$ and U is a $\delta-open$ set containing x, then $U \cap [D_\delta(A) - \{x\}] \neq \phi$. Let $y \in U \cap [D_\delta(A) - \{x\}]$. Then, since $y \in D_\delta(A)$ and $y \in U$, so $U \cap [A - \{y\}] \neq \phi$. Let $z \in U \cap [A - \{y\}]$. Then, $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap [A - \{x\}] \neq \phi$. Therefore, $x \in D_\delta(A)$.

(5) Let $x \in D_\delta[A \cup D_\delta(A)]$. If $x \in A$, the result is obvious. So, let $x \in [D_\delta(A \cup D_\delta(A)) - A]$, then, for $\delta-open$ set U containing x, $U \cap [A \cup D_\delta(A) - \{x\}] \neq \phi$. Thus, $U \cap [A - \{x\}] \neq \phi$ or $U \cap [D_\delta(A) - \{x\}] \neq \phi$. Now, it follows similarly from (4) that $U \cap [A - \{x\}] \neq \phi$. Hence, $x \in D_\delta(A)$.

Therefore, in any case, $D_\delta[A \cup D_\delta(A)] \subseteq [A \cup D_\delta(A)]$.

Theorem 1.3. For any subset A of a space X, $Cl_\delta(A) = A \cup D_\delta(A)$.

Proof. Since $D_\delta(A) \subseteq Cl_\delta(A)$, $A \cup D_\delta(A) \subseteq Cl_\delta(A)$. On the other hand, let $x \in Cl_\delta(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each $\delta-open$ set U containing x intersects A at a point distinct from x; so $x \in D_\delta(A)$. Thus, $Cl_\delta(A) \subseteq [A \cup D_\delta(A)]$, which completes the proof.

Corollary 1.4. A subset A is $\delta-closed$ if and only if it contains the set of its $\delta-limit$ it points.

Definition 1.5. A point $x \in X$ is said to be a $\delta-interior$ point of A if there exists a $\delta-open$ set U containing x such that $U \subseteq A$. The set of all $\delta-interior$ points of A is said to be $\delta-interior$ of A and is denoted by $Int_\delta(A)$.

Theorem 1.6. For subsets A, B of a space X, the following statements are true:

(1) $Int_\delta(A)$ is the largest $\delta-open$ set contained in A;

(2) A **is** $\delta-open$ if and only if $A = Int_\delta(A)$;

(3) $Int_\delta[Int_\delta(A)] = Int_\delta(A)$;

(4) $Int_\delta(A) = [A - D_\delta(X-A)]$;

(5) $[X - Int_\delta(A)] = Cl_\delta(X-A)$;

(6) $[X - Cl_\delta(A)] = Int_\delta(X-A)$;

(7) $A \subseteq B$, then $Int_\delta(A) \subseteq Int_\delta(B)$.

(8) $Int_\delta(A) \cup Int_\delta(B) \subseteq Int_\delta(A \cup B)$;

(9) $Int_\delta(A \cap B) = Int_\delta(A) \cap Int_\delta(B)$;

Proof. (4) If $x \in [A - D_\delta(X-A)]$, then $x \notin D_\delta(X-A)$ and so there exists a $\delta-open$ set U containing x such that $U \cap (X-A) = \phi$. Then, $x \in U \subseteq A$ and hence $x \in Int_\delta(A)$, that is, $[A - D_\delta(X-A)] \subseteq Int_\delta(A)$. On the other hand, if $x \in Int_\delta(A)$, then $x \notin D_\delta(X-A)$ since $Int_\delta(A)$ is $\delta-open$ and $[Int_\delta(A) \cap (X-A)] = \phi$. Hence, $Int_\delta(A) = [A - D_\delta(X-A)]$.

(5) $X - Int_\delta(A) = X - [A - D_\delta(X-A)] = (X-A) \cup D_\delta(X-A) = Cl_\delta(X-A)$

Definition 1.7. $Bd_\delta(A) = A - Int_\delta(A)$ is said to be the $\delta-border$ of A.

Theorem 1.8. For a subset A of a space X, the following statements hold:

(1) $Bd(A) \subseteq Bd_\delta(A)$ where $Bd(A)$ denotes the border of A;

(2) $A = Int_\delta(A) \cup Bd_\delta(A)$;

(3) $Int_\delta(A) \cap Bd_\delta(A) = \phi$;

(4) A is a $\delta-open$ set if and only if $Bd_\delta(A) = \phi$;

(5) $Bd_\delta[Int_\delta(A)] = \phi$;

(6) $Int_\delta[Bd_\delta(A)] = \phi$;

(7) $Bd_\delta[Bd_\delta(A)] = Bd_\delta(A)$;

(8) $Bd_\delta(A) = A \cap [Cl_\delta(X-A)]$;

(9) $Bd_\delta(A) = D_\delta(X-A)$.

Proof. (6) If $x \in Int_\delta[Bd_\delta(A)]$, then $x \in Bd_\delta(A)$. On the other hand, since $Bd_\delta(A) \subseteq A$, $x \in Int_\delta[Bd_\delta(A)] \subseteq Int_\delta(A)$. Hence, $x \in Int_\delta(A) \cap Bd_\delta(A)$, which contradicts (3). Thus, $Int_\delta[Bd_\delta(A)] = \phi$.

(8) $Bd_\delta(A) = A - Int_\delta(A) = A - [X - Cl_\delta(X - A)] = A \cap Cl_\delta(X - A)$.

(9) $Bd_\delta(A) = A - Int_\delta(A) = A - [A - D_\delta(X - A)] = D_\delta(X - A)$.

Definition 1.9. $Fr_\delta(A) = Cl_\delta(A) - Int_\delta(A)$ is said to be the $\delta-$*frontier* of A.

Theorem 1.10. For a subset A of a space X, the following statements hold:

(1) $Fr(A) \subseteq Fr_\delta(A)$ where $Fr(A)$ denotes the frontier of A;

(2) $Cl_\delta(A) = Int_\delta(A) \cup Fr_\delta(A)$;

(3) $Int_\delta(A) \cap Fr_\delta(A) = \phi$;

(4) $Bd_\delta(A) \subseteq Fr_\delta(A)$;

(5) $Fr_\delta(A) = Bd_\delta(A) \cup D_\delta(A)$;

(6) A is a $\delta-$*open* set if and only if $Fr_\delta(A) = D_\delta(A)$;

(7) $Fr_\delta(A) = Cl_\delta(A) \cap Cl_\delta(X - A)$;

(8) $Fr_\delta(A) = Fr_\delta(X - A)$;

(9) $Fr_\delta(A)$ is $\delta-$*closed*;

(10) $Fr_\delta[Fr_\delta(A)] \subseteq Fr_\delta(A)$;

(11) $Fr_\delta[Int_\delta(A)] \subseteq Fr_\delta(A)$;

(12) $Fr_\delta[Cl_\delta(A)] \subseteq Fr_\delta(A)$;

(13) $Int_\delta(A) = A - Fr_\delta(A)$.

Proof. (2) $Int_\delta(A) \cup Fr_\delta(A) = Int_\delta(A) \cup [Cl_\delta(A) - Int_\delta(A)] = Cl_\delta(A)$

(3) $Int_\delta(A) \cap Fr_\delta(A) = Int_\delta(A) \cap [Cl_\delta(A) - Int_\delta(A)] = \phi$.

(5) Since $Int_\delta(A) \cup Fr_\delta(A) = Int_\delta(A) \cup Bd_\delta(A) \cup D_\delta(A)$, $Fr_\delta(A) = Bd_\delta(A) \cup D_\delta(A)$

(7) $Fr_\delta(A) = Cl_\delta(A) - Int_\delta(A) = Cl_\delta(A) \cap Cl_\delta(X - A)$.

(9) $Cl_\delta[Fr_\delta(A)] = Cl_\delta[Cl_\delta(A) \cap Cl_\delta(X - A)] \subseteq Cl_\delta[Cl_\delta(A)] \cap Cl_\delta[Cl_\delta(X - A)]$

$= Cl_\delta(A) \cap Cl_\delta(X - A) = Fr_\delta(A)$. Hence $Fr_\delta(A)$ is $\delta-$*closed*.

(10) $Fr_\delta[Fr_\delta(A)] = Cl_\delta[Fr_\delta(A)] \cap Cl_\delta[X - Fr_\delta(A)] \subseteq Cl_\delta[Fr_\delta(A)] = Fr_\delta(A)$

(12) $Fr_\delta[Cl_\delta(A)] = Cl_\delta[Cl_\delta(A)] - Int_\delta[Cl_\delta(A)] = Cl_\delta(A) - Int_\delta[Cl_\delta(A)] \subseteq [Cl_\delta(A) - Int_\delta(A)] = Fr_\delta(A)$.

(13) $A - Fr_\delta(A) = A - [Cl_\delta(A) - Int_\delta(A)] = Int_\delta(A)$.

Remark 1.11. Let A and B be subsets of X. Then $A \subseteq B$ does not imply that either $Fr_\delta(B) \subseteq Fr_\delta(A)$ or $Fr_\delta(A) \subseteq Fr_\delta(B)$.

Definition 1.12. $Ext_\delta(A) = Int_\delta(X - A)$ is said to be a $\delta-$*exterior* of A.

Theorem 1.13. For a subset A of a space X, the following statements hold:

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(1) $Ext_{\delta}(A) \subseteq Ext(A)$ where $Ext(A)$ denotes the exterior of A;

(2) $Ext_{\delta}(A)$ is $\delta-opn$;

(3) $Ext_{\delta}(A) = Int_{\delta}(X-A) = X - Cl_{\delta}(A)$;

(4) $Ext_{\delta}[Ext_{\delta}(A)] = Int_{\delta}[Cl_{\delta}(A)]$;

(5) If $A \subseteq B$, then $Ext_{\delta}(B) \subseteq Ext_{\delta}(A)$;

(6) $Ext_{\delta}(A \cup B) = Ext_{\delta}(A) \cup Ext_{\delta}(B)$;

(7) $Ext_{\delta}(A) \cap Ext_{\delta}(B) \subseteq Ext_{\delta}(A \cap B)$;

(8) $Ext_{\delta}(X) = \phi$;

(9) $Ext_{\delta}(\phi) = X$;

(10) $Ext_{\delta}(A) = Ext_{\delta}[X - Ext_{\delta}(A)]$;

(11) $Int_{\delta}(A) \subseteq Ext_{\delta}[Ext_{\delta}(A)]$;

(12) $X = Int_{\delta}(A) \cup Ext_{\delta}(A) \cup Fr_{\delta}(A)$;

(13) $Ext_{\delta}(A) \cup Ext_{\delta}(B) \subseteq Ext_{\delta}(A \cap B)$.

Proof. (4) $Ext_{\delta}[Ext_{\delta}(A)] = Ext_{\delta}[X - Cl_{\delta}(A)] =$
$Int_{\delta}[X - (X - Cl_{\delta}(A))] = Int_{\delta}[Cl_{\delta}(A)]$

(10) $Ext_{\delta}[X - Ext_{\delta}(A)] = Ext_{\delta}[X - Int_{\delta}(X-A)]$
$= Int_{\delta}[X - (X - Int_{\delta}(X-A))]$
$= Int_{\delta}[Int_{\delta}(X-A)] = Int_{\delta}(X-A) = Ext_{\delta}(A)$.

(11) $Int_{\delta}(A) \subseteq Int_{\delta}[Cl_{\delta}(A)] = Int_{\delta}[X - Int_{\delta}(X-A)]$
$= Int_{\delta}[X - Ext_{\delta}(A)] = Ext_{\delta}[Ext_{\delta}(A)]$.

(13)
$Ext_{\delta}(A) \cup Ext_{\delta}(B) = Int_{\delta}(X-A) \cup Int_{\delta}(X-B)$
$\subseteq Int_{\delta}[(X-A) \cup (X-B)]$
$= Int_{\delta}[X - (A \cap B)] = Ext_{\delta}(A \cap B)$.

Definition 1.14. Let X be a topological space. A set $A \subseteq X$ is said to be $\delta-saturated$ if for every $x \in A$ it implies that $Cl_{\delta}(\{x\}) \subseteq A$. The class of all $\delta-saturated$ sets in X will be denoted by $B_{\delta}(X)$.

Theorem 1.15. Let X be a topological space. Then $B_{\delta}(X)$ is a complete Boolean set Algebra.

Proof. We will prove that all the unions and complements of elements of $B_{\delta}(X)$ are members of $B_{\delta}(X)$. Obviously, only the proof regarding the complements is not trivial. Let $A \in B_{\delta}(X)$ and suppose that $Cl_{\delta}(\{x\}) \not\subset (X-A)$ for some $x \in (X-A)$. Then there exists $y \in A$ such that $y \in Cl_{\delta}(\{x\})$. It follows that x and y have no disjoint neighborhoods. Then $x \in Cl_{\delta}(\{y\})$. But this is a contradiction, because by the definition of $B_{\delta}(X)$ we have $Cl_{\delta}(\{y\}) \subseteq A$. Hence, $Cl_{\delta}(\{x\}) \subseteq (X-A)$ for every $x \in (X-A)$, which implies $(X-A) \in B_{\delta}(X)$.

Corrolary 1.16. $B_{\delta}(X)$ contains every union and every intersection of $\delta-closed$ and $\delta-open$ sets in X.

Definition 1.17. A space X is said to be $\delta-$Hausdorff if for every $x \neq y \in X$, there exist $\delta-open$ sets U_x, V_y such that $x \in U_x$, $y \in V_v$ and $U_x \cap V_y = \phi$.

Theorem 1.18. Let (X, τ) be a topological space. Then the following statements are equivalent:

(1) X is $\delta - T_2$;

(2) Let $x \in X$. For each $y \neq x$, there exists a $\delta-open$ set U such that $x \in U$ and $y \notin Cl_{\delta}(U)$;

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(3) For each $x \in X$, $\bigcap \{Cl_\delta(U) | U \in \tau_\delta \text{ and } x \in U\} = \{x\}$;

(4) The diagonal $\Delta = \{(x,x) | x \in X\}$ is a $\delta-closed$ set in $X \times X$.

Proof. $(1) \Rightarrow (2)$: Let $x \in X$ and $y \neq x$. Then there are disjoint $\delta-open$ sets U and V such that $x \in U$ and $y \in V$. Clearly, V^c is $\delta-closed$, $Cl_\delta(U) \subseteq V^c$, $y \notin V^c$ and therefore $y \notin Cl_\delta(U)$.

$(2) \Rightarrow (3)$. If $y \neq x$, there exists a $\delta-open$ set U such that $x \in U$ and $y \notin Cl_\delta(U)$. So $y \notin \bigcap \{Cl_\delta(U) | U \in \tau_\delta \text{ and } x \in U\}$.

$(3) \Rightarrow (4)$. We prove that Δ^c is $\delta-open$. Let $(x,y) \notin \Delta$. Then $y \neq x$ and since $\bigcap \{Cl_\delta(U) | U \in \tau_\delta \text{ and } x \in U\} = \{x\}$ there is some $U \in \tau_\delta$ with $x \in U$ and $y \notin Cl_\delta(U)$. Since $U \cap [Cl_\delta(U)]^c = \phi$, $U \times [Cl_\delta(U)]^c$ is a $\delta-open$ set such that $(x,y) \in U \times [Cl_\delta(U)]^c \subseteq \Delta^c$.

$(4) \Rightarrow (1)$. If $y \neq x$, then $(x,y) \notin \Delta$ and thus there exist $\delta-open$ sets U and V such that $(x,y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the $\delta-open$ sets U and V we have: $x \in U$, $y \in V$ and $U \cap V = \phi$.

Definition 1.19. A subset A of a space X is said to be $\delta-compact$ if every cover of X by $\delta-open$ sets has a finite subcover.

It is well-known that every closed subset of a compact space is compact. The next theorem approximates this result for $\delta-compactness$.

Theorem 1.20. A $\delta-compact$ subset of a $\delta-$Hausdorff space is $\delta-closed$.

Proof. Let A be a $\delta-compact$ subset of a $\delta-$Hausdorff space X. We will show that $(X-A)$ is $\delta-open$. Let $x \in (X-A)$. Then for each $a \in A$ there exist $\delta-open$ sets $U_{x,a}$ and V_a such that $x \in U_{x,a}$ and $a \in V_a$ and $U_{x,a} \cap V_a = \phi$. The collection $\{V_a | a \in A\}$ is a $\delta-open$ cover of A. Therefore, there exists a finite subcollection $\{V_{a_k} | k = 1,2,3,...,n\}$ that covers A. Let $U_x = \bigcap \{U_{x,a_k} | k = 1,2,3,...,n\}$. Then $x \in U_x$, U_x is δ-open and $U_x \cap A = \phi$. This proves that A is $\delta-closed$.

Theorem 1.21. A $\delta-closed$ subset of a $\delta-$Hausdorff space is $\delta-compact$.

Proof. Let X be $\delta-compact$ and let A be a $\delta-closed$ subset of X. Let Γ be a $\delta-open$ cover of A. Then $\Gamma^* = \Gamma \cup \{X-A\}$ is a $\delta-open$ cover of X. Since X is $\delta-compact$, this collection Γ^* has a finite collection Λ^* that covers X. But then Γ ha a finite subcollection $\Lambda = \Lambda^* - \{X-A\}$ that covers A as we need.

Definition 1.22. Let A be a subset of a topological space X. Then $\delta-kernel$ of A, denoted by $Ker_\delta(A) = \bigcap \{O \in \tau_\delta | A \subseteq O\}$.

Definition 1.23. Let x be a point of a topological space X. Then $\delta-kernel$ of x, denoted by $Ker_\delta(\{x\})$ is defined to be the set $Ker_\delta(\{x\}) = \bigcap \{O \in \tau_\delta | x \in O\}$.

Lemma 1.24. Let (X, τ) be a topological space and $x \in X$. Then $Ker_\delta(A) = \{x \in X | Cl_\delta(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in Ker_\delta(A)$ and $Cl_\delta(\{x\}) \cap A = \phi$. Hence $x \notin [X - Ker_\delta(\{x\})]$ which is a $\delta-open$ set containing A. This is impossible, since $x \in Ker_\delta(A)$. Consequently, $Cl_\delta(\{x\}) \cap A \neq \phi$. Let $Cl_\delta(\{x\}) \cap A \neq \phi$ and $x \notin Ker_\delta(A)$. Then there exists a $\delta-open$ set D containing A and

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$x \notin D$. Let $y \in Cl_\delta(\{x\}) \cap A$. Hence, D is a $\delta-open$ neighborhood of y with $x \notin D$. By this contradiction, $x \in Ker_\delta(A)$ and the claim.

Definition 1.25. A topological space (X, τ) is said to be a $\delta-R_0$ space if every $\delta-open$ set contains the $\delta-closure$ of each of its singletons.

Lemma 1.26. Let (X, τ) be a topological space and $x \in X$. Then $y \in Ker_\delta(\{x\})$ if and only if $x \in Ker_\delta(\{y\})$.

Proof. Suppose that $y \notin Ker_\delta(\{x\})$. Then there exists a $\delta-open$ set V containing x such that $y \notin V$. Therefore we have $x \notin Cl_\delta(\{y\})$. The proof of the converse case can be done similarly.

Lemma 1.27. The following statements are equivalent for any points x and y in a topological space (X, τ):

(1) $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$;

(2) $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

Proof. $(1) \Rightarrow (2)$: Suppose that $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$. Then there exists a point z in X such that $z \in Ker_\delta(\{x\})$ and $z \notin Ker_\delta(\{y\})$. It follows from $z \in Ker_\delta(\{x\})$ that $\{x\} \cap Cl_\delta(\{x\}) \neq \phi$. This implies that $x \in Cl_\delta(\{z\})$. By $z \notin Ker_\delta(\{y\})$, we have $\{y\} \cap Cl_\delta(\{z\}) = \phi$. Since $x \in Cl_\delta(\{z\})$ and $Cl_\delta(\{x\}) \subseteq Cl_\delta(\{z\})$. Hence $\{y\} \cap Cl_\delta(\{x\}) = \phi$. Therefore, $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

$(2) \Rightarrow (1)$: Suppose that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Then there exists a point $z \in X$ such that $z \in Cl_\delta(\{x\})$ and $z \notin Cl_\delta(\{y\})$. Then, there exists a $\delta-open$ set containing z and therefore x but not y, i.e., $y \notin Ker_\delta(\{x\})$. Hence $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$.

Theorem 1.28. A topological space (X, τ) is a $\delta-R_0$ space if and only if for every x and y in X. $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$ implies $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$.

Proof. Necessity. Suppose that (X, τ) is $\delta-R_0$ and $x, y \in X$ such that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Then there exists $z \in Cl_\delta(\{x\})$ such that $z \notin Cl_\delta(\{y\})$ (or $z \in Cl_\delta(\{y\})$ such that $z \notin Cl_\delta(\{x\})$. There exists $V \in \tau_\delta$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin Cl_\delta(\{y\})$. Thus $x \in [X - Cl_\delta(\{y\})] \in \tau_\delta$, which implies $Cl_\delta(\{x\}) \subseteq [X - Cl_\delta(\{y\})]$ and $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. The proof for otherwise is similar.

Sufficiency. Let $V \in \tau_\delta$ and let $x \in V$. We will show that $Cl_\delta(\{x\}) \subseteq V$. Let $y \notin V$, i.e., $y \in (X - V)$. Then $x \neq y$ and $x \notin Cl_\delta(\{y\})$. This shows that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. By assumption, $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. Hence $y \notin Cl_\delta(\{x\})$ and therefore $Cl_\delta(\{x\}) \subseteq V$.

Theorem 1.29. A topological space (X, τ) is a $\delta-R_0$ space if and only if for any points x and y in X. $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ implies $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$.

Proof. Suppose that (X, τ) is a $\delta-R_0$ space. Thus by Lemma 1.27, for any points x and y in X if $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ then $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Now we prove that $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$. Assume that $z \in Ker_\delta(\{x\}) \cap Ker_\delta(\{y\})$. By $z \in Ker_\delta(\{x\})$ and

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Lemma 1.26, it follows that $x \in Ker_{\delta}(\{z\})$. Since $x \in Ker_{\delta}(\{x\})$, by Theorem 1.28, $Cl_{\delta}(\{x\}) = Cl_{\delta}(\{z\})$. Similarly, we have $Cl_{\delta}(\{y\}) = Cl_{\delta}(\{z\}) = Cl_{\delta}(\{x\})$. This is a contradiction. Therefore, we have $Ker_{\delta}(\{x\}) \cap Ker_{\delta}(\{y\}) = \phi$.

Conversely, let (X, τ) be a topological space such that for any points x and y in X, $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$ implies $Ker_{\delta}(\{x\}) \cap Ker_{\delta}(\{y\}) = \phi$. If $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(\{y\})$, then by Lemma 1.27, $Ker_{\delta}(\{x\}) \neq Ker_{\delta}(\{y\})$. Hence $Ker_{\delta}(\{x\}) \cap Ker_{\delta}(\{y\}) = \phi$ which implies $Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\}) = \phi$. Because $z \in Ker_{\delta}(\{x\})$ implies that $x \in Ker_{\delta}(\{z\})$. Therefore $Ker_{\delta}(\{x\}) \cap Ker_{\delta}(\{y\}) \neq \phi$. By hypothesis, we have $Ker_{\delta}(\{x\}) = Ker_{\delta}(\{z\})$. Then $z \in Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\})$ implies that $Ker_{\delta}(\{x\}) = Ker_{\delta}(\{z\}) = Ker_{\delta}(\{y\})$. This is a contradiction. Hence, $Cl_{\delta}(\{x\}) \cap Cl_{\delta}(\{y\}) = \phi$. By Theorem 1.28 (X, τ) is a $\delta - R_0$ space.

Theorem 1.30. For a topological space (X, τ), the following properties are equivalent:

(1) (X, τ) is a $\delta - R_0$ space;

(2) For any $A \neq \phi$ and $G \in \tau_{\delta}$ such that $A \cap G \neq \phi$, there exists $F \in C_{\delta}(X, \tau)$ such that $A \cap F \neq \phi$ and $F \subseteq G$;

(3) Any $G \in \tau_{\delta}$, $G = \cup\{F \in C_{\delta}(X, \tau) | F \subseteq G\}$;

(4) Any $F \in C_{\delta}(X, \tau_{\delta})$, $F = \cap\{G \in \tau_{\delta} | F \subseteq G\}$;

(5) For any $x \in X$, $Cl_{\delta}(\{x\}) \subseteq Ker_{\delta}(\{x\})$.

Proof. $(1) \Rightarrow (2)$: Let A be a nonempty subset of X and $G \in \tau_{\delta}$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \tau_{\delta}$, $Cl_{\delta}(\{x\}) \subseteq G$. Set $F = Cl_{\delta}(\{x\})$. Then F is a $\delta -$ *closed* subset of X such that $F \subseteq G$ and $A \cap F \neq \phi$.

$(2) \Rightarrow (3)$: Let $G \in \tau_{\delta}$. Then $\cup\{F \in C_{\delta}(X, \tau) | F \subseteq G\} \subseteq G$. Let x be any point of G. There exists $F \in C_{\delta}(X, \tau)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in C_{\delta}(X, \tau) | F \subseteq G\}$ and hence $G = \cup\{F \in C_{\delta}(X, \tau) | F \subseteq G\}$.

$(3) \Rightarrow (4)$: This is obvious.

$(4) \Rightarrow (5)$: Let x be any point of X and $y \notin Ker_{\delta}(\{x\})$. There exists $V \in \tau_{\delta}$ such that $x \in V$ and $y \notin V$; hence $Cl_{\delta}(\{x\}) \cap V = \phi$. By (4) $\left(\cap\{G \in \tau_{\delta} | Cl_{\delta}(\{y\}) \subseteq G\}\right) \cap V = \phi$. There exists $G \in \tau_{\delta}$ such that $x \notin G$ and $Cl_{\delta}(\{y\}) \subseteq G$. Therefore $Cl_{\delta}(\{x\}) \cap G = \phi$ and $y \notin Cl_{\delta}(\{x\})$. Consequently, we obtain $Cl_{\delta}(\{x\}) \subseteq Ker_{\delta}(\{x\})$.

$(5) \Rightarrow (1)$: Let $G \in \tau_{\delta}$ and $x \in G$. Suppose $y \in Ker_{\delta}(\{x\})$. Then $x \in Cl_{\delta}(\{y\})$ and $y \in G$. This implies that $Cl_{\delta}(\{x\}) \subseteq Ker_{\delta}(\{x\}) \subseteq G$. Therefore, (X, τ) is a $\delta - R_0$ space.

Corollary 1.31. For a topological space (X, τ), the following properties are equivalent:

(1) (X, τ) is a $\delta - R_0$ space;

(2) $Cl_{\delta}(\{x\}) = Ker_{\delta}(\{x\})$ for all $x \in X$.

Proof. $(1) \Rightarrow (2)$: Suppose that (X, τ) is a $\delta - R_0$ space. By Theorem 1.30, $Cl_{\delta}(\{x\}) \subseteq Ker_{\delta}(\{x\})$ for each $x \in X$. Let $y \in Ker_{\delta}(\{x\})$. Then $x \in Cl_{\delta}(\{y\})$ and so $Cl_{\delta}(\{x\}) = Cl_{\delta}(\{y\})$. Therefore, $y \in Cl_{\delta}(\{x\})$ and hence $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$. This shows that $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$.

$(2) \Rightarrow (1)$: This is obvious by Theorem 1.30.

Theorem 1.32. For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) $x \in Cl_\delta(\{y\})$ if and only if $y \in Cl_\delta(\{x\})$, for any points x and y in X.

Proof. $(1) \Rightarrow (2)$: Assume that X is $\delta - R_0$. Let $x \in Cl_\delta(\{y\})$ and D be any $\delta - open$ set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every $\delta - open$ set containing y contains x. Hence $y \in Cl_\delta(\{x\})$.

$(2) \Rightarrow (1)$: Let U be a $\delta - open$ set and $x \in U$. If $y \notin U$, then $x \notin Cl_\delta(\{y\})$ and hence $y \notin Cl_\delta(\{x\})$. This implies that $Cl_\delta(\{x\}) \subseteq U$. Hence (X,τ) is $\delta - R_0$.

Theorem 1.33. For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) If F is $\delta - closed$, then $F = Ker_\delta(F)$.

(3) If F is $\delta - closed$ and $x \in F$, then $Ker_\delta(F) \subseteq F$.

(4) If $x \in X$, then $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$.

Proof. $(1) \Rightarrow (2)$: Let F be a $\delta - closed$ set and $x \notin F$. Thus $(X - F)$ is a $\delta - open$ set containing x. Since (X,τ) is $\delta - R_0$. $Cl_\delta(\{x\}) \subseteq (X - F)$. Thus $Cl_\delta(\{x\}) \cap F = \phi$ and by Lemma 1.24 $x \notin Ker_\delta(F)$. Therefore $Ker_\delta(F) = F$.

$(2) \Rightarrow (3)$: In general, $A \subseteq B$ implies $Ker_\delta(A) \subseteq Cl_\delta(B)$. Therefore, it follows from (2) that $Ker_\delta(\{x\}) \subseteq Ker_\delta(F) = F$.

$(3) \Rightarrow (4)$: *Since* $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is $\delta - closed$, by (3), $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$.

$(4) \Rightarrow (1)$ We show the implication by using Theorem 3.19. Let $x \in Cl_\delta(\{y\})$. Then by Lemma 1.26, $y \in Ker_\delta(\{x\})$. Since $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is a $\delta - closed$ set, by (4) we obtain $y \in Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$. Therefore $x \in Cl_\delta(\{y\})$ implies $y \in Cl_\delta(\{x\})$. The converse is obvious and (X,τ) is $\delta - R_0$.

Theorem 1.34. Let (X,τ) be a topological space. Then $\bigcap \{Cl_\delta(\{x\}) | x \in X\} = \phi$ if and only if $Ker_\delta(\{x\}) \neq X$ for every $x \in X$.

Proof. Necessity. Suppose that $\bigcap \{Cl_\delta(\{x\}) | x \in X\} = \phi$. Assume that there is a point y in X such that $Ker_\delta(\{y\}) = X$. Then $y \notin O$, where O is some proper $\delta - open$ subset of X. This implies that $y \in \bigcap \{Cl_\delta(\{x\}) | x \in X\}$. But this is a contradiction.

Sufficiency. Assume that $Ker_\delta(\{x\}) \neq X$ for every $x \in X$. If there exists a point $y \in X$ such that $y \in \bigcap \{Cl_\delta(\{x\}) | x \in X\}$, then every $\delta - open$ set containing y must contain every point of X. This implies that the space X is the unique $\delta - open$ set containing y. Hence

$Ker_\delta(\{x\}) = X$ which is a contradiction. Therefore, $\bigcap\{Cl_\delta(\{x\}) | x \in X\} = \phi$.

Definition 1.35. A filter base F is called $\delta-$*convergent* to a point x in X, if for any $\delta-$*open* set U of X containing x, there exists B in F such that B is a subset of U.

Lemma 1.36. Let (X, τ) be a topological space and x and y be any two points in X such that every net in X $\delta-$*converging* to y $\delta-$*converges* to x. Then $x \in Cl_\delta(\{y\})$.

Proof. Suppose that $x_\alpha = y$ for $\alpha \in I$. Then $\{x_\alpha : \alpha \in I\}$ is a net in $Cl_\delta(\{y\})$. Since $\{x_\alpha : \alpha \in I\}$ $\delta-$*converges* to y, so $\{x_\alpha : \alpha \in I\}$ $\delta-$*converges* to x and this implies that $x \in Cl_\delta(\{y\})$.

Theorem 1.37. For a topological space (X, τ), the following statements are equivalent:

(1) (X, τ) is $\delta-R_0$ space;

(2) If $x, y \in X$, then $y \in Cl_\delta(\{x\})$ if and only if every net in X $\delta-$*converging* to y $\delta-$*converges* to x.

Proof. $(1) \Rightarrow (2)$: Let $x, y \in X$ such that $y \in Cl_\delta(\{x\})$. Suppose that $\{x_\alpha : \alpha \in I\}$ is a net in X such that this net $\delta-$*converges* to y. Since $y \in Cl_\delta(\{x\})$ so by Theorem 1.28 we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $x \in Cl_\delta(\{y\})$. This means that the net $\{x_\alpha : \alpha \in I\}$ $\delta-$*converges* to x.

Conversely, let $x, y \in X$ such that every net in X $\delta-$*converging* to y $\delta-$*converges* to x. Then $x \in Cl_\delta(\{y\})$ by Lemma 1.36. By Theorem 1.28, we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $y \in Cl_\delta(\{x\})$.

$(2) \Rightarrow (1)$: Assume that x and y are any two points of X such that $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) \neq \phi$. Let $z \in Cl_\delta(\{x\}) \cap Cl_\delta(\{y\})$. So there exists a net $\{x_\alpha : \alpha \in I\}$ in $Cl_\delta(\{x\})$ such that $\{x_\alpha : \alpha \in I\}$ $\delta-$*converges* to z. Since $z \in Cl_\delta(\{y\})$. So by hypothesis $\{x_\alpha : \alpha \in I\}$ $\delta-$*converges* to y. It follows that $y \in Cl_\delta(\{x\})$. Similarly we obtain $x \in Cl_\delta(\{y\})$. Therefore $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$ and by Theorem 1.28, (X, τ) is $\delta-R_0$.

2. CHARACTERIZATIONS OF MAPPINGS

The purpose of this part is to explore properties and characterizations of $\delta-$*continuous*, $\delta-$*irresolute*, $\delta-$*open*, $\delta-$*closed*, *pre*$-\delta-$*open*, and *pre*$-\delta-$*closed* functions.

2.1. DELTA – CONTINUOUS FUNCTIONS

The purpose of this section is to investigate properties and characterizations of $\delta-$continuous functions.

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\delta-$continuous if $f^{-1}(V) \in \tau_\delta$ for every $V \in \sigma$.

Theorem 2.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:

(1) f is $\delta-$continuous;

(2) The inverse image of each closed set in Y is a $\delta-$closed set in X;

(3) $Cl_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Cl(V)\right]$, for every $V\subseteq Y$;

(4) $f\left[Cl_\delta(U)\right]\subseteq Cl\left[f(U)\right]$, for every $U\subseteq X$;

(5) For any point $x\in X$ and any open set V of Y containing $f(x)$, there exists $U\in\tau_\delta$ such that $x\in U$ and $f(U)\subseteq V$;

(6) $Bd_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Bd(V)\right]$, for every $V\subseteq Y$;

(7) $f\left[D_\delta(U)\right]\subseteq Cl\left[f(U)\right]$, for every $U\subseteq X$;

(8) $f^{-1}\left[Int(V)\right]\subseteq Int_\delta\left[f^{-1}(V)\right]$, for every $V\subseteq Y$;

Proof. $(1)\Rightarrow(2)$: Let $F\subseteq Y$ be closed. Since f is $\delta-$continuous, $f^{-1}(Y-F)=X-f^{-1}(F)$ is $\delta-open$. Therefore, $f^{-1}(F)$ is $\delta-closed$ in X.

$(2)\Rightarrow(3)$: Since $Cl(V)$ is closed for every $V\subseteq Y$, then $f^{-1}\left[Cl(V)\right]$ is $\delta-closed$. Therefore $f^{-1}\left[Cl(V)\right]=Cl_\delta\left[f^{-1}(Cl(V))\right]\supseteq Cl_\delta\left[f^{-1}(V)\right]$.

$(3)\Rightarrow(4)$: Let $U\subseteq X$ and $f(U)=V$. Then $Cl_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Cl(V)\right]$. Thus $Cl_\delta(U)\subseteq Cl_\delta\left[f^{-1}(f(U))\right]\subseteq f^{-1}\left[Cl(f(U))\right]$ and $f\left[Cl_\delta(U)\right]\subseteq Cl\left[f(U)\right]$.

$(4)\Rightarrow(2)$: Let $W\subseteq Y$ be a closed set, and $U=f^{-1}(W)$. Then $f\left[Cl_\delta(U)\right]\subseteq Cl\left[f(U)\right]$ $=Cl\left[f(f^{-1}(W))\right]\subseteq Cl(W)=W$. Thus $Cl_\delta(U)\subseteq f^{-1}\left[f(Cl_\delta(U))\right]\subseteq f^{-1}(W)=U$. So U is $\delta-closed$.

$(2)\Rightarrow(1)$: Let $V\subseteq Y$ be an open set. Then $Y-V$ is closed. Then $f^{-1}(Y-V)=X-f^{-1}(V)$ is $\delta-closed$ in X and hence $f^{-1}(V)$ *is* $\delta-open$ *in* X.

$(1)\Rightarrow(5)$: Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be $\delta-continuous.$ For any $x\in X$ and any open set V of Y containing $f(x)$, $U=f^{-1}(V)\in\tau_\delta$, and $f(U)=f\left[f^{-1}(V)\right]\subseteq V$.

$(5)\Rightarrow(1)$: Let $V\in\sigma$. We prove $f^{-1}(V)\in\tau_\delta$. Let $x\in f^{-1}(V)$. Then $f(x)\in V$ and there exists $U\in\tau_\delta$ such that $x\in U$ and $f(x)\in f(U)\subseteq V$. Hence $x\in U\subseteq f^{-1}\left[f(U)\right]\subseteq f^{-1}(V)$. It shows that $f^{-1}(V)$ is a $\delta-neighborhood$ of each of its points. Therefore $f^{-1}(V)\in\tau_\delta$.

$(6)\Rightarrow(8)$: Let $V\subseteq Y$. Then by hypothesis, $Bd_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Bd(V)\right]$

$\Rightarrow f^{-1}(V)-Int_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[V-Int(V)\right]$
$=f^{-1}(V)-f^{-1}\left[Int(V)\right]$

$\Rightarrow f^{-1}\left[Int(V)\right]\subseteq Int_\delta\left[f^{-1}(V)\right]$.

$(8)\Rightarrow(6)$: Let $V\subseteq Y$. Then by hypothesis, $f^{-1}\left[Int(V)\right]\subseteq Int_\delta\left[f^{-1}(V)\right]$

$\Rightarrow$
$f^{-1}(V)-Int_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}(V)-f^{-1}\left[Int(V)\right]=f^{-1}\left[V-Int(V)\right]$

$\Rightarrow Bd_\delta\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Bd(V)\right]$.

$(1)\Rightarrow(7)$: It is obvious, since f *is* $\delta-continuous$ and by (4) $f\left[Cl_\delta(U)\right]\subseteq Cl\left[f(U)\right]$ for each $U\subseteq X$. So $f\left[D_\delta(U)\right]\subseteq Cl\left[f(U)\right]$.

$(7)\Rightarrow(1)$: Let $U\subseteq Y$ be an open set, $V=Y-U$ and $f^{-1}(V)=W$. Then by hypothesis $f\left[D_\delta(W)\right]\subseteq Cl\left[f(W)\right]$. Thus

$f\left[D_{\delta}\left(f^{-1}(V)\right)\right] \subseteq Cl\left[f\left(f^{-1}(V)\right)\right] \subseteq Cl(V)=V$. Then $D_{\delta}\left[f^{-1}(V)\right] \subseteq f^{-1}(V)$ and $f^{-1}(V)$ *is* $\delta-closed$. Therefore, f *is* $\delta-continuous$.

$(1) \Rightarrow (8)$: Let $V \subseteq Y$. Then $f^{-1}\left[Int(V)\right]$ is $\delta-open$ *in* X. Thus $f^{-1}\left[Int(V)\right] = Int_{\delta}\left[f^{-1}\left(Int(V)\right)\right] \subseteq Int_{\delta}\left[f^{-1}(V)\right]$. Therefore $f^{-1}\left[Int(V)\right] \subseteq Int_{\delta}\left[f^{-1}(V)\right]$.

$(8) \Rightarrow (1)$: Let $V \subseteq Y$ be an open set. Then $f^{-1}(V) = f^{-1}\left[Int(V)\right] \subseteq Int_{\delta}\left[f^{-1}(V)\right]$. Therefore, $f^{-1}(V)$ *is* $\delta-open$. Hence f *is* $\delta-continuous$.

In the next Theorem, $\#\delta-c.$ denotes the set of points x of X for which a function $f:(X,\tau)\rightarrow(Y,\sigma)$ is not $\delta-continuous$.

Theorem 2.3. $\#\delta-c.$ is identical with the union of the $\delta-$ *frontiers* of the inverse images of $\delta-open$ sets containing $f(x)$.

Proof. Suppose that f is not $\delta-continuous$ at a point x of X. Then there exists an open set $V \subseteq Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \tau_{\delta}$ containing x. Hence, we have $U \cap f^{-1}\left(X - f^{-1}(V)\right) \neq \phi$ for every $U \in \tau_{\delta}$ containing x. It follows that $x \in Cl_{\delta}\left[X - f^{-1}(V)\right]$. We also have $x \in f^{-1}(V) \subseteq Cl_{\delta}\left[f^{-1}(V)\right]$. This means that $x \in Fr_{\delta}\left[f^{-1}(V)\right]$. Now, let f be $\delta-continuous$ at $x \in X$ and $V \subseteq Y$ any open set containing $f(x)$. Then, $x \in f^{-1}(V)$ is a $\delta-open$ set of X. Thus. $x \in Int_{\delta}\left[f^{-1}(V)\right]$ and therefore $x \notin Fr_{\delta}\left[f^{-1}(V)\right]$ for every open set V containing $f(x)$.

Remarks 2.4. (1) Every $\delta-continuous$ function is continuous but the converse may not be true.

(2) If a function $f:(X,\tau)\rightarrow(Y,\sigma)$ is $\delta-continuous$ and a function $g:(Y,\sigma)\rightarrow(Z,\vartheta)$ is $\delta-continuous$, then $g o f:(X,\tau)\rightarrow(Z,\vartheta)$ is $\delta-continuous$.

(3) If a function $f:(X,\tau)\rightarrow(Y,\sigma)$ is $\delta-continuous$ and a function $g:(Y,\sigma)\rightarrow(Z,\vartheta)$ is continuous, then $g o f:(X,\tau)\rightarrow(Z,\vartheta)$ is $\delta-continuous$.

(4) Let (X,τ) and (Y,σ) be topological spaces. If $f:(X,\tau)\rightarrow(Y,\sigma)$ is a function, and one of the following

(a) $f^{-1}\left[Int(B)\right] \subseteq Int_{\delta}\left[f^{-1}(B)\right]$ for each $B \subseteq Y$.

(b) $Cl_{\delta}\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Cl(B)\right]$ for each $B \subseteq Y$.

(c) $f\left[Cl_{\delta}(A)\right] \subseteq Cl\left[f(A)\right]$ for each $A \subseteq X$

holds, then f is continuous.

Lemma 2.5. Let $A \subseteq Y \subseteq X$, Y is $\delta-open$ in X and A is $\delta-open$ in Y. Then A is $\delta-open$ in X.

Proof. Since A is $\delta-open$ in Y, there exists a $\delta-open$ set $U \subseteq X$ such that $A = Y \cap U$. Thus A being the intersection of two $\delta-open$ sets in X, is $\delta-open$ in X.

Theorem 2.6. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a mapping and $\{U_i : i \in I\}$ be a cover of X such that $U_i \in \tau_{\delta}$ for each $i \in I$. Then prove that f is $\delta-continuous$.

Proof. Let $V \subseteq Y$ be an open set, then $\left(f|U_i\right)^{-1}(V)$ is $\delta-open$ in U_i for each $i \in I$.

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Since U_i is $\delta-open$ in X for each $i \in I$. So by Lemma 2.5, $\left(f|U_i\right)^{-1}(V)$ is $\delta-open$ in X for each $i \in I$. But, $f^{-1}(V)=\cup\left\{\left(f|U_i\right)^{-1}(V): i \in I\right\}$, then $f^{-1}(V) \in \tau_\delta$ because τ_δ is a topology on X. This implies that f is $\delta-continuous$.

2.2. DELTA – IRRESOLUTE FUNCTIONS

In this section, the functions to be considered are those for which inverses of $\delta-open$ sets are $\delta-open$. We investigate some properties and characterizations of such functions.

Definition 2.7. Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau)\to(Y,\sigma)$ is called $\delta-irresolute$ if the inverse image of each $\delta-open$ set of Y is a $\delta-open$ set in X.

Theorem 2.8. Let $f:(X,\tau)\to(Y,\sigma)$ be a function between topological spaces. Then the following are equivalent:

(1) f is $\delta-irresolute$;

(2) the inverse image of each $\delta-closed$ set in Y is a $\delta-closed$ set in X;

(3) $Cl_\delta\left[f^{-1}(V)\right] \subseteq f^{-1}\left[Cl_\delta(V)\right]$ for every $V \subseteq Y$;

(4) $f\left[Cl_\delta(U)\right] \subseteq Cl_\delta\left[f(U)\right]$ for every $U \subseteq X$;

(5) $f^{-1}\left[Int_\delta(B)\right] \subseteq Int_\delta\left[f^{-1}(B)\right]$ for every $B \subseteq Y$.

Theorem 2.9. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $\delta-irresolute$ if and only if for each point p in X and each $\delta-open$ set B in Y with $f(p)\in B$, there is a $\delta-open$ set A in X such that $p \in A$, $f(A)\subseteq B$.

Proof. Necessity. Let $p \in X$ and $B \in \sigma_\delta$ such that $f(p)\in B$. Let $A=f^{-1}(B)$. Since f is $\delta-irresolute$, A is $\delta-open$ in X. Also $p \in f^{-1}(B)=A$ as $f(p)\in B$. Thus we have $f(A)=f\left[f^{-1}(B)\right]\subseteq B$.

Sufficiency. Let $B \in \sigma_\delta$, let $A=f^{-1}(B)$. We show that A is $\delta-open$ in X. For this let $x \in A$. It implies that $f(x)\in B$. Then by hypothesis, there exists $A_x \in \tau_\delta$ such that $x \in A_x$ and $f(A_x)\subseteq B$. Then $A_x \subseteq f^{-1}\left[f(A_x)\right]\subseteq f^{-1}(B)=A$. Thus $A=\cup\{A_x : x \in A\}$. It follows that A is $\delta-open$ in X. Hence f is $\delta-irresolute$.

Definition 2.10. Let (X,τ) be a topological space. Let $x \in X$ and $N \subseteq X$. We say that N is a $\delta-neighborhood$ of x if there exists a $\delta-open$ set M of X such that $x \in M \subseteq N$.

Theorem 2.11. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $\delta-irresolute$ if and only if for each x in X, the inverse image of every $\delta-neighborhood$ of $f(x)$, is a $\delta-neighborhood$ of x.

Proof. Necessity. Let $x \in X$ and let B be a $\delta-neighborhood$ of $f(x)$. Then there exists $U \in \sigma_\delta$ such that $f(x)\in U \subseteq B$. This implies that $x \in f^{-1}(U)\subseteq f^{-1}(B)$. Since f is $\delta-irresolute$, so $f^{-1}(U)\in\tau_\delta$. Hence $f^{-1}(B)$ is a $\delta-neighborhood$ of x.

Sufficiency. Let $B \in \sigma_\delta$. Put $A=f^{-1}(B)$. Let $x \in A$. Then $f(x)\in B$. But then, B being $\delta-open$ set, is a $\delta-neighborhood$ of $f(x)$. So by hypothesis, $A=f^{-1}(B)$ is a $\delta-neighborhood$ of x. Hence by definition, there exists $A_x \in \tau_\delta$ such that $x \in A_x \subseteq A$. Thus

$A = \cup\{A_x : x \in A\}$. It follows that A is a $\delta-open$ set in X. Therefore f is $\delta-irresolute.$

Theorem 2.12. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $\delta-irresolute$ if and only if for each x in X. and each $\delta-neighborhood$ U of $f(x)$, there is a $\delta-neighborhood$ V of x such that $f(V)\subseteq U$.

Proof. Necessity. Let $x\in X$ and let U be a $\delta-neighborhood$ of $f(x)$. Then there exists $O_{f(x)}\in\sigma_\delta$ such that $f(x)\in O_{f(x)}\subseteq U$. It follows that $x\in f^{-1}\left[O_{f(x)}\right]\subseteq f^{-1}(U)$. By hypothesis, $f^{-1}\left[O_{f(x)}\right]\in\tau_\delta$. Let $V=f^{-1}(U)$. Then it follows that V is a $\delta-neighborhood$ of x and $f(V)=f\left[f^{-1}(U)\right]\subseteq U$.

Sufficiency. Let $B\in\sigma_\delta$. Put $O=f^{-1}(B)$. Let $x\in O$. Then $f(x)\in B$. Thus B is a $\delta-neighborhood$ of $f(x)$. So by hypothesis, there exists a $\delta-neighborhood$ V_x of x such that $f(V_x)\subseteq B$. Thus it follows that $x\in V_x\subseteq f^{-1}\left[f(V_x)\right]\subseteq f^{-1}(B)=O$. Since V_x is a $\delta-neighborhood$ of x, so there exists an $O_x\in\tau_\delta$ such that $x\in O_x\subseteq V_x$. Hence $x\in O_x\subseteq O$, $O_x\in\tau_\delta$. Thus $O=\cup\{O_x : x\in O\}$. It follows that O is $\delta-open$ in X. Therefore, f is $\delta-irresolute.$

Theorem 2.13. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $\delta-irresolute$ if and only if $f\left[D_\delta(A)\right]\subseteq f(A)\cup D_\delta\left[f(A)\right]$, for all $A\subseteq X$.

Proof. Necessity. Let $f:(X,\tau)\to(Y,\sigma)$ be $\delta-irresolute.$ Let $A\subseteq X$, and $a_0\in D_\delta(A)$. Assume that $f(a_0)\notin f(A)$ and let V denote a $\delta-neighborhood$ of $f(a_0)$. Since f is $\delta-irresolute$, so by Theorem 2.12, there exists a $\delta-neighborhood$ U of a_0 such that $f(U)\subseteq V$. From $a_0\in D_\delta(A)$, it follows that $U\cap A\neq\phi;$ there exists, therefore, at least one element $a\in U\cap A$ such that $f(a)\in f(A)$ and $f(a)\in f(V)$. Since $f(a_0)\notin f(A)$, we have $f(a)\neq f(a_0)$. Thus every $\delta-neighborhood$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$, consequently, $f(a_0)\in D_\delta\left[f(A)\right]$. This proves necessity of the condition.

Sufficiency. Assume that f is not $\delta-irresolute.$ Then by Theorem 2.12, there exists $a_0\in X$ and a $\delta-neighborhood$ V of $f(a_0)$ such that every $\delta-neighborhood$ U of a_0 contains at least one element $a\in U$ for which $f(a)\notin V$. Put $A=\{a\in X : f(a)\notin V\}$. Then $a_0\notin A$ since $f(a_0)\in V$, and therefore $f(a_0)\notin A;$ also $f(a_0)\notin D_\delta\left[f(A)\right]$ since $V\cap\left(V-\{f(a_0)\}\right)=\phi$. It follows that $f(a_0)\in f\left[D_\delta(A)\right]-\left[f(A)\cup D_\delta\left(f(A)\right)\right]\neq\phi$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.

Theorem 2.14. Let $f:(X,\tau)\to(Y,\sigma)$ be a one-to-one function. Then f is $\delta-irresolute$ if and only if $f\left[D_\delta(A)\right]\subseteq D_\delta\left[f(A)\right]$, for all $A\subseteq X$.

Proof. Necessity. Let f be $\delta-irresolute.$ Let $A\subseteq X$, $a_0\in D_\delta(A)$ and V be a $\delta-neighborhood$ of $f(a_0)$. Since f is $\delta-irresolute,$ so by Theorem 2.12, there exists a $\delta-neighborhood$ U of a_0 such that $f(U)\subseteq V$. But $a_0\in D_\delta(A);$ hence there exists an element $a\in U\cap A$ such that $a\neq a_0;$ then $f(a)\in f(A)$ and, since f is one to one, $f(a)\neq f(a_0)$. Thus every $\delta-neighborhood$ V of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0);$ consequently $f(a_0)\in D_\delta\left[f(A)\right]$. We have therefore $f\left[D_\delta(A)\right]\subseteq D_\delta\left[f(A)\right]$.

Sufficiency. Follows from Theorem 2.13.

2.3. DELTA – OPEN FUNCTIONS

The purpose of this section is to investigate some characterizations of $\delta-open$ mappings.

Definition 2.15. Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau)\to(Y,\sigma)$ is called $\delta-open$ if for every open set G in X, $f(G)$ is a $\delta-open$ set in Y.

Theorem 2.16. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $\delta-open$ if and only if for each $x\in X$, and $U\in\tau$ such that $x\in U$, there exists a $\delta-open$ set $W\subseteq Y$ containing $f(x)$ such that $W\subseteq f(U)$.

Proof. Follows immediately from Definition 2.15.

Theorem 2.17. Let $f:(X,\tau)\to(Y,\sigma)$ be $\delta-open.$ If $W\subseteq Y$ and $F\subseteq X$ is a closed set containing $f^{-1}(W)$, then there exists a $\delta-closed$ $H\subseteq Y$ containing W such that $f^{-1}(H)\subseteq F$.

Proof. Let $H=Y-f(Y-F)$. Since $f^{-1}(W)\subseteq F$, we have $f^{-1}(Y-F)\subseteq(Y-W)$. Since f is $\delta-open,$ then H is $\delta-closed$ and $f^{-1}(H)=X-f^{-1}\left[f(X-F)\right]\subseteq X-(X-F)=F$.

Theorem 2.18. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta-open$ function and let $B\subseteq Y$. Then $f^{-1}\left[Cl_\delta\left(Int_\delta\left(Cl_\delta(B)\right)\right)\right]\subseteq Cl\left[f^{-1}(B)\right]$.

Proof. $Cl\left[f^{-1}(B)\right]$ is closed in X containing $f^{-1}(B)$. By Theorem 2.17, there exists a $\delta-closed$ set $B\subseteq H\subseteq Y$ such that $f^{-1}(H)\subseteq Cl\left[f^{-1}(B)\right]$. Thus,

$$f^{-1}\left[Cl_\delta\left(Int_\delta\left(Cl_\delta(B)\right)\right)\right]\subseteq f^{-1}\left[Cl_\delta\left(Int_\delta\left(Cl_\delta(H)\right)\right)\right]$$
$$\subseteq f^{-1}[H]\subseteq Cl\left[f^{-1}(B)\right].$$

Theorem 2.19. Prove that a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is $\delta-open$ if and only if $f\left[Int(A)\right]\subseteq Int_\delta\left[f(A)\right]$, *for all* $A\subseteq X$.

Proof. Necessity. Let $A\subseteq X$. Let $x\in Int(A)$. Then there exists $U_x\in\tau$ such that $x\in U_x\subseteq A$. So $f(x)\in f(U_x)\subseteq f(A)$. and by hypothesis, $f(U_x)\in\sigma_\delta$. Hence $f(x)\in Int_\delta\left[f(A)\right]$. Thus $f\left[Int(A)\right]\subseteq Int_\delta\left[f(A)\right]$.

Sufficiency. Let $U\in\tau$. Then by hypothesis, $f\left[Int(U)\right]\subseteq Int_\delta\left[f(U)\right]$. Since $Int(U)=U$ as U is open. Also $Int_\delta\left[f(U)\right]\subseteq f(U)$. Hence $f(U)=Int_\delta\left[f(U)\right]$. Thus $f(U)$ is $\delta-open$ open in Y. So f is $\delta-open.$

Remark 2.20. The equality may not hold in the preceding Theorem.

Theorem 2.21. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $\delta-open$ if and only if $Int\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Int_\delta(B)\right]$, for all $B\subseteq Y$.

Proof. Necessity. Let $B\subseteq Y$. Since $Int\left[f^{-1}(B)\right]$ is open in X and f is $\delta-open,$ $f\left[Int\left(f^{-1}(B)\right)\right]$ is $\delta-open$ in Y. Also we have $f\left[Int\left(f^{-1}(B)\right)\right]\subseteq f\left[f^{-1}(B)\right]\subseteq B.$ Hence, $f\left[Int\left(f^{-1}(B)\right)\right]\subseteq Int_\delta(B)$. Therefore $Int\left(f^{-1}(B)\right)\subseteq f^{-1}\left[Int_\delta(B)\right]$.

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Sufficiency. Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $Int(A) \subseteq Int\left[f^{-1}\left(f(A)\right)\right] \subseteq f^{-1}\left[Int_{\delta}\left(f(A)\right)\right]$. Thus $f\left[int(A)\right] \subseteq Int_{\delta}\left[f(A)\right]$, for all $A \subseteq X$. Hence, by Theorem 2.19, f is $\delta-open$.

Theorem 2.22. Let $f:(X,\tau) \to (Y,\sigma)$ be a mapping. Then a necessary and sufficient condition for f to be $\delta-open$ is that $f^{-1}\left[Cl_{\delta}(B)\right] \subseteq Cl\left[f^{-1}(B)\right]$ for every subset B of Y.

Proof. Necessity. Assume f is $\delta-open$. Let $B \subseteq Y$. Let $x \in f^{-1}\left[Cl_{\delta}(B)\right]$. Then $f(x) \in Cl_{\delta}(B)$. Let $U \in \tau$ such that $x \in U$. Since f is $\delta-open$, then $f(U)$ is a $\delta-open$ set in Y. Therefore, $B \cap f(U) \neq \phi$. Then $U \cap f^{-1}(B) \neq \phi$. Hence $x \in Cl\left[f^{-1}(B)\right]$. We conclude that $f^{-1}\left[Cl_{\delta}(B)\right] \subseteq Cl\left[f^{-1}(B)\right]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}\left[Cl_{\delta}(Y-B)\right] \subseteq Cl\left[f^{-1}(Y-B)\right]$. This implies that $X - Cl\left[f^{-1}(Y-B)\right] \subseteq X - f^{-1}\left[Cl_{\delta}(Y-B)\right]$. Hence $X - Cl\left[X - f^{-1}(B)\right] \subseteq f^{-1}\left[Y - Cl_{\delta}(Y-B)\right]$. By applying Theorem 10[18], $Int\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Int_{\delta}(B)\right]$. Now form Theorem 2.21, it follows that f is $\delta-open$.

2.4. DELTA – CLOSED FUNCTIONS

In this section we introduce $\delta-closed$ functions and study certain properties and characterizations of this type of functions.

Definition 2.23. A mapping $f:(X,\tau) \to (Y,\sigma)$ is called $\delta-closed$ if the image of each closed set in X is a $\delta-closed$ set in Y.

Theorem 2.24. Prove that a mapping $f:(X,\tau) \to (Y,\sigma)$ is $\delta-closed$ if and only if $Cl_{\delta}\left[f(A)\right] \subseteq f\left[Cl(A)\right]$ for each $A \subseteq X$.

Proof. Necessity. Let f be $\delta-closed$ and let $A \subseteq X$. Then $f(A) \subseteq f\left[Cl(A)\right]$ and $f\left[Cl(A)\right]$ is a $\delta-closed$ set in Y. Thus $Cl_{\delta}\left[f(A)\right] \subseteq f\left[Cl(A)\right]$.

Sufficiency. Suppose that $Cl_{\delta}\left[f(A)\right] \subseteq f\left[Cl(A)\right]$, for each $A \subseteq X$. Let $A \subseteq X$ be a closed set. Then $Cl_{\delta}\left[f(A)\right] \subseteq f\left[Cl(A)\right] = f(A)$. This shows that $f(A)$ is a $\delta-closed$ set. Hence f is $\delta-closed$.

Theorem 2.25. Let $f:(X,\tau) \to (Y,\sigma)$ be $\delta-closed$. If $V \subseteq Y$ and $E \subseteq X$ is an open set containing $f^{-1}(V)$, then there exists a $\delta-open$ set $G \subseteq Y$ containing V such that $f^{-1}(G) \subseteq E$.

Proof. Let $G = Y - f(X-E)$. Since $f^{-1}(V) \subseteq E$, we have $f(X-E) \subseteq Y - V$. Since f is $\delta-closed$, then G is a $\delta-open$ set and $f^{-1}(G) = X - f^{-1}\left[f(X-E)\right] \subseteq X - (X-E) = E$.

Theorem 2.26. Suppose that $f:(X,\tau) \to (Y,\sigma)$ is a $\delta-closed$ mapping. Then $Int_{\delta}\left[Cl_{\delta}\left(f(A)\right)\right] \subseteq f\left[Cl(A)\right]$ for every subset A of X.

Proof. Suppose f is a $\delta-closed$ mapping and A is an arbitrary subset of X. Then $f\left[Cl(A)\right]$ is $\delta-closed$ in Y. Then $Int_{\delta}\left[Cl_{\delta}\left(f(Cl(A))\right)\right] \subseteq f\left[Cl(A)\right]$. But also $Int_{\delta}\left[Cl_{\delta}\left(f(A)\right)\right] \subseteq Int_{\delta}\left[Cl_{\delta}\left(f(Cl(A))\right)\right]$. Hence $Int_{\delta}\left[Cl_{\delta}\left(f(A)\right)\right] \subseteq f\left[Cl(A)\right]$.

Theorem 2.27. Let $f:(X,\tau) \to (Y,\sigma)$ be a $\delta-closed$ function, and $B, C \subseteq Y$.

Proof. (1) If U is an open neighborhood of $f^{-1}(B)$, then there exists a $\delta-open$ neighborhood V of B such that $f^{-1}(B)\subseteq f^{-1}(V)\subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods, so have B and C.

Proof. (1) Let $V=Y-f(X-U)$. Then $V^c=Y-V=f(U^c)$. Since f is $\delta-closed$, so V is a $\delta-open$ set. Since $f^{-1}(B)\subseteq U$, we have $V^c=f(U^c)\subseteq f\left[f^{-1}(B^c)\right]\subseteq B^c$. Hence, $B\subseteq V$, and thus V is a $\delta-open$ neighborhood of B. Further $U^c\subseteq f^{-1}\left[f(U^c)\right]=f^{-1}(V^c)=\left[f^{-1}(V)\right]^c$. This proves that $f^{-1}(V)\subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods M and N, then by (1), we have $\delta-open$ neighborhoods U and V of B and C respectively such that $f^{-1}(B)\subseteq f^{-1}(U)\subseteq Int_\delta(M)$ and $f^{-1}(C)\subseteq f^{-1}(V)\subseteq Int_\delta(N)$. Since M and N are disjoint, so are $Int_\delta(M)$ and $Int_\delta(N)$, hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 2.28. Prove that a surjective mapping $f:(X,\tau)\to(Y,\sigma)$ is $\delta-closed$, if and only if for each subset B of Y and each open set U in X containing $f^{-1}(B)$, there exists a $\delta-open$ set V in Y containing B such that $f^{-1}(V)\subseteq U$.

Proof. Necessity. This follows from (1) of Theorem 2.27.

Sufficiency. Suppose F is an arbitrary closed set in X. Let y be an arbitrary point in $Y-f(F)$. Then $f^{-1}(y)\subseteq X-f^{-1}\left[f(F)\right]\subseteq(X-F)$ and $(X-F)$ is open in X. Hence by hypothesis, there exists a $\delta-open$ set V_y containing y such that $f^{-1}(V_y)\subseteq(X-F)$. This implies that $y\in V_y\subseteq\left[Y-f(F)\right]$. Thus $Y-f(F)=\bigcup\{V_y:y\in Y-f(F)\}$. Hence $Y-f(F)$, being a union of $\delta-open$ sets, is $\delta-open$. Thus its complement $f(F)$ is $\delta-closed$. This shows that f is $\delta-closed$.

Theorem 2.29. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijection. Then the following are equivalent:

(a) f is $\delta-closed$.

(b) f is $\delta-open.$

(c) f^{-1} is $\delta-continuous.$

Proof. $(a)\Rightarrow(b)$: Let $U\in\tau$. Then $X-U$ is closed in X. By (a), $f(X-U)$ is $\delta-closed$ in Y. But $f(X-U)=f(X)-f(U)=Y-f(U)$. Thus $f(U)$ is $\delta-open$ in Y. This shows that f is $\delta-open.$

$(b)\Rightarrow(c)$: Let $U\subseteq X$. be an open set. Since f is $\delta-open.$ So $f(U)=\left(f^{-1}\right)^{-1}(U)$ is $\delta-open$ in Y. Hence f^{-1} is $\delta-continuous.$

$(c)\Rightarrow(a)$: Let A be an arbitrary closed set in X. Then $X-A$ is open in X. Since f^{-1} is $\delta-continuous$, $\left(f^{-1}\right)^{-1}(X-A)$ is $\delta-open$ in Y. But $\left(f^{-1}\right)^{-1}(X-A)=f(X-A)=Y-f(A)$. Thus $f(A)$ is $\delta-closed$ in Y. This shows that f is $\delta-closed$.

Remark 2.30. A bijection $f:(X,\tau)\to(Y,\sigma)$ may be open and closed but neither $\delta-open$ nor $\delta-closed$.

2.5. PRE – DELTA – OPEN FUNCTIONS

The purpose of this section is to introduce and discuss certain properties and characterizations of $pre-\delta-open$ functions.

Definition 2.31. Let (X,τ) and (Y,σ) be topological spaces. Then a function $f:(X,\tau)\to(Y,\sigma)$ is said to be $pre-\delta-open$ if and only if for each $A\in\tau_\delta$, $f(A)\in\sigma_\delta$.

Theorem 2.32. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be any two $pre-\delta-open$ functions. Then the composition function $g\circ f:(X,\tau)\to(Z,\mu)$ is a $pre-\delta-open$ function.

Proof. Let $U\in\tau_\delta$. Then $f(U)\in\sigma_\delta$. Since f is $pre-\delta-open.$ But then $g(f(U))\in\mu_\delta$ as g is $pre-\delta-open.$ Hence, $g\circ f$ is $pre-\delta-open.$

Theorem 2.33. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-open$ if and only if for each $x\in X$ and for any $U\in\tau_\delta$ such that $x\in U$, there exists $V\in\sigma_\delta$ such that $f(x)\in V$ and $V\subseteq f(U)$.

Proof. Routine.

Theorem 2.34. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-open$ if and only if for each $x\in X$ and for any $\delta-neighborhood$ U of x in X, there exists a $\delta-neighborhood$ V of $f(x)$ in Y such that $V\subseteq f(U)$.

Proof. Necessity. Let $x\in X$ and let U be a $\delta-neighborhood$ of x. Then there exists $W\in\tau_\delta$ such that $x\in W\subseteq U$. Then $f(x)\in f(W)\subseteq f(U)$. But $f(W)\in\sigma_\delta$ as f is $pre-\delta-open.$ Hence $V=f(W)$ is a $\delta-neighborhood$ of $f(x)$ and $V\subseteq f(U)$.

Sufficiency. Let $U\in\tau_\delta$. Let $x\in U$. Then U is a $\delta-neighborhood$ of x. So by hypothesis, there exists a $\delta-neighborhood$ $V_{f(x)}$ of $f(x)$ such that $f(x)\in V_{f(x)}\subseteq f(U)$. It follows at once that $f(U)$ is a $\delta-neighborhood$ of each of its points. Therefore $f(U)$ is $\delta-open.$ Hence f is $pre-\delta-open.$

Theorem 2.35. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-open$ if and only if $f\left[Int_\delta(A)\right]\subseteq Int_\delta\left[f(A)\right]$, for all $A\subseteq X$.

Proof. Necessity. Let $A\subseteq X$. Let $x\in Int_\delta(A)$. Then there exists $U_x\in\tau_\delta$ such that $x\in U_x\subseteq A.$ So $f(x)\in f(U_x)\subseteq f(A)$ and by hypothesis, $f(U_x)\in\sigma_\delta$. Hence $f(x)\in Int_\delta\left[f(A)\right]$. Thus $f\left[Int_\delta(A)\right]\subseteq Int_\delta\left[f(A)\right]$.

Sufficiency. Let $U\in\tau_\delta$. Then by hypothesis, $f\left[Int_\delta(U)\right]\subseteq Int_\delta\left[f(U)\right]$. Since $Int_\delta(U)=U$ as U is $\delta-open.$ Also $Int_\delta\left[f(U)\right]\subseteq f(U)$. Hence $f(U)=Int_\delta\left[f(U)\right]$. Thus $f(U)$ is $\delta-open$ in Y. So f is $pre-\delta-open.$

We remark that the equality does not hold in Theorem 2.35 as the following example shows.

Example 2.36. Let $X=Y=\{1,2\}$. suppose X is antidiscrete and Y is discrete. Let $f=Id.$, $A=\{1\}$. Then $\phi=f\left[Int_\delta(A)\right]\neq Int_\delta\left[f(A)\right]=\{1\}$.

Theorem 2.37. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-open$ if and only if $Int_\delta\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Int_\delta(B)\right]$, for all $B\subseteq Y$.

Proof. Necessity. Let $B\subseteq Y$. Since $Int_\delta\left[f^{-1}(B)\right]$ is $\delta-open$ in X and f is $pre-\delta-open,$ $f\left[Int_\delta\left(f^{-1}(B)\right)\right]$ is $\delta-open$ in Y. Also we have $f\left[Int_\delta\left(f^{-1}(B)\right)\right]\subseteq f\left[f^{-1}(B)\right]$ $\subseteq B.$ Hence, $f\left[Int_\delta\left(f^{-1}(B)\right)\right]\subseteq Int_\delta(B)$. Therefore $Int_\delta\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Int_\delta(B)\right]$.

Sufficiency. Let $A\subseteq X$. Then $f(A)\subseteq Y$. Hence by hypothesis, we obtain $Int_\delta(A)\subseteq Int_\delta\left[f^{-1}\left(f(A)\right)\right]\subseteq f^{-1}\left[Int_\delta\left(f(A)\right)\right]$.

This implies that $f\left[Int_\delta(A)\right] \subseteq f\left[f^{-1}\left(Int_\delta\left(f(A)\right)\right)\right] \subseteq Int_\delta\left[f(A)\right]$. Thus $f\left[Int_\delta(A)\right] \subseteq Int_\delta\left[f(A)\right]$, for all $A \subseteq X$. Hence, by Theorem 2.35, f is $pre-\delta-open$.

Theorem 2.38. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-open$ if and only if $f^{-1}\left[Cl_\delta(B)\right] \subseteq Cl_\delta\left[f^{-1}(B)\right]$, for every subset B of Y.

Proof. Necessity. Let $B \subseteq Y$. Let $x \in f^{-1}\left[Cl_\delta(B)\right]$. Then $f(x) \in Cl_\delta(B)$. Let $U \in \tau_\delta$ such that $x \in U$. By hypothesis, $f(U) \in \sigma_\delta$ and $f(x) \in f(U)$. Thus $f(U) \cap B \neq \phi$. Hence $U \cap f^{-1}(B) \neq \phi$. Therefore, $x \in Cl_\delta\left[f^{-1}(B)\right]$, So we obtain $f^{-1}\left[Cl_\delta(B)\right] \subseteq Cl_\delta\left[f^{-1}(B)\right]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}\left[Cl_\delta(Y-B)\right] \subseteq Cl_\delta\left[f^{-1}(Y-B)\right]$. This implies that $X - Cl_\delta\left[f^{-1}(Y-B)\right] \subseteq X - f^{-1}\left[Cl_\delta(Y-B)\right]$. Hence $X - Cl_\delta\left[X - f^{-1}(B)\right] \subseteq f^{-1}\left[Y - Cl_\delta(Y-B)\right]$. By Theorem 2.7(6)[20], $Int_\delta\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Int_\delta(B)\right]$. Now by Theorem 2.37, it follows that f is $pre-\delta-open$.

Theorem 2.39. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be two mappings such that $g\circ f:(X,\tau)\to(Z,\mu)$ is $\delta-irresolute$. Then

(1) If g is a $pre-\delta-open$ injection, then f is $\delta-irresolute$.

(2) If f is a $pre-\delta-open$ surjection, then g is $\delta-irresolute$.

Proof. (1) Let $U \in \sigma_\delta$. Then $g(U) \in \mu_\delta$ since g is $pre-\delta-open$. Also $g\circ f$ is $\delta-irresolute$. Therefore, we have $(g\circ f)^{-1}\left[g(U)\right] \in \tau_\delta$. Since g is an injection, so we have : $(g\circ f)^{-1}\left[g(U)\right] = \left(f^{-1}\circ g^{-1}\right)\left[g(U)\right] = f^{-1}\left[g^{-1}\left(g(U)\right)\right] = f^{-1}(U)$. Consequently $f^{-1}(U)$ is $\delta-open$ in X. This proves that f is $\delta-irresolute$.

(2) Let $V \in \mu_\delta$. Then $(g\circ f)^{-1}(V) \in \tau_\delta$ since $g\circ f$ is $\delta-irresolute$. Also f is $pre-\delta-open$, $f\left[(g\circ f)^{-1}(V)\right]$ is $\delta-open$ in Y. Since f is surjective, we note that $f\left[(g\circ f)^{-1}(V)\right] = \left[f\circ(g\circ f)^{-1}\right](V) = \left[f\circ\left(f^{-1}\circ g^{-1}\right)\right](V) = \left[\left(f\circ f^{-1}\right)\circ g^{-1}(V)\right] = g^{-1}(V)$. Hence g is $\delta-irresolute$.

2.6. PRE – DELTA – CLOSED FUNCTIONS

In this last section, we introduce and explore several properties and characterizations of $pre-\delta-closed$ functions.

Definition 2.40. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $pre-\delta-closed$ if and only if the image set $f(A)$ is $\delta-closed$ for each $\delta-closed$ subset A of X.

Theorem 2.41. The composition of two $pre-\delta-closed$ mappings is a $pre-\delta-closed$ mapping.

Proof. The straight forward proof is omitted.

Theorem 2.42. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $pre-\delta-closed$ if and only if $Cl_\delta\left[f(A)\right] \subseteq f\left[Cl_\delta(A)\right]$ for every subset A of X.

Proof. Necessity. Suppose f is a $pre-\delta-closed$ mapping and A is an arbitrary subset of X. Then $f\left[Cl_\delta(A)\right]$ is $\delta-closed$ in Y. Since $f(A) \subseteq f\left[Cl_\delta(A)\right]$, we obtain $Cl_\delta\left[f(A)\right] \subseteq f\left[Cl_\delta(A)\right]$.

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Sufficiency. Suppose F is an arbitrary $\delta-closed$ set in X. By hypothesis, we obtain $f(F)\subseteq Cl_\delta[f(F)]\subseteq f[Cl_\delta(F)]=f(F)$. Hence $f(F)=Cl_\delta[f(F)]$. Thus $f(F)$ is $\delta-closed$ in Y. It follows that f is $pre-\delta-closed$.

Theorem 2.43. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a $pre-\delta-closed$ function, and $B,C\subseteq Y$.

(1) If U is a $\delta-open$ neighborhood of $f^{-1}(B)$, then there exists a $\delta-open$ neighborhood V of B such that $f^{-1}(B)\subseteq f^{-1}(V)\subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $\delta-open$ neighborhoods, so have B and C.

Proof. (1) Let $V=Y-f(X-U)$. Then $V^c=Y-V=f(U^c)$. Since f is $pre-\delta-closed$, so V is $\delta-open$. Since $f^{-1}(B)\subseteq U$, we have $V^c=f(U^c)\subseteq f[f^{-1}(B^c)]\subseteq B^c$. Hence, $B\subseteq V$, and thus V is a $\delta-open$ neighborhood of B. Further $U^c\subseteq f^{-1}[f(U^c)]=f^{-1}(V^c)=[f^{-1}(V)]^c$. This proves that $f^{-1}(V)\subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $\delta-open$ neighborhoods M and N, then by (1), we have $\delta-open$ neighborhoods U and V of B and C respectively such that $f^{-1}(B)\subseteq f^{-1}(U)\subseteq Int_\delta(M)$ and $f^{-1}(C)\subseteq f^{-1}(V)\subseteq Int_\delta(N)$. Since M and N are disjoint, so are $Int_\delta(M)$ and $Int_\delta(N)$, and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 2.44. Prove that a surjective mapping $f:(X,\tau)\longrightarrow(Y,\sigma)$ is $pre-\delta-closed$ if and only if for each subset B of Y and each $\delta-open$ set U in X containing $f^{-1}(B)$, there exists a $\delta-open$ set V in Y containing B such that $f^{-1}(V)\subseteq U$.

Proof. Necessity. This follows from (1) of Theorem 2.43.

Sufficiency. Suppose F is an arbitrary $\delta-closed$ set in X. Let y be an arbitrary point in $Y-f(F)$. Then $f^{-1}(y)\subseteq X-f^{-1}[f(F)]\subseteq(X-F)$ and $(X-F)$ is $\delta-open$ in X. Hence by hypothesis, there exists a $\delta-open$ set V_y containing y such that $f^{-1}(V_y)\subseteq(X-F)$. This implies that $y\in V_y\subseteq[Y-f(F)]$. Thus $Y-f(F)=\cup\{V_y \mid y\in Y-f(F)\}$. Hence $Y-f(F)$, being a union of $\delta-open$ sets is $\delta-open$. Thus its complement $f(F)$ is $\delta-closed$. This shows that f is $\delta-closed$.

Theorem 2.45. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a bijection. Then the following are equivalent:

(1) f is $pre-\delta-closed$.

(2) f is $pre-\delta-open$.

(3) f^{-1} is $\delta-irresolute$.

Proof. $(1)\Rightarrow(2)$: Let $U\in\tau_\delta$. Then $X-U$ is $\delta-closed$ in X. By (1), $f(X-U)$ is $\delta-closed$ in Y. But $f(X-U)=f(X)-f(U)=Y-f(U)$. Thus $f(U)$ is $\delta-open$ in Y. This shows that f is $pre-\delta-open$.

$(2)\Rightarrow(3)$: Let $A\subseteq X$. Since f is $pre-\delta-open$, so by Theorem 2.38, $f^{-1}\left[Cl_{\delta}\left(f\left(A\right)\right)\right]\subseteq Cl_{\delta}\left[f^{-1}\left(f\left(A\right)\right)\right]$. It implies that $Cl_{\delta}\left[f\left(A\right)\right]\subseteq f\left[Cl_{\delta}\left(A\right)\right]$.

Thus $Cl_{\delta}\left[\left(f^{-1}\right)^{-1}\left(A\right)\right]\subseteq\left(f^{-1}\right)^{-1}\left[Cl_{\delta}\left(A\right)\right]$, for all $A\subseteq X$. Then by Theorem 2.8, it follows that f^{-1} is $\delta-irresolute$.

$(3)\Rightarrow(1)$: Let A be an arbitrary $\delta-closed$ set in X. Then $X-A$ is $\delta-open$ in X. Since f^{-1} is $\delta-irresolute$, $\left(f^{-1}\right)^{-1}\left(X-A\right)$ is $\delta-open$ in Y. But $\left(f^{-1}\right)^{-1}\left(X-A\right)=f\left(X-A\right)=Y-f\left(A\right)$. Thus $f\left(A\right)$ is $\delta-closed$ in Y. This shows that f is $pre-\delta-closed$.

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